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ON EXACT ORBIT SEQUENCES

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Introduction

This note unifies and generalizes the exact orbit sequences of [3], [4] and [5], which have been used in connection with problems of localization in homotopy theory.

Our approach lies between Hilton and Roitberg's use of elementary calculation in each individual case, on the one hand, and Steiner's relatively complicated topological fibrations, on the other. Our middle line uses groupoid theoretical arguments which are more conceptual than Hilton and Roitberg's; are simpler than Steiner's; and have the advantage of providing generalizations without extra cost.

In Section 1 we define the groupoid $G \tilde{\times} M$ associated with a group G acting on a set M. In Section 2, we derive, under a mild condition, the exact orbit sequence (EOS) associated with a commutative diagram (2.1) of group actions; examples are given.

Finally Section 3 deals with the situation where a sequence of groups acts on another sequence of groups. This situation is specialized to complete the unification and generalizations mentioned above and initiated in Section 2.

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1. The groupoid $G \tilde{\times} M$

The following construction goes back to Ehresmann (see [2]).

Let G be a group, M a G-set, i.e. we are given an action $G \times M \to M$ of the group G on the set M. We then have a groupoid $G \tilde{\times} M$ with objects $\mathrm{Ob}\,(G \tilde{\times} M) = M$, morphisms $(g, a)\colon a \to g \cdot a$ for $g \in G$, $a \in M$, and composition

$$(g', a')(g, a) = (g'g, a),$$

where $g, g' \in G$, $a, a' \in M$ with $a' = ga$.

The set of components of $G \tilde{\times} M$, $\pi_0(G \tilde{\times} M)$, is the orbit set M/G of M under the action of G. For any $a \in M$ there is a canonical isomorphism

$$(G \tilde{\times} M)\{a\} \xrightarrow{\cong} G(a), \qquad (g, a) \mapsto g,$$

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of the object group of $G \tilde{\times} M$ at a and the stability (isotropy) group $G(a)$ of a under the action of G, $G(a) = \{g \in G \mid ga = a\}$. We note that if the action is trivial then $\pi_0(G \tilde{\times} M) = M$ and $(G \tilde{\times} M)\{a\} \cong G$.

2. The exact orbit sequence

Throughout this section we assume that G and H are groups, M is a G-set, N an H-set, $\beta\colon G \to H$ a group homomorphism, $\kappa\colon M \to N$ a β-map, i.e. the diagram

$$
\text{(2.1)} \qquad \begin{array}{ccc} G \times M & \xrightarrow{\;\cdot\;} & M \\ {\scriptstyle \beta \times \kappa} \downarrow & & \downarrow {\scriptstyle \kappa} \\ H \times N & \xrightarrow{\;\cdot\;} & N \end{array}
$$

is commutative.

Define $\beta \tilde{\times} \kappa\colon G \tilde{\times} M \to H \tilde{\times} N$ by

$$\kappa\colon M \to N \quad \text{on objects}$$

and

$$(\beta \tilde{\times} \kappa)(g, a) = (\beta(g), \kappa(a)) \quad \text{on morphisms.}$$

$\beta \tilde{\times} \kappa$ is a functor.

2.2. PROPOSITION. *If β is an epimorphism, then $\beta \tilde{\times} \kappa$ is a fibration of groupoids (see [1], 2).*

Proof. Let $a \in M$, $(h, \kappa a)\colon \kappa a \to h \cdot (\kappa a)$ be a morphism of $H \tilde{\times} N$. Since β is surjective, we can choose $g \in G$ with $\beta(g) = h$. Then $(g, a)\colon a \to ga$ lifts $(h, \kappa a)$. ■

Let $a \in M$. Define $F := \kappa^{-1}(a)$, $K := \operatorname{Ker} \beta$.

2.3. PROPOSITION. (i) *F is a K-set by restricting the G-action on M.*
(ii) *The fibre of $\beta \tilde{\times} \kappa$ over κa is $K \tilde{\times} F$.*

Proof. (i) If $k \in K$, $a' \in F$, then $\kappa(ka') = \beta(k) \cdot \kappa(a') = 1 \cdot \kappa(a) = \kappa(a)$. Thus $ka' \in F$.

(ii) Let L denote the fibre of $\beta \tilde{\times} \kappa$ over κa. Then $\operatorname{Ob}(L) = \kappa^{-1}\kappa(a) = F$. The morphisms of L are those pairs (g, a'), $g \in G$, $a' \in M$, such that

$$(\beta \tilde{\times} \kappa)(g, a') = 1_{\kappa(a)}.$$

This is equivalent to $(\beta(g), \kappa(a')) = (1, \kappa(a))$ which means $g \in K$, $a' \in F$. Thus $L = K \tilde{\times} F$. ■

2.4. *Remark.* If β is an epimorphism then by 2.3 we can say that the short exact sequence of groups

$$K \overset{i}{\rightarrowtail} G \overset{\beta}{\twoheadrightarrow} H$$

acts on the exact sequence of pointed sets

$$F \overset{j}{\rightarrow} M \overset{\kappa}{\rightarrow} N$$

with base points $a \in F$, $a \in M$, $\kappa a \in N$, yielding

$$K \tilde{\times} F \xrightarrow{i \tilde{\times} j} G \tilde{\times} M \xrightarrow{\beta \tilde{\times} \kappa} H \tilde{\times} N.$$

The exact sequence of a fibration of groupoids [1, (4.2)] applied to the fibration $\beta \tilde{\times} \kappa$ and the object $a \in M = \mathrm{Ob}\,(G \tilde{\times} M)$ now gives the following result.

2.5. THEOREM. *Let $\beta\colon G \rightarrow H$ be an epimorphism. Then for each $a \in M$ there is an exact orbit sequence of groups and pointed sets*

(EOS)

$$1 \longrightarrow K(a) \xrightarrow{i_*} G(a) \xrightarrow{\beta_*} H(\kappa a) \xrightarrow{\partial} F/K \xrightarrow{j_*} M/G \xrightarrow{\kappa_*} N/H$$

with base points $\mathrm{orb}_K\,(a)$, $\mathrm{orb}_G\,(a)$, $\mathrm{orb}_H\,(\kappa a)$, *where* $\mathrm{orb}_G\,(a)$, *for example, is the G-orbit of M containing a.*

Moreover there is an operation $\circ$ of $H(\kappa a)$ on F/K, given by

$$h \circ \mathrm{orb}_K\,(a') = \mathrm{orb}_K\,(g \cdot a'),$$

where $a' \in F$, $g \in G$ and $h = \beta(g) \in H(\kappa a)$, such that $\partial(h) = h \circ \mathrm{orb}_K\,(a)$.

Further:

(a) *If $h, h' \in H(\kappa a)$, then $\partial(h) = \partial(h')$ if and only if $h^{-1}h' \in \beta_* G(a)$.*

(b) *If $f, f' \in F/K$, then $j_*(f) = j_*(f')$ if and only if there exists $h \in H(\kappa a)$ such that $f' = h \circ f$.*

2.6. *Remark.* (i) If κ is surjective, then so is $\kappa_*\colon M/G \rightarrow N/H$ and the exact orbit sequence can be extended by 1 at the right-hand end.

(ii) By 2.5(a), (b) there is a bijection between $j_*^{-1}(\mathrm{orb}_G\,(a))$ and the set of left cosets $H(\kappa a)/\beta_* G(a)$.

2.7. *Examples.* (1) Let

$$K \overset{i}{\rightarrowtail} G \overset{\beta}{\twoheadrightarrow} H$$

be a short exact sequence of groups, M a G-set, $a \in M$. Apply 2.5 to the constant map $\kappa\colon M \rightarrow 1$. Then we get the exact sequence of [4, Theorem 2.1],

$$1 \rightarrow K(a) \rightarrow G(a) \rightarrow H \rightarrow M/K \rightarrow M/G \rightarrow 1,$$

under the weaker assumption that M is merely a G-set and not a G-group.

(2) Let

$$K \overset{i}{\rightarrowtail} G \overset{\beta}{\twoheadrightarrow} H$$

be a short exact sequence of groups, $a \in K$. Let each of the groups G, H, K act on itself by conjugation. Apply 2·5 to $\kappa = \beta$. Note that β is a β-map. Then we obtain the exact sequence 1 of [5],

$$1 \to C_K(a) \to C_G(a) \to H \to [K] \to [G] \to [H] \to 1,$$

where $C_G(a)$ denotes the centralizer of a in G and $[G]$ is the set of conjugacy classes of G.

3. The case of a sequence of groups acting on a sequence of groups

In this section let

$$K \overset{\alpha}{\rightarrowtail} G \overset{\beta}{\twoheadrightarrow} H$$

be a short exact sequence of groups, M a G-group, N an H-group, $\kappa: M \to N$ a β-homomorphism (i.e. κ is a group homomorphism and a β-map).

We assume from now on that

(3.1) *K operates trivially on M.*

Then there is an induced operation $\circ$ of H on M given by $h \circ b = g \cdot b$ where $h = \beta(g)$ and $b \in M$.

Let $a \in M$; then $F = \kappa^{-1}\kappa(a)$ is the left coset aE of a with respect to $E := \operatorname{Ker} \kappa$. Theorem 2·5 gives us an exact sequence

$$\text{(3.2)} \quad 1 \longrightarrow K \xrightarrow{\subset} G(a) \xrightarrow{\beta_*} H(\kappa a) \xrightarrow{\partial} aE \xrightarrow{j_*} M/G \xrightarrow{\kappa_*} N/H.$$

The operation of $H(\kappa a)$ on aE is the restriction of the operation $\circ$ of H on M.

In (3.2) we replace ∂ by $\partial' = l_a^{-1} \circ \partial$, j_* by $j'_* = j_* \circ l_a$, where $l_a: E \to aE$ is the bijection $l_a(e) = ae$. Then we have the following result.

3.3. THEOREM. *There is an exact 6-term sequence*

$$\text{(3.4)} \quad 1 \longrightarrow K \xrightarrow{\subset} G(a) \xrightarrow{\beta_*} H(\kappa a) \xrightarrow{\partial'} E \xrightarrow{j'_*} M/G \xrightarrow{\kappa_*} N/H$$

with base points $1 \in E$, $\operatorname{orb}_G(a) \in M/G$, $\operatorname{orb}_H(\kappa a) \in N/H$ in which:

(i) *β_* is induced by β, κ_* is induced by κ;*

(ii) *$\partial'(x) = a^{-1} \cdot (x \circ a)$ for $x \in H(\kappa a)$;*

(iii) *$j'_*(b) = \operatorname{orb}_G(ab)$ for $b \in E$;*

(iv) *∂' is a crossed homomorphism; that is*

$$\partial'(xy) = (\partial' x)(x \circ \partial' y) \quad \text{for } x, y \in H(\kappa a);$$

(v) *If $y, y' \in E$, then $j'_*(y) = j'_*(y')$ if and only if there exists $x \in H(\kappa a)$ such that $y' = (\partial' x)(x \circ y)$.*

Proof. Parts (i), (ii), (iii) are immediate from the definitions.

(v) Property (b) of 2.5 carries over to j'_* and the operation $\circ'$ of $H(\kappa a)$ on E given by

$$x \circ' y = l_a^{-1}(x \circ (l_a y)) = a^{-1}(x \circ (ay)) = a^{-1}(x \circ a)(x \circ y) = (\partial' x)(x \circ y).$$

This proves (v).

(iv)
$$\begin{aligned}(\partial' x)(x \circ \partial' y) &= a^{-1}(x \circ a)(x \circ (a^{-1}(y \circ a))) \\ &= a^{-1}(x \circ a)(x \circ a)^{-1}(x \circ (y \circ a)) \\ &= a^{-1}((xy) \circ a) = \partial'(xy). \quad \blacksquare\end{aligned}$$

3.5. *Remark.* Note that the following property (iv)′ which is an immediate consequence of 2.5(a) can also be deduced from 3.3(iv).

(iv)′ *If* $x, y \in H(\kappa a)$, *then* $\partial'(x) = \partial'(y)$ *if and only if*

$$x^{-1}y \in \partial'^{-1}(1) = \beta_* G(a).$$

3.6. *Remark.* Suppose the following condition holds:

(3.7) The action of G on M restricts to an action of G on E and this action is trivial.

Then the induced action $\circ$ of H on M restricts to a trivial action of H on E. It follows that ∂' in (3.4) is a homomorphism and 3.3(v) takes the form:

(v)′ *If* $y, y' \in E$, *then* $j'_*(y) = j'_*(y')$ *if and only if* $y'y^{-1} \in \partial' H(\kappa a)$.

Elementary algebraic arguments show that in this case there is an operation $\circ$ of E on M/G, given by

$$y \circ \mathrm{orb}_G\,(c) = \mathrm{orb}_G\,(yc) \quad \text{for } y \in E,\ c \in M,$$

such that:

(vi) *If* $\gamma, \gamma' \in M/G$, *then* $\kappa_* \gamma = \kappa_* \gamma'$ *if and only if there exists* $y \in E$ *with* $\gamma' = y \circ \gamma$.

3.8. *Examples.* (1) Let M, N be G-groups, $\kappa\colon M \to N$ a G-homomorphism (i.e. 1_G-homomorphism), $a \in M$. Apply Theorem 3.3 to $\beta = 1_G$; hence $K = 1$, and the element $a^{-1} \in M$, and replace $j'_*\colon E \to M/G$ by the composition

$$j'_*\colon E \xrightarrow{\ \iota\ } E \xrightarrow{\ j'_*\ } M/G \xrightarrow{\ \iota'\ } M/G$$

where ι and ι' are the bijections $\iota(x) = x^{-1}$, $\iota'(\mathrm{orb}_G\,(c)) = \mathrm{orb}_G\,(c^{-1})$. We thus obtain the sequence

$$1 \to G(a) \to G(\kappa a) \to E \to M/G \to N/G$$

(Theorem 2.5 of [4]). Note that $G(a^{-1}) = G(a)$ and $G(\kappa a^{-1}) = G(\kappa a)$.

(2) As in 2.7(2) let

$$K \overset{\alpha}{\rightarrowtail} G \overset{\beta}{\twoheadrightarrow} H$$

be a short exact sequence of groups and let G and H act on themselves by conjugation. Let $\gamma \in G$ and K be central in G; then (3.1) and (3.7) hold. We apply Theorem 3.3 and Remark 3.6 to the situation $\kappa = \beta$ and obtain the exact sequence 2 of [5]:

$$1 \longrightarrow K \xrightarrow{\alpha} C_G(y) \xrightarrow{\beta} C_H(\beta y) \xrightarrow{\delta y} K \xrightarrow{\alpha y} [G] \longrightarrow [H] \longrightarrow 1.$$

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