ILLINOIS JOURNAL OF MATHEMATICS
Volume 26, Number 3, Fall 1982

LOCALLY LIPSCHITZ CONTINUOUS FUNCTIONAL DIFFERENTIAL EQUATIONS AND NONLINEAR SEMIGROUPS

BY

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1. Introduction

We will consider the nonlinear functional differential equation

$$\begin{cases} u'(t) + \alpha u(t) + Bu(t) = F(u_t),\ t \geq 0, \\ u(t) = \phi(t),\ t \leq 0, \end{cases} \tag{1}$$

where $u_t(x) = u(t + x)$, $t \geq 0$, $x \leq 0$, $\alpha > 0$, and B is a single-valued, densely-defined, m-accretive operator in a Banach space E with norm $\|\cdot\|$. The initial data function ϕ is a mapping from $(-\infty, 0]$ into E and is taken from a Banach space X of such functions with norm denoted by $\|\cdot\|_X$.

In [1] and [2] the author developed nonlinear semigroup representations for solutions of equations of the form (1) and used these representations to obtain approximation, continuous dependence, stability, and asymptotic stability results. Several other authors have used the theory of nonlinear semigroups or evolution systems to obtain information about the solutions of nonlinear functional differential equations and related Volterra equations in various initial data spaces [4], [8], [9], [10], [11], [12]. An hypothesis of many of the results in this area is global Lipschitz continuity of the nonlinear functional involved. An exception to this condition is given in [10]. In [6], the global Lipschitzian condition is weakened to local by the use of averaging approximations. The purpose of this paper is to develop a simple technique for weakening from global to local the Lipschitz continuity condition of [1] for a nonlinear semigroup representation of solutions of (1).

2. Preliminaries

Rather than establish immediately a specific space of initial functions, only the basic properties which we require for the semigroup representation will be

Received August 4, 1980.

[1] This research was done while on leave at Virginia Polytechnic Institute and State University. The author wishes to express his thanks to both this university and the University of Arkansas for this opportunity.

assumed. Let X be a Banach space of functions $h\colon (-\infty, 0] \to E$ which has the following properties:

(HX) (i) for every $h \in X$, $h(0)$ is defined and $\|h(0)\| \le \|h\|_X$.
(ii) if $h \in X$, $\lambda > 0$, $\theta \in E$, and ϕ is the solution of

$$\begin{cases} \phi(x) - \lambda\phi'(x) = h(x), \; x \le 0, \\ \phi(0) = \theta, \end{cases} \tag{2}$$

then $\phi \in X$ and $\|\phi\|_X \le \max\{\|\theta\|, \|h\|_X\}$.

It will be shown in Section 4 that a modification of the fading memory space employed in [1] and [2] satisfies (HX).

An operator A on $D(A)$ in X associated with (1) is now defined in a standard way. Let

$$A\phi = -\phi',$$
$$D(A) = \{\phi \in X\colon \phi' \in X, \phi(0) \in D(B), \phi'(0) = F(\phi) - \alpha\phi(0) - B\phi(0)\}.$$

In [1] it is shown that $-A$ generates a nonlinear semigroup on X under conditions which include the hypothesis that F is uniformly Lipschitz continuous on X. In this paper we weaken this assumption to Lipschitz continuity on a bounded subset of X at the expense of similarly restricting the domain of the semigroup. Fix $\beta > 0$ and let $X_\beta = \{\phi \in X\colon \|\phi\|_X \le \beta\}$. We assume $F\colon X \to E$, $F(0) = 0$, and there is a number $M > 0$ (depending on β) such that

$$(HF) \qquad \|F(h_1) - F(h_2)\| \le M\|h_1 - h_2\|_X$$

whenever $h_1, h_2 \in X_\beta$.

The main result of this paper will be an application of the following special case of the basic result in [3].

THEOREM 1 (M. Crandall and T. Liggett). *Let A be an accretive operator in a Banach space and let $\lambda_0 > 0$. If $R(I + \lambda A)$ contains $\overline{D(A)}$ for $0 < \lambda < \lambda_0$, then $-A$ generates a nonlinear contraction semigroup on $\overline{D(A)}$.*

An operator A is accretive if $(I + \lambda A)^{-1}$ is a contraction on its domain for every $\lambda > 0$, and is m-accretive if, in addition, $R(I + \lambda A)$ is the entire space for every $\lambda > 0$.

3. The main result

THEOREM 2. *Let A be as defined above with B an m-accretive, single-valued operator on $D(B)$ in E with $B(0) = 0$ and $\overline{D(B)} = E$. Suppose (HX) and (HF) hold and $\alpha \ge M$. Let A_β be the restriction of A to X_β. Then $-A_\beta$ generates a nonlinear contraction semigroup $S_\beta(t)$ on $\overline{D(A_\beta)}$.*

Proof. According to Theorem 1, we need to show that for every $\lambda > 0$, $(I + \lambda A)^{-1}$ is defined and is a contraction on X_β. Given $h \in X_\beta$, we wish to find $\phi \in D(A_\beta)$ with the property that $(I + \lambda A_\beta)\phi = h$. By definition of A, ϕ must satisfy

$$\phi(x) - \lambda\phi'(x) = h(x),\ x \leq 0, \tag{3}$$

$$\phi'(0) = F(\phi) - \alpha\phi(0) - B\phi(0), \tag{4}$$

$$\phi(0) \in D(B),\ \phi \in X_\beta.$$

Let $E_\beta = \{\theta \in E\colon \|\theta\| \leq \beta\}$. For $\theta \in E_\beta$, let $\phi(x;\, \theta)$ be the unique solution of (2). Note that ϕ is given by the variation of constants formula (7). Since $h \in X_\beta$, (HX) implies that $\phi(\cdot;\ \theta) \in X_\beta$ for every $\theta \in E_\beta$. Clearly $\phi'(0;\ \theta) = (1/\lambda)(\theta - h(0))$, so according to (4) we wish to find $\theta \in D(B) \cap E_\beta$ satisfying

$$\theta - h(0) = \lambda F(\phi(\cdot\,;\theta)) - \alpha\lambda\theta - \lambda B\theta,$$

or, equivalently,

$$\theta = \left(I + \frac{\lambda}{1 + \lambda\alpha} B\right)^{-1}\left(\frac{h(0)}{1 + \lambda\alpha} + \frac{\lambda}{1 + \lambda\alpha} F(\phi(\cdot\,;\theta))\right). \tag{5}$$

Let a mapping $T\colon E_\beta \to E$ be defined by the right-hand side of (5). By (HF), since $B(0) = 0$, B is m-accretive, and $\phi \in X_\beta$ we have

$$\begin{aligned}\|T\theta\| &\leq \frac{\|h(0)\|}{1 + \alpha\lambda} + \frac{\lambda M}{1 + \alpha\lambda}\|\phi(\cdot\,;\theta)\|_X \\ &\leq \frac{\beta}{1 + \alpha\lambda} + \frac{\lambda M\beta}{1 + \alpha\lambda} \\ &\leq \frac{\beta}{1 + \alpha\lambda} + \frac{\lambda\alpha\beta}{1 + \alpha\lambda} \\ &= \beta.\end{aligned}$$

Therefore $T\colon E_\beta \to E_\beta$. For $\theta_1, \theta_2 \in E_\beta$, by (HF),

$$\|T\theta_1 - T\theta_2\| \leq \frac{\lambda M}{1 + \alpha\lambda}\|\psi\|_X$$

where $\psi(x) = \phi(x;\, \theta_1) - \phi(x;\, \theta_2)$ satisfies

$$\begin{cases}\psi(x) - \lambda\psi'(x) = 0 \\ \psi(0) = \theta_1 - \theta_2.\end{cases}$$

So by (HX), $\psi \in X$ and $\|\psi\|_X \leq \max\{\|\theta_1 - \theta_2\|, 0\} = \|\theta_1 - \theta_2\|$. Therefore,

$$\|T\theta_1 - T\theta_2\| \leq \frac{\lambda\alpha}{1 + \lambda\alpha}\|\theta_1 - \theta_2\|.$$

Hence T is a strict contraction from E_β into E_β for every $\lambda > 0$. Let $\bar{\theta}$ be the unique fixed point of T in E_β. Then $\phi(\cdot\,; \bar{\theta}) = (I + \lambda A_\beta)^{-1}h$ as desired. This shows that $(I + \lambda A_\beta)^{-1}$ is defined on X_β.

Choose $h_1, h_2 \in X_\beta$ and let $\phi_i = (I + \lambda A_\beta)^{-1}h_i$, $i = 1, 2$. Note that ϕ_1, $\phi_2 \in X_\beta$. Let $\psi = \phi_1 - \phi_2$. Then ψ satisfies

$$\begin{cases} \psi(x) - \lambda\psi'(x) = h_1(x) - h_2(x) \\ \psi(0) = \phi_1(0) - \phi_2(0). \end{cases}$$

By (HX),

$$\|\phi_1 - \phi_2\|_X = \|\psi\|_X \le \max\{\|\phi_1(0) - \phi_2(0)\|, \|h_1 - h_2\|_X\}.$$

If $\|\phi_1(0) - \phi_2(0)\| \le \|h_1 - h_2\|_X$, then $\|\phi_1 - \phi_2\|_X \le \|h_1 - h_2\|_X$ and we are done. So suppose $\|h_1 - h_2\|_X \le \|\phi_1(0) - \phi_2(0)\|$. Then $\|\phi_1 - \phi_2\|_X \le \|\phi_1(0) - \phi_2(0)\| \le \|\phi_1 - \phi_2\|_X$. From (5) we have

$$\phi_i(0) = \left(I + \frac{\lambda}{1 + \alpha\lambda} B\right)^{-1}\left(\frac{h_i(0)}{1 + \alpha\lambda} + \frac{\lambda}{1 + \alpha\lambda} F(\phi_i)\right), \quad i = 1, 2.$$

Therefore by (HF) since $\phi_1, \phi_2 \in X_\beta$,

$$\begin{aligned} \|\phi_1 - \phi_2\|_X &= \|\phi_1(0) - \phi_2(0)\| \\ &\le \frac{1}{1 + \alpha\lambda}\|h_1(0) - h_2(0)\| + \frac{\lambda M}{1 + \alpha\lambda}\|\phi_1 - \phi_2\|_X \\ &\le \frac{1}{1 + \alpha\lambda}\|h_1 - h_2\|_X + \frac{\alpha\lambda}{1 + \alpha\lambda}\|\phi_1 - \phi_2\|_X, \end{aligned}$$

which yields $\|\phi_1 - \phi_2\|_X \le \|h_1 - h_2\|_X$. Therefore $(I + \lambda A_\beta)^{-1}$ is a contraction on X_β for every $\lambda > 0$. Theorem 1 now implies that $-A_\beta$ generates a nonlinear semigroup of contractions on $\overline{D(A_\beta)}$.

COROLLARY 3. *Suppose in addition to the hypotheses of Theorem 2 that the set*

$$Y = \{h \in X_\beta : h' \in X\}$$

is dense in X_β. Then $-A_\beta$ generates a nonlinear semigroup of contractions on X_β.

Proof. It only remains to show that $\overline{D(A_\beta)} = X_\beta$. Clearly $\overline{D(A_\beta)}$ is contained in X_β. We wish to approximate every element h of X_β by an element of $D(A_\beta)$. By hypothesis we may assume $h \in Y$. Let $\phi_\lambda = (I + \lambda A_\beta)^{-1}h$. Then by the proof of Theorem 2, ϕ_λ exists in $D(A_\beta)$ for every $\lambda > 0$ and satisfies

$$\begin{aligned} &\phi_\lambda(x) - \lambda\phi_\lambda'(x) = h(x), \ x \le 0, \\ &\phi_\lambda(0) = \left(I + \frac{\lambda}{1 + \alpha\lambda} B\right)^{-1}\left(\frac{h(0)}{1 + \alpha\lambda} + \frac{1}{1 + \alpha\lambda} F(\phi_\lambda)\right). \end{aligned} \tag{6}$$

Let $\psi_\lambda = \phi_\lambda - h$. Then ψ_λ satisfies

$$\begin{cases} \psi_\lambda(x) - \lambda\psi'_\lambda(x) = \lambda h'(x), \ x \le 0, \\ \psi_\lambda(0) = \phi_\lambda(0) - h(0). \end{cases}$$

Since $h' \in X$, we have, by (HX),

$$\|\phi_\lambda - h\|_X \le \max\{\|\phi_\lambda(0) - h(0)\|, \lambda\|h'\|_X\}$$

for every $\lambda > 0$. Therefore $\|\phi_\lambda - h\|_X \to 0$ as $\lambda \to 0$ if $\|\phi_\lambda(0) - h(0)\| \to 0$ as $\lambda \to 0$. By (6) and the fact that B is m-accretive,

$$\begin{aligned}\|\phi_\lambda(0) - h(0)\| \le &\left\|\left(I + \frac{\lambda}{1+\alpha\lambda} B\right)^{-1}\left(\frac{h(0)}{1+\alpha\lambda} + \frac{\lambda}{1+\alpha\lambda} F(\phi_\lambda)\right)\right. \\ &\left. - \left(I + \frac{\lambda}{1+\alpha\lambda} B\right)^{-1}(h(0))\right\| \\ &+ \left\|\left(I + \frac{\lambda}{1+\alpha\lambda} B\right)^{-1}(h(0)) - h(0)\right\| \\ \le &\frac{\alpha\lambda}{1+\alpha\lambda}\|h(0)\| + \frac{\lambda M\beta}{1+\alpha\lambda} + \left\|\left(I + \frac{\lambda}{1+\alpha\lambda} B\right)^{-1}(h(0)) - h(0)\right\|.\end{aligned}$$

Since $\overline{D(B)} = E$, the last term goes to zero as $\lambda \to 0$. Therefore this inequality yields the desired result.

Remark 4. There remains the question of whether $S_\beta(t)\phi = u_t$ where u is a "classical" solution of (1). As this question seems best approached on a case by case basis depending on the initial data space, we will here indicate only some approaches to its solution. If a solution of (1) is known to exist by classical methods (e.g., Picard iterates) then the theory of nonlinear semigroups may be invoked as in [1] and [2] to conclude that $S_\beta(t)\phi = u_t$. A more direct approach using the semigroup and its generator is given in [5] and generalized to infinite intervals in [7].

4. The initial data space

We conclude this paper by considering some examples of function spaces which satisfy (HX). Note that if (2) holds then by the variation of constants formula

$$\phi(x) = \theta e^{\gamma x} + \gamma\int_x^0 e^{\gamma(x-\eta)}h(\eta)\,d\eta, \quad x \le 0, \tag{7}$$

where $\gamma = 1/\lambda$. Therefore

$$\|\phi(x)\| \leq \|\theta\| e^{\gamma x} + (1 - e^{\gamma x}) \sup_{x \leq \eta \leq 0} \|h(\eta)\|$$

$$\leq \max \left\{ \|\theta\|, \sup_{x \leq \eta \leq 0} \|h(\eta)\| \right\}.$$

Hence

$$\sup_{x \leq 0} \|\phi(x)\| \leq \max \left\{ \|\theta\|, \sup_{x \leq 0} \|h(x)\| \right\}.$$

This shows that the space C of continuous functions on $(-\infty, 0]$ with supremum norm satisfies (HX) and so also does any closed subspace of C. Note however that C does not satisfy the additional hypothesis of Corollary 3. This is to be expected since translation is not a continuous operator on C. The subspaces

$$\{h \in C\colon h \text{ has a limit at } -\infty\} \quad \text{and} \quad \{h \in C\colon h(x) = 0 \text{ for } x < -r\}$$

satisfy the hypotheses of Corollary 3.

Another example is provided by a modification of the "fading memory space" used in [1] and [2]. Fix $r > 0$ and let X be the space of all functions ϕ: $(-\infty, 0] \to E$ which are continuous on $[-r, 0]$ and for which $e^x \phi(x)$ is Bochner integrable on $(-\infty, -r)$. Then X is a Banach space under the norm $\|\phi\|_X = \max\{\|\phi\|_\infty, \|\phi\|_1\}$, where

$$\|\phi\|_\infty = \sup_{-r \leq x \leq 0} e^x \|\phi(x)\| \quad \text{and} \quad \|\phi\|_1 = \int_{-\infty}^{-r} e^x \|\phi(x)\|\, dx.$$

Clearly X satisfies (HX) (i). The following technical lemma implies that the results of Section 3 may be applied to this space.

LEMMA 5. *Let X be as defined above. Suppose $h \in X$, $\lambda > 0$, and ϕ satisfies* (2). *Then $\phi \in X$ and $\|\phi\|_X \leq \max\{\|\theta\|, \|h\|_X\}$.*

Proof. Clearly $\phi \in X$ if $\|\phi\|_X < \infty$. From (7) it follows that

$$e^x \|\phi(x)\| \leq \|\theta\| e^{(\gamma+1)x} + \frac{\gamma}{\gamma + 1} (1 - e^{(\gamma+1)x}) \sup_{x \leq \eta \leq 0} e^\eta \|h(\eta)\|,$$

for $-r \leq x \leq 0$. Therefore

$$\|\phi\|_\infty \leq \max \left\{ \|\theta\|, \frac{\gamma}{\gamma + 1} \|h\|_\infty \right\}. \tag{8}$$

Furthermore, by (7),

$$\begin{aligned}
\|\phi\|_1 &\le \int_{-\infty}^{-r} e^{(\gamma+1)x}\|\theta\|\,dx + \gamma\int_{-\infty}^{-r}\int_x^0 e^{(\gamma+1)(x-\eta)}e^{\eta}\|h(\eta)\|\,d\eta\,dx\\
&\le \frac{1}{\gamma+1}\|\theta\|e^{-(\gamma+1)r} + \frac{\gamma}{\gamma+1}\int_{-r}^0 e^{\eta}\|h(\eta)\|e^{-(\gamma+1)(r+\eta)}\,d\eta\\
&\quad + \frac{\gamma}{\gamma+1}\int_{-\infty}^{-r} e^{\eta}\|h(\eta)\|\,d\eta\\
&\le \frac{1}{\gamma+1}\|\theta\|e^{-(\gamma+1)r} + \frac{\gamma}{(\gamma+1)^2}(1-e^{-(\gamma+1)r})\|h\|_\infty + \frac{\gamma}{\gamma+1}\|h\|_1\\
&\le \frac{1}{\gamma+1}\max\left\{\|\theta\|, \frac{\gamma}{\gamma+1}\|h\|_\infty\right\} + \frac{\gamma}{\gamma+1}\|h\|_1. \qquad (9)
\end{aligned}$$

Let $K = \max\left\{\|\theta\|, \frac{\gamma}{\gamma+1}\|h\|_\infty\right\}$. Then (8) and (9) imply that

$$\|\phi\|_X \le \max\left\{K, \frac{K}{\gamma+1} + \frac{\gamma}{\gamma+1}\|h\|_1\right\}. \qquad (10)$$

If $K = \|\theta\|$, then (10) yields

$$\begin{aligned}
\|\phi\|_X &\le \max\left\{\|\theta\|, \frac{\|\theta\|}{\gamma+1} + \frac{\gamma}{\gamma+1}\|h\|_X\right\}\\
&\le \max\{\|\theta\|, \|h\|_X\},
\end{aligned}$$

which is the desired inequality. On the other hand if

$$K = \frac{\gamma}{\gamma+1}\|h\|_\infty,$$

then (10) yields

$$\begin{aligned}
\|\phi\|_X &\le \max\left\{\frac{\gamma}{\gamma+1}\|h\|_\infty, \frac{\gamma}{\gamma+1}\left(\frac{1}{\gamma+1}+1\right)\|h\|_X\right\}\\
&\le \max\left\{\frac{\gamma}{\gamma+1}\|h\|_X, \frac{\gamma(\gamma+2)}{(\gamma+1)^2}\|h\|_X\right\}\\
&\le \|h\|_X \le \max\{\|\theta\|, \|h\|_X\}.
\end{aligned}$$

This completes the proof of Lemma 5.

It may be shown by similar arguments that the weight function e^x may be replaced by any nonnegative function $p(x)$ with the property that $p(0) = 1$ and $p(x)e^{-x}$ is nondecreasing on $(-\infty, 0]$. It is interesting to note that unlike the situation in [1], the weight function is independent of the Lipschitz constant M.

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