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ORDERS OF CONTACT OF REAL AND COMPLEX SUBVARIETIES

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Introduction

Let X' be a complex manifold. Suppose Ω is an open subset of X'. An important question in several complex variables and partial differential equations is what role the geometry of bd Ω plays in the regularity theory for the Cauchy-Riemann operator. One approach to this regularity theory is the $\bar{\partial}$-Neumann problem. We recall for the reader those aspects of this theory which motivate the basic geometric question discussed here. See [5] for a thorough discussion.

Suppose that $M = \text{bd}\,\Omega$ is a smooth manifold. Consider the differential equation

$$\bar{\partial} u = f \tag{1}$$

where f is a $(0, q + 1)$ form satisfying the necessary compatibility conditions and $\bar{\partial}$ is defined in the sense of distributions. The $\bar{\partial}$-Neumann problem constructs an operator N so that

$$u = \bar{\partial}^* N f \tag{2}$$

solves (1). Suppose that N is a pseudo-local operator, i.e.

$$\text{sing supp }(Nf) \subset \text{sing supp }(f).$$

(Nf is smooth where f is.) Then the solution (2) also satisfies

$$\text{sing supp }(u) \subset \text{sing supp }(f).$$

Pseudolocality for N follows from the following type of subelliptic estimate.

DEFINITION. The $\bar{\partial}$-Neumann problem is ε-subelliptic at p on $(0, q)$ forms if there are constants C, $\varepsilon > 0$ and a neighborhood U of p so that

$$\|u\|_\varepsilon^2 \le C(\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2 + \|u\|^2) \tag{3}$$

for all $(0, q)$ forms $u \in C_0^\infty(U)$. Kohn, aided by a geometric result of Diederich-Fornaess in [3], has proved the following great theorem. See [7] and [8].

THEOREM (Kohn, 1977). *Let M be a real analytic pseudoconvex manifold. The $\bar{\partial}$-Neumann problem is ε-subelliptic at p on $(0, q)$ forms whenever there is no*

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germ of a complex analytic q-dimensional subvariety of X' based at p and contained in M. (Kohn has also proved the converse under additional hypotheses.)

Recently, Catlin has proved the converse of this theorem, and independently Diederich and Pflug have proved the analog of the converse for hypoellipticity. These results will appear in the Proceedings of the Conference on Several Complex Variables held at Princeton in April, 1979.

It is therefore important to consider the following question. Let M be any real subvariety of X'. Let $p \in M$, and suppose that r is a local defining function for M near p. What conditions on r force all q dimensional analytic varieties to have finite order of contact with M at p? An example of such a condition is that the Levi form of r have n-q eigenvalues of one sign at p. Of course this condition depends only on the second order Taylor polynomial of r. We generalize this idea to Taylor polynomials of arbitrary order. Stated imprecisely (See Definition 2), we say that condition F_q holds at p if some Taylor polynomial of a defining function prevents q dimensional complex analytic varieties from having high order of contact with M at p.

The author has considered this question in case $q = 1$ and M is a manifold. See [2]. Related questions were studied by Diederich-Fornaess in [3]. In this paper we give a systematic method for studying condition F_q, although we cannot answer all the difficult algebraic questions. Suppose first that M is the zero set of a real-valued polynomial. We find necessary and sufficient conditions for the non-existence of q-dimensional complex varieties in M. These conditions involve only ideals generated by holomorphic polynomials and parametrized by a unitary group. We then determine when such conditions on a truncation of the Taylor series of the defining function imply condition F_q.

The main results here are Theorems 10 and 12. In Theorem 10, we find necessary and sufficient conditions for F_q in case M is the zero set of a (real-analytic) function with the form

$$p(z, \bar{z}) = 2 \operatorname{Re} (h(z)) + \|f(z)\|^2 - \|g(z)\|^2$$

Here h is a holomorphic function, and f and g are holomorphic maps. By Proposition 7, any real-valued polynomial has this form. Therefore we can apply Theorem 10 to the Taylor polynomials of any defining function. This leads to Theorem 12.

We conclude with some examples and by showing that if the Levi form has n-q eigenvalues of the same sign, then F_q must hold.

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Section 1

Suppose M is a real subvariety of $\mathbf{C}^n$ and that 0 lies in M. We would like to relate the local geometry of M near 0 to the behavior (in terms of order of contact with M) of complex analytic varieties in the ambient space $\mathbf{C}^n$. Let $M = Z(r)$ be the zero set of a smooth real-valued function r. We write $j_k r$ for the k-th order Taylor polynomial of r at 0. Our interest is what analytic and algebraic conditions on some $j_k r$ prevent complex analytic varieties from having infinite order of contact with $Z(r)$. Suppose that $(V, 0)$ is the germ of a q-dimensional complex analytic variety. Then we can find a holomorphic map $z: (\mathbf{C}^q, 0) \to (V, 0)$ so that rank $dz(p) = q$ for p in a dense open subset of a neighborhood of 0 in $\mathbf{C}^q$. (Such a map is guaranteed by the resolution of singularities for example.) We call such a map q-regular. z is a q-regular if and only if some $q \times q$ minor determinant of dz is not the zero power series. Using this definition we extend this notion to formal power series maps. Write t for variables in $\mathbf{C}^q$ and let $z^*r = r(z(t))$ be the pullback map.

Suppose V is actually contained in $Z(r)$. Then z^*r vanishes identically. Given the Taylor coefficients for r at 0, we can consider the equations

$$D^a \bar{D}^b z^* r \,|_{t=0} = 0 \quad \text{for all multi-indices } a \text{ and } b \tag{1}$$

as multi-linear equations for the unknown Taylor coefficients of the map z. We are interested in whether the system of equations (1) has any formal power series solutions, any convergent solutions, and (mainly) when the answer depends on only finitely many derivatives of r. In case $Z(r)$ is a strongly pseudo-convex hypersurface (see Def. 2.2) the author shows in [2] that the system (1) has no non-constant solutions. The discussion there motives the following basic definition.

2. Definition. Suppose $r: \mathbf{C}^n \to \mathbf{R}$ is a smooth function, and $0 \in Z(r)$. We say that 0 *is a point of q-finite type for* $Z(r)$, or that F_q *holds at* 0, if there is an integer k with the following property: Whenever r' is any smooth function satisfying $j_k r = j_k r'$, and z is any q-regular formal power series map, then z^*r' vanishes to finite order. In other words, the system of equations (1) has no solution whenever r is replaced by any r' satisfying $j_k r = j_k r'$.

3. *Example.* Let

$$r(z) = 2 \operatorname{Re}(z_n) + \sum_1^{n-q} |z_i|^2 + O(|z|^3).$$

Then F_q holds at 0. We can take $k = 2$. See Proposition 2.3.

4. *Remarks.* (1) It is essential that we consider q-regular maps rather than simply formal maps which have rank q at 0. It is possible for there to be singular solutions to equations (1), but no non-singular solutions. See Example 14.2.

(2). For the definition of F_q we could also allow only holomorphic maps, since everything vanishes to finite order. In this paper we will not attempt to distinguish the various infinite order possibilities. However, the reader should observe (see [1]) that there may be complex analytic varieties tangent to $Z(r)$ to arbitrarily high order, but none tangent to infinite order.

(3) If M is a given subvariety, the existence of such a number k for some defining function r does not depend upon r. However the integer k does depend on r, since we do not assume that $dr \neq 0$. Therefore we will always think of the defining function as fixed.

An obvious necessary condition for F_q is the existence of an integer k such that F_q holds for $Z(j_k r)$. Therefore we must study what F_q means for polynomially defined surfaces. We first need the following propositions. Let $\mathscr{U}(N)$ denote the group of unitary matrices on $\mathbf{C}^N$.

5. PROPOSITION. *Let $f, g\colon B \subset \mathbf{C}^n \to \mathbf{C}^N$ be holomorphic maps of some ball about* 0. *Suppose that $\|f(z)\| = \|g(z)\|$ for every $z \in B$. Then there is some $U \in \mathscr{U}(N)$ for which $f(z) = U \cdot g(z)$ for all $z \in B$. Here of course, $\|f\|^2 = \sum |f_j|^2$.*

Proof. We expand f and g in convergent power seies in B:

$$f(z) = \sum f_a z^a \quad \text{and} \quad g(z) = \sum g_a z^a,$$

where a is a multi-index and the coefficients are elements of $\mathbf{C}^N$. Let $\langle\ ,\ \rangle$ denote the usual Hermitian inner product on $\mathbf{C}^N$. Since $\|f\|^2$ and $\|g\|^2$ are real analytic functions, we equate their Taylor coefficients. This gives

$$(*) \qquad \langle f_a, f_b\rangle = \langle g_a, g_b\rangle \quad \text{for all } a, b.$$

Consider a maximal linearly independent set $G \subset \{g_a\}$. We define a map U by $U \cdot g_a = f_a$ if $g_a \in G$, and otherwise extend by linearity. We claim that U restricted to the span of G is unitary. It is enough to show that $\langle Ug_a, Ug_b\rangle = \langle g_a, g_b\rangle$ for all a, b. This is immediate from (*) and the definition of U. Finally we extend U to be unitary on all of $\mathbf{C}^N$. We claim that $f(z) = U \cdot g(z)$ on B. It is enough to show that $Ug_a = f_a$ for all a. This is true by definition on G, and we can express any other g_a as a linear combination of those on G. The coefficients are completely determined by the inner product, which is preserved by (*). Therefore the same coefficients work for the f_a, and $Ug_a = f_a$. Therefore $f(z) = U \cdot g(z)$ for some U.

Remark. It has been pointed out to the author that Calabi [9] and later Cowen and Douglas [10] have proved versions of the above proposition. The proofs are essentially the same.

6. COROLLARY. *Suppose that f, g are holomorphic or formal power series maps and that $j_{2k}\|f\|^2 = j_{2k}\|g\|^2$. Then there is some $U \in \mathscr{U}(N)$ so that $j_k f = U \cdot j_k g$.*

Proof. The hypothesis implies that $\|j_k f\|^2 = \|j_k g\|^2$, so we can apply Proposition 5.

7. PROPOSITION (Holomorphic decomposition). *Suppose $p\colon \mathbf{C}^n \to \mathbf{R}$ is a real-valued polynomial with $p(0) = 0$. Then there are holomorphic polynomials h, $f_1, \ldots, f_N, g_1, \ldots, g_N$ so that all of the following hold:*

(1) $p(z, \bar{z}) = 2 \operatorname{Re} h(z) + \|f(z)\|^2 - \|g(z)\|^2$

(2) *h is unique, and if $\{f'\} \subset \{f\}$ and $\{g'\} \subset \{g\}$ are such that $\|f'\| = \|g'\|$, then $f'_k = g'_k = 0$ for all k.*

(3) *h and the components of f and g all vanish at 0.*

(4) *N depends only on n and the degree of p.*

Proof. Let $p(z, \bar{z}) = \sum c_{ab} z^a \bar{z}^b$. Let $h(z) = \sum c_{ao} z^a$, and let

$$\begin{aligned} w(z, \bar{z}) &= p(z, \bar{z}) - 2 \operatorname{Re} h(z) \\ &= \sum_1 c_{ab} z^a \bar{z}^b \\ &= \sum \mu_b(z) \bar{z}^b \\ &= \sum |\gamma_b \mu_b(z) + \eta_b z^b|^2 - |\gamma_b \mu_b(z)|^2 - |\eta_b z^b|^2 \end{aligned}$$

where $\mu_b(z)$ is holomorphic, and $\gamma_b \bar{\eta}_b = \frac{1}{2}$. Equality holds since w is real valued. We discard all those f's and g's for which the conclusion of (2) is false. Since h is clearly unique we see that (2) holds. (3) is obvious. By choosing some of the functions to be identically 0 we arrange that there are the same number of f's and g's. Given the second part of (2), we see that (4) holds. When all the above properties hold we call $\langle h, f, g \rangle$ a holomorphic decomposition for p.

The idea of Proposition 7 is to reduce everything to questions about holomorphic functions. In case r is a real analytic function for which there are finitely many holomorphic functions so that

$$r(z, \bar{z}) = 2 \operatorname{Re} h(z) + \|f(z)\|^2 - \|g(z)\|^2,$$

we also call this a holomorphic decomposition.

8. Definitions. Let $\mathcal{O}_n$ denote the ring of germs of holomorphic functions at 0 in $\mathbf{C}^n$. Let $\mathcal{M}$ be the maximal ideal of non units, and $\mathcal{M}^k$ the ideal of germs which vanish to order at least k. If $J \subset \mathcal{M}$ is an ideal, we write $V(J)$ for the variety of the ideal. Suppose $\langle h, f, g \rangle$ is a holomorphic decomposition of some (real analytic) real-valued r. We let $I_U = (h, f - Ug)$ be the ideal in $\mathcal{O}_n$ generated by h and the components of $f - Ug$, where $U \in \mathcal{U}(N)$. We denote by $\mathcal{I}$ the family of such ideals. Notice that we need part (2) of Proposition 7 for $\mathcal{I}$ to make sense.

9. Proposition. *Let $\langle h, f, g \rangle$ be a holomorphic decomposition for a real analytic function $r\colon \mathbf{C}^n \to \mathbf{R}$. Suppose $z\colon (\mathbf{C}^q, 0) \to (\mathbf{C}^n, 0)$ satisfies $z^*r = 0$. Then there is some $U \in \mathcal{U}(N)$ for which $z^*h = z^*(f - Ug) = 0$.*

Proof. We have

$$0 = z^*h + \overline{z^*h} + \|z^*f\| - \|z^*g\|^2.$$

The last three terms all contain anti-holomorphic factors. Apply any holomorphic derivative D^a and evaluate at 0. We get $0 = D^a(z^*h)|_{t=0}$ for all a. Since z^*h is the germ of a holomorphic function, it must be 0. Therefore $\|z^*f\|^2 = \|z^*g\|^2$.

By Proposition 5, there is some U for which $z^*f = U \cdot z^*g$. Since U is constant, $z^*(f - Ug) = 0$.

We are now in a position to determine necessary and sufficient conditions for F_q in case r has the form in Proposition 9. By Proposition 7 any real-valued polynomial has this form. From the proof we also derive several sufficient conditions on $j_k r$ implying that F_q holds for $Z(r)$.

10. THEOREM. *Suppose $\langle h, f, g\rangle$ is a holomorphic decomposition for r. Then the following are equivalent:*

(a) *F_q holds at* 0.
(b) *$z^*r = 0$ has no q-regular formal or holomorphic solutions.*
(c) *$\dim V(I_U) < q$ for every $I_U \in \mathscr{I}$.*

Proof. By Proposition 10 we see that $z^*r = 0$ if and only if $z^*h = 0$ and $z^*(f - Ug) = 0$ for some $U \in \mathscr{U}(N)$. If some dim $V(I_U)$ were greater than or equal to q, we could select a q-regular z with $z^*r = 0$. Conversely if $z^*h = 0$ and $z^*(f - Ug) = 0$ for some q-regular z, then, dim $(I_U) \geq q$. Therefore (b) and (c) are equivalent. Also, that (a) implies (b) is obvious. The hard part is to show that (c) implies (a). We first assume that $q = 1$. Then c says that $V(I_U) = \{0\}$ for all U. Therefore $\mathcal{O}/I_U$ is a finite-dimensional complex vector space, say of dimension $d(U)$. Since $d(U)$ is the degree of the finite analytic mapping defined by $(h, f - Ug)$, it is an upper semi-continuous function on $\mathscr{U}(N)$. Since $\mathscr{U}(N)$ is compact, $d(U)$ attains a maximum on $\mathscr{U}(N)$. Now let $k(U)$ denote the smallest integer for which $\mathscr{M}^{k(U)} \subset I_U$. $k(U)$ is finite if and only if $d(U)$ is, by the Nullstellensatz. Furthermore if $k(U)$ were not uniformly bounded on $\mathscr{U}(N)$, then neither would be $d(U)$; therefore we must have an integer k such that $\mathscr{M}^k \subset I_U$ for every U. (We remark at this point that $k(U)$ is not semi-continuous. For example, put

$$f(z) = (z_1^2 + z_2, z_2^3) \quad \text{and} \quad g(z) = (z_2, 0).$$

For unitary matrices of the form

$$U = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}$$

we have that $k(U) = 4$ if $e^{i\theta} = 1$, but $k(U) = 6$ otherwise. Even simpler examples show that $k(U)$ can jump the other way.)

We claim that $2k$ is the integer needed in the definition of finite type. Let $z: (\mathbf{C}, 0) \to (\mathbf{C}^n, 0)$ be 1-regular, i.e. non-constant. Then there is a smallest integer m for which $j_m z$ is non-zero. We say that z is of order m. We will show that $j_{2km} z^*r' \neq 0$, whenever $j_{2k} r' = j_{2k} r$. Since z has order m we clearly have that

$$j_{2km}(z^*r') = j_{2km}(z^*(j_{2km} r)) = j_{2km}(z^*r).$$

Suppose that $j_{2km} z^* r = 0$. Then $j_{2km}(z^* h) = 0$ and $j_{2km} \|z^* f\|^2 = j_{2km} \|z^* g\|^2$, since pure and mixed terms are independent. By Corollary 6, $j_{km} h = 0$, and for some U, $j_{km}(z^*(f - Ug)) = 0$. Recall however that $\mathcal{M}^k \subset I_U$ for every U. Since h and $f - Ug$ generate I_U, and z has order m, we must have that $z^* h$ or $z^*(f - Ug)$ vanishes to order less that or equal to km. This contradiction proves that F_1 holds. For the case of general q we no longer have $j_{2km}(z^* r^1) = j_{2km}(z^* r)$, if m is the smallest integer for which $j_m z$ is q-regular. However the left side cannot vanish when the right side does not, since we simply restrict both sides to an appropriate 1-dimensional complex line, and apply the previous argument. By the same reasoning as in the case $q = 1$, we claim $j_{2km} z^* r$ cannot vanish. We need only apply the following lemma.

11. LEMMA. *Suppose* $w\colon (\mathbf{C}^n, 0) \to (\mathbf{C}^N, 0)$ *and* $\dim V(w) = q - 1$. *Then there is an integer* k *so that if* $j_k \eta = 0$, *then* $\dim V(w + \eta) \le q - 1$.

Proof. When $q = 1$ the integer k is the smallest integer for which $\mathcal{M}^k \subset (w)$. Namely we have $\mathcal{M}^k \subset (w) \subset (w + \eta) + \mathcal{M}^k$. By Nakayama's lemma, $\mathcal{M}^k \subset (w + \eta)$. By symmetry, $(w) = (w + \eta)$, and $\dim V(w + \eta) = 0$ also. For general q we choose a projection $p\colon \mathbf{C}^n \to \mathbf{C}^{n-q+1}$ so that $\dim V(p^* w) = 0$, when considered as a subvariety of $\mathbf{C}^{n-q+1}$. Then $\dim V(p^*(w + \eta)) = 0$ by the case $q = 1$, and $\dim V(w + \eta) \le q - 1$.

By Lemma 11, we see that the conditions (a), (b), (c) in Theorem 10 depend on only sufficiently high polynomial truncations of the functions h, f, and g.

12. THEOREM. *Suppose* r *is a* C^∞ *function and that* $r(0) = 0$. *Let* $\mathcal{I}_k$ *denote the family of ideals coming from a holomorphic decomposition of* $j_k r$. *For* F_q *to hold at* 0, *it is necessary that there be an integer* k_o *so that for all* k *with* $k \ge k_o$, $\dim V(I_U) < q$ *for every* $I_U \in \mathcal{I}_k$. *Furthermore, for* F_q *to hold at* 0, *it is sufficient that there be an integer* k *with the following property: Whenever* $z\colon (\mathbf{C}^q, 0) \to (\mathbf{C}^n, 0)$, *and* m *is the smallest integer for which* $j_m z$ *is* q-*regular, we have* $j_{mk} z^* w \ne 0$ *for some* w. *Here* w *is one of the holomorphic polynomials* h *or* (*a component of*) $f - Ug$, *coming from a holomorphic decomposition of* $j_k r$.

Proof. The necessity follows from Theorem 10 and the fact that in the definition of F_q we demand control of all C^∞ functions r' satisfying $j_{k_o} r' = j_{k_o} r$. On the other hand, whenever the alleged sufficient condition holds, we can apply the proof of Theorem 10 to see that F_q holds.

13. COROLLARY. *Suppose* r *is as in Theorem* 10. *Then* F_q *holds at* 0 *for* $Z(r)$ *if and only if* F_q *holds at* 0 *for* $Z(r_U)$ *for every* U, *where*

$$r_U = 2\,\mathrm{Re}\,(h) + \|f - Ug\|^2.$$

Furthermore suppose that $Z(r)$ *is a pseudoconvex hypersurface. Then any of the conditions of Theorem* 10 *imply that the* $\bar\partial$-*Neumann of Kohn is* ε-*subelliptic on* $(0, q)$ *forms.*

Proof. The first statement follows from Theorem 10. The second statement follows from Theorem 10 and the result of Kohn mentioned in the introduction. However, in [8, Section 7], Kohn indicates a considerably simpler proof for surfaces defined by

$$s(z) = 2 \operatorname{Re} (z_n) + \|f(z)\|^2.$$

We remark that this simplication extends to this case. First of all, since $Z(r)$ is a hypersurface, we can assume that $h = z_n$. Secondly, by the first part of the corollary it is enough to show that one can obtain the estimates uniformly in the parameter U. This can be done, again using the compactness of $\mathscr{U}(N)$. We omit the details.

Suppose now that r is a C^∞ function, and that for some k, $j_k r$ satisfies the hypotheses of Theorem 10. It need not be true that F_q holds for $Z(r)$. The following examples illustrate the theory quite nicely.

14. *Example 1.* $r(z) = 2 \operatorname{Re} (z_3) + |z_1 - z_2^3|^2 + |z_1^2 - z_2^6|^2$. Consider the 3 cases:

$$\begin{aligned} a &= j_6 r = 2 \operatorname{Re} (z_3) + |z_1 - z_2^3|^2 + |z_1^2|^2 \\ b &= j_{11} r = r(z) - |z_2^6|^2 \\ c &= j_{12} r = r. \end{aligned}$$

Then $z^*a = 0 \Leftrightarrow z = 0$, and $z^*b = 0 \Leftrightarrow z = 0$. However $z^*r = 0$ for $z(t) = (t^3, t, 0)$. For case (a) we see that $I = I_U = (z_3, z_1 - z_2^3, z_1^2)$ for every U. An elementary computation shows that $\mathscr{M}^6 \subset I$. According to the proof of Theorem 10, we see that the 12 jet of a acts as an obstruction. Thus if $j_{12} r' = a$, then z^*r' has no constant solutions. Notice however that r itself satisfies $j_6 r = j_6 a$, but that $z^*r = 0$ has a non-trivial solution.

Let

$$s(z) = 2 \operatorname{Re} (z_3^m) + |z_1^2 - z_2^3|^2 + |z_1^2|^2 - |z_2^3|^2.$$

Then for any integer $m > 1$, $Z(s)$ is not a manifold. For all integers $m \geq 1$, $z(t) = (t^3, t^2, 0)$ is a solution to $z^*s = 0$. This curve is singular; there are no non-singular solutions even when $Z(s)$ is a manifold. Also the reader can verify that $V(I_U) = \{0\}$ for all U except the following 3 cases:

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

Section 2

Given a smooth function r, we now find some conditions on $j_k r$ which imply that F_q holds at 0. All the ideas are present in the case $q = 1$, so we state explicitly the following Corollary of Theorem 12.

1. PROPOSITION. *Suppose r is C^∞, $0 \in Z(r)$, and that for some k, $\mathcal{M}^k \subset I_U$ for every $I_U \in \mathcal{I}_{2k}$. Then F_1 holds at 0.*

Proof. The containment $\mathcal{M}^k \subset I_U$ implies the condition of Theorem 12.

2. DEFINITION. Let $Z(r)$ be a real hypersurface in $\mathbf{C}^n$. The Levi form λ is the Hermitian form $\lambda(L, M) = \langle \partial\bar{\partial} r, L \wedge \bar{M} \rangle$ restricted to those type (1, 0) vector fields tangent to $Z(r)$. $Z(r)$ is called strongly pseudoconvex if λ is positive definite, and pseudoconvex if λ is positive semi-definite. (If λ is negative definite, the Levi form corresponding to $-r$ will be positive definite).

3. PROPOSITION. *F_q holds at any point where λ has $n - q$ eigenvalues of the same sign.*

Proof. After a change of coordinates, and by considering $(-r)$ if necessary we can assume that

$$j_2 r = 2 \operatorname{Re}(z_n) + \sum_1^{n-q} |z_k|^2 + O(|z_n z|) + O(|z'|).$$

where $z' = (z_{n-q+1}, \ldots, z_{n-1})$. It is sufficient to set $z' = 0$, and verify that F_1 holds for the corresponding surface in $\mathbf{C}^{n-q}$. But we see immediately that $I_U = (z_1, \ldots, z_{n-q}, z_n)$, (by taking $k = 1$) for every U. Therefore the result follows from Proposition 2.1.

We remark that the integer k in Proposition 2.1 is not always the best possible. Suppose for example that $p_i \leq p$, and that

$$j_{2p} r = 2 \operatorname{Re}(z_n) + \sum_{i=1}^{n-1} |z_i|^{2p_i}.$$

Then F_1 certainly holds at 0. On the other hand, if $k \leq \sum p_i + 1 - n$, then

$$\mathcal{M}^k \not\subset (z_1^{p_1}, \ldots, z_{n-1}^{p_{n-1}}, z_n) = I_U \quad \text{for all } U.$$

Of course, the more scrambled the variables, the closer to k we must take in general. Also if $Z(r)$ is a hypersurface we can assume h is a coordinate function. Our theory indicates algebraically why the complex tangential and the "bad" tangential directions have different weights. In particular, since h appears to the first power, we need look only at $j_{km} z^* h$; for the $\|f\|^2 - \|g\|^2$ part we must study $j_{2km}(\|z^* f\|^2 - \|z^* g\|^2)$ to obtain information about $j_{km} z^*(f - Ug)$.

We have seen that if $Z(r)$ is pseudoconvex, the condition F_1 is a generalization of strong pseudoconvexity. Neverthless there is no one generalization which is appropriate in all contexts. The following example shows that pseudoconvexity does not depend in general on some $j_k r$.

5. *Example.* Let

$$r(z) = 2 \operatorname{Re}(z_1) + |z_2|^4 + 2/3(z_2 \bar{z}_2^3 + z_2^3 \bar{z}_2).$$

Then F_1 holds at 0 with $k = 2$. The Levi form vanishes when $|z_2|^2 + \operatorname{Re} z_2^2 = 0$. Let f be a smooth function for which $f_{z_2 \bar{z}_2}$ is negative at points on this line arbitrarily close to 0. Then $r + f$ does not define a pseudoconvex surface. We could even take f to have a zero of infinite order.

Pseudoconvexity actually gives very little useful information about condition F_q. Therefore it is not clear whether F_q is really related to the estimates of Kohn, which presently require pseudoconvexity.

6. *Remarks.* To fully understand when some jet of r actually defines an obstruction to solving $j_k(z^*r) = 0$ for all k, is a very difficult question. Consider the following special case. Let $w\colon (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ have the following properties:

$$j_k w = w, \tag{1}$$

$$\mathcal{M}^s \subset (w). \tag{2}$$

Suppose first that $s \leq k$. Then by the proof of Lemma 11, the part using Nakayama's lemma, we have that $(w + \eta) = (w)$ whenever $j_k \eta = 0$. On the other hand, if $s > k$, it may or may not happen that $(w + \eta)$ defines the same ideal. The variety $V(w + \eta)$ may even be positive dimensional. We ask, when is there an η with $j_k \eta = 0$ such that $\dim V(w + \eta) > 0$? This involves understanding the partial differential congruence

$$\det d(w + \eta) \equiv 0 \bmod (w + \eta)$$

$$j_k \eta = 0.$$

The author hopes to investigate this question in later work.

References

1. T. Bloom and I. Graham, *A geometric characterization of points of type m on real submanifolds of C^n*, J. Diff. Geom., vol. 12 (1977), pp. 171–182.
2. E. Calabi, *Isometric imbeddings of complex manifolds*, Ann. of Math., vol. 58 (1953), pp. 1–23.
3. D. Catlin, *Necessary conditions for subellipticity and hypoellipticity of the $\bar{\partial}$-Neumann problem on pseudoconvex domains*, Proc. Conference on Several Complex Variables, Princeton, April, 1979.
4. M. J. Cowen and R. G. Douglas, *Complex geometry and operator theory*, Acta Math., vol. 141 (1978), pp. 187–261.
5. J. D'Angelo, *Finite type conditions for real hypersurfaces*, J. Diff. Geom., vol. 14 (1979), pp. 59–66.
6. K. Diederich and J. Fornaess, *Pseudoconvex domains with real analytic boundary*, Ann. of Math., vol. 107 (1978), pp. 371–384.
7. D. Eisenbud and H. Levine, *An algebraic formula for the degree of a smooth map germ*, Ann. of Math., vol. 106 (1977), pp. 19–44.
8. G. Folland and J. Kohn, *The Neumann problem for the Cauchy Riemann complex*, Ann. of Math. Studies, no. 75, Princeton, N.J., 1972.
9. R. Gunning, *Lectures on complex analytic varieties: finite analytic mappings*, Princeton Univ. Press, Princeton, N.J., 1974.

10. J. Kohn, *Sufficient conditions for subellipticity on weakly pseudoconvex domains*, Proc. Nat. Acad. Sci., vol. 74 (1977), pp. 2214–2216.
11. ———, *Subellipticity of the $\bar{\partial}$-Neumann problem on weakly pseudoconvex domains: sufficient conditions*, Acta Math., vol. 142 (1979), pp. 79–122.

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