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# POSITIVE WEIGHT HOMOTOPY TYPES

BY

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## Introduction

Automorphism groups of minimal models in various algebraic categories are investigated in this paper, in order to apply the observation that all notions of "positive weight" for a rational homotopy type (RHT) coincide. This equivalence contrasts with the distinction between the set of "formal" RHT's and the set of "coformal" RHT's. However, both of these sets are proper subsets of the set $\Delta$, consisting of the positive weight RHT's. The notions of formal and coformal are dual to one another, in the sense of Eckmann-Hilton duality [8], while the notion of positive weight is self-dual. $\Delta$ is closed under the formation of products, coproducts (i.e., one-point-unions), retracts, and Postnikov sections; moreover, the elements of $\Delta$ decompose uniquely as products, and also as coproducts.

Essentially, the positive weight property (for an object of an "algebraic" category) can be understood to mean that the "zero" (or basepoint) endomorphism lies in the Zariski closure of the group of automorphisms (within the monoid of all endomorphisms). However, this paper will apply several other interesting characterizations of "positive weight" for RHT's.

We observe the behavior of the "positive weight" property under a field extension. If $K$ is an arbitrary field of characteristic zero, then "$K$-positive weight" may be defined similarly for "$K$-homotopy types"; however, the concept of positive weight is independent of the (characteristic zero) field of definition, in the sense that the $K$-homotopy type of a RHT is $K$-positive weight, if and only if the RHT itself is positive weight.

The general discussion of positive weight in section one of this paper is applied in section two to obtain a simpler proof of the uniqueness of coproduct decompositions for positive weight RHT's. The proof for products in [3] (using minimal associative algebras) is dualized here to give a proof of uniqueness for coproducts which is significantly easier than that given in [7], by working in the category of minimal Lie algebras, rather than the homotopy category of minimal associative algebras. This illustrates the way

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in which the differences between these two types of minimal models may be exploited, in spite of the equivalence of their homotopy categories.

In the proofs contained in this paper we consider only finitely generated, minimal models of simply-connected RHT's; however, the "finitely generated" hypothesis will have two different geometric meanings (depending on the choice of category, to which the minimal model belongs), and in any case, can be replaced by the much more general assumption that the Hurewicz homomorphism of the RHT has a finite dimensional image. The techniques required for this more general hypothesis are developed in section 3 of [7], in the context of limits of inverse systems of algebraic groups. As explained in the introduction to [7], the question of coproduct decompositions for nilpotent RHT's reduces at once to the simply-connected case. For this reason, and also for simplicity, we restrict ourselves to simply-connected RHT's and their associated minimal models, throughout this paper, unless the more general hypothesis, "nilpotent" is explicitly mentioned.

Finally, in section three, "weight criteria" equivalent to formality (respectively, coformality) are given, which complement those in [9], and clarify the relationship between formal, coformal, and positive weight.

## 1. Positive Weight Generalities

Let $K$ be an arbitrary field of characteristic zero, and let $Q$ denote the field of rational numbers. Soon we shall need to refer to "positive weight" objects in many distinct types of categories (and with respect to various fields $K$). For this reason, our discussion will require the following general definitions.

Let $\Omega$ be a category with zero-object (i.e., an object which is both terminal and coterminal). Suppose that the objects of $\Omega$ are $K$-vector spaces with an "algebraic structure", where an *algebraic structure* on $K$-vector space $M$ (underlying an object, also denoted $M$, in the category $\Omega$), is a set of homomorphisms, whose domains and ranges are elements of the set $\{K, M, M \otimes M, M \otimes M \otimes M, \ldots\}$. Here, "$\otimes$" is the tensor product of vector spaces, over $K$. For example, a "grading" may be considered as a sequence of orthogonal idempotent endomorphisms of $M$; a "product" (for a $K$-algebra structure on $M$) is a homomorphism: $M \otimes M \to M$; a "coproduct" (for a coalgebra) is a homomorphism: $M \to M \otimes M$; of course, a differential is a homomorphism: $M \to M$.

Suppose further that $E(M)$, the set of endomorphisms of an object $M$, is the set of $K$-rational points of an affine variety, defined over $Q$ (see [4] for definitions), and the composition of endomorphisms is a morphism of varieties. We shall refer to such a category as an *algebraic category*.

If $M$ is an object of an algebraic category $\Omega$, then (loosely speaking, omitting reference to "$K$-rational points") it follows that $E(M)$ is an algebraic monoid, and $G(M)$, the algebraic group of automorphisms of $M$ in $\Omega$, is an open subset of $E(M)$. Let $F(M)$ be a Zariski closure of $G(M)$ in $E(M)$. Of course, if $M$ is not the zero-object, then $\mathbf{0}$, the basepoint morphism in

$E(M)$, is never in $G(M)$. We shall say that $M$ is a *positive weight* object of $\Omega$, if $\mathbf{0}$ is in $F(M)$.

Recall that $E(M)$, $G(M)$, and $F(M)$ are actually "functors of rational points", in the sense that their values for $K$ (a field of characteristic zero) will be their respective sets of $K$-rational points, which may be viewed as subsets of some affine spaces. Moreover, for each such field $K$, there is a topology (and closure operator) on each affine space, called the $K$-Zariski topology (and the $K$-Zariski closure, respectively). Notice that (in spite of these options) the definition of positive weight is not ambiguous, because of the following.

LEMMA 0. *If* $\mathbf{0}$ *is in some Zariski closure of some set of rational points of* $G(M)$, *then* $\mathbf{0}$ *is in all such closures of each of the sets of rational points of* $G(M)$.

Of course, it suffices to show (for $K$, an arbitrary field of characteristic zero) that $\mathbf{0}$ is in the $K$-Zariski closure of the set of $Q$-rational points of $G(M)$, if $\mathbf{0}$ is in the $Q$-Zariski closure of the set of $K$-rational points of $G(M)$. This is easily seen by using condition (b) in Proposition 1 (below), together with an argument similar to that given in the last two paragraphs of the proof of Lemma 3 in [2].

Proposition 1 gives four equivalent characterizations of positive weight; the first of these is our definition, the second is used to prove Lemma 0, and the third is required for an elementary proof of Theorem 2 (and is the definition of positive weight in [7]). The fourth characterization is included primarily for perspective. However, before stating this proposition, we must define the general concept of a "weight splitting" on an object in an arbitrary algebraic category.

A "weight splitting" of object $M$ in algebraic category $\Omega$ is a direct sum decomposition (indexed by the group of all integers) of the underlying $K$-vector space (also denoted $M$), which is "compatible" with the algebraic structure supported by $M$. Of course, it remains to explain the meaning of "compatible".

*Compatibility* of a grading with another algebraic structure means that the homomorphisms (which constitute that structure) preserve grading degrees (i.e., form certain obvious commutative diagrams with the idempotent endomorphisms of the appropriate gradings), where it is understood that a grading on $M$ induces the usual (summed) grading on a tensor product of copies of $M$. If an integer-indexed grading of $M$ is compatible with every structure of category $\Omega$ on object $M$, then the grading is said to be a *weight splitting* of $M$. If all the non-positive components of a weight splitting lie in the image of the (unique) morphism from the zero-object to $M$, then we say that the weight splitting is a *positive weight splitting*.

PROPOSITION 1. *If* $M$ *is an object of an algebraic category, then the following are equivalent:*

(a) $\mathbf{0}$ *is in* $F(M)$ (*definition of positive weight*).

(b) *There exists a 1-parameter subgroup in* $G(M)$, *which converges to* $\mathbf{0}$ *in* $E(M)$.

(c) $M$ *possesses a positive weight splitting.*

(d) *All irreducible components of the* (*affine*) *algebraic variety* $E(M)$ *contain* $\mathbf{0}$.

The remarks in sections (3.1) and (3.2) in [7] are easily generalized to give a proof of the equivalence of (a), (b), and (c).

(b) implies (d). For an arbitrary irreducible component $E'$ of $E(M)$, there is an element $x$ in $E'$ which is not in any other component. The closure of the 1-parameter subgroup obtained from (b) is an irreducible subset of $E(M)$, as is $J$, its image under translation by $x$. Since $x$ is in $J$, it follows that $J$ is a subset of $E'$. Of course, $\mathbf{0}$ is in $J$, and so $\mathbf{0}$ is in $E'$.

(d) implies (a). Let $\mathbf{1}$ be the unit of $G(M)$, and suppose both $\mathbf{0}$ and $\mathbf{1}$ are in the same component, $E'$. Then $E'$ is the closure of the component of $G(M)$ containing $\mathbf{1}$. Thus, $\mathbf{0}$ is in $F(M)$.

The equivalent definitions of positive weight in Proposition 1, are general enough to be applicable in the following categories:

1. Simply-connected minimal differential graded $K$-algebras, and their morphisms.
2. Simply-connected minimal differential graded $K$-coalgebras, and their morphisms.
3. Connected minimal differential graded Lie $K$-algebras, and their morphisms.
4. Connected minimal differential graded Lie $K$-coalgebras, and their morphisms.
5. Simply-connected "zero-local" spaces, and homotopy classes of maps (the rational homotopy category).
6. Non-negatively graded $K$-vector spaces with some (specific) degree-preserving, "pointed" algebraic structure, and their morphisms.

Descriptions of categories one, two, three, and five (with $K = Q$), and several of the constructions and results required here, may be found in [11]. In each of these six "algebraic categories", it is of interest to know which objects are "positive weight".

In the case of category six, *all* objects are positive weight, by condition (c) in Proposition 1. Of course, "pointed" means that category six has a zero-object; however, there can be no differentials (or other structure morphisms) which change degree. In [2] (for perfect fields), and more generally in [6] (for arbitrary fields), there are uniqueness theorems for coproduct decompositions of algebras. These are in the context of a category of algebras over a field; moreover, these results are generalized by the "Remark" at the end of Section 2.

The first five categories have equivalent "homotopy categories" (see

[11]); moreover, the positive weight property is preserved by this correspondence of objects, as can be seen from the following result.

THEOREM 2. *All four types of minimal models for an arbitrary specific RHT will be positive weight, if any one of them is.*

*Proof.* Objects of category one (respectively, three) are strictly dual to objects of category two (respectively, four). Moreover, it is clear that condition (c) in Proposition 1 is preserved by both of these bijective correspondences. Preservation of the "positive weight" conditions by the correspondence between categories one and five, may be found (implicitly) in [12]. Related to this is the equivalence between the two properties, "positive weight" and "$p$-Universal" (see [7], p. 169, Remark 2.2). Historically, the earliest of these equivalent properties is "$p$-Universal", which is studied in [10]. Thus, it remains only to check that condition (c) is preserved under the bijective correspondence between the sets of isomorphism classes of the objects of categories two and three; however, this is clear from the construction of the bijection from (isomorphism classes of) minimal coalgebras to (isomorphism classes of) minimal Lie algebras (see [11]). Q.E.D.

Let $M$ and $N$ be two minimal algebra models over $Q$ (objects of category one, with field $Q$), representing two RHT's, and let $K$ be an arbitrary field of characteristic zero. Just as we may refer to $M$ as the RHT which it uniquely represents, so we may define $M \otimes K$ to be the *$K$-homotopy-type* of the RHT $M$ (with all structure homomorphisms extended in the obvious manner). We shall say that $M$ and $N$ share the *same $K$-homotopy-type* if $M \otimes K$ and $N \otimes K$ are isomorphic objects.

Recall that the endomorphisms of $M$, $E(M)$ is the set of $Q$-rational points in $E(M \otimes K)$, and $G(M)$ is the set of $Q$-rational points in $G(M \otimes K)$. Now we see that Lemma 0 also implies the following.

PROPOSITION 3. *The $K$-homotopy type, $M \otimes K$ (of the minimal model, $M$, of a RHT) is positive weight, if and only if $M$ is positive weight.*

Thus, there is no ambiguity in referring to a RHT as being "positive weight".

## 2. Unique Decomposition Results

One aim of this section is to simplify the proof of a result (recorded here as Theorem 5), which appears in [7]. This is accomplished by dualizing an earlier proof (in [3]) of the dual result (whose generalization in [7] is recorded here as Theorem 4).

THEOREM 4. *Positive weight RHT's (which are spherically finite dimensional and nilpotent) decompose uniquely as categorical* **products** *(of irreducible factors).*

THEOREM 5. *Positive weight RHT's (which are spherically finite dimensional and nilpotent) decompose uniquely as categorical* **coproducts** *(of irreducible factors).*

It is only for the class of positive weight RHT's that uniqueness of product and coproduct decomposition has been proved; however, we *conjecture* (in [7]) that the positive weight hypothesis is not necessary.

As pointed out in the introduction, Theorem 4 is proved for "finitary" positive weight RHT's in [3]. This proof, extended by the limit techniques of [7], proves Theorem 4 without passing to the homotopy category. The very same procedure just outlined in category one (Sullivan minimal models), can be used in category three (minimal Lie algebras) to prove Theorem 5 without passing to the homotopy category.

The coproduct (i.e., the graded tensor product) of objects in category one, corresponds to the product of corresponding RHT's (objects of category five). "Dually", the coproduct of minimal Lie algebras (category three objects) corresponds to the coproduct (one-point-union) of corresponding RHT's (see Lemmas 8.5 and 8.6 in [11]). Thus, Theorems 4 and 5 both follow from Proposition 6 (below), for RHT's whose minimal models are finitely generated; moreover, this line of argument provides proofs for both Theorems 4 and 5, when extended by the inverse limit results of [7].

Recall from Section 1 of [7], that a category is said to be *I-split* (for coproduct (resp., products)), if the following three conditions are satisfied:

(i) there exists a zero-object;

(ii) idempotent endomorphisms "split" (i.e., if $e^2 = e$, then morphisms $p$ and $h$ exist, such that $hp = e$ and $ph = 1$);

(iii) the category is closed under formation of finite coproducts (resp., products), and contains no infinite coproducts (resp., products).

For example, all six algebraic categories in the preceding section are $I$-split (categories one, three, and six for coproducts; categories two and four for products; and category five for both).

Recall further that coproduct decompositions of objects in $I$-split categories correspond to "splittings", bijectively. Now the (overworked) term "splitting" refers (as in [7]) to the set of idempotent endomorphisms, categorically determining (as well as determined by) the corresponding coproduct decomposition. Of course, similar remarks hold for products.

PROPOSITION 6. *Each positive weight object in an I-split, algebraic category has an unique irreducible decomposition as a coproduct (resp., product).*

*Proof.* Given a splitting of positive weight object $M$, each coproduct (respectively, product) factor is positive weight (as it is a retract of $M$, and the proof of Lemma 3 in [3] shows that retraction preserves positive weight). By using condition (c) of Proposition 1 on each factor of the coproduct (respectively, product) decomposition, it is easy to directly construct a $Q$-split torus $T$ in $G(M)$, such that the Zariski closure of $T$ contains the endomorphisms of the splitting (see [1] for the explicit construction of $T$). Of course, without loss of generality, we can assume that $T$ is a maximal $Q$-split torus in $G(M)$.

Any two maximal $Q$-split tori are conjugate, by a $Q$-rational point of $G(M)$ (i.e., an automorphism of $M$). Moreover, any closure of a commutative set of endomorphisms (such as $T$) is again commutative. Of course, $G(M)$ acts by conjugation on (the endomorphisms of) the splittings of $M$. Thus, given any two splittings of $M$, there is an automorphism of $M$, such that this automorphism's conjugates of the endomorphisms in one splitting and the endomorphisms of the other splitting, all lie in the closure of the same maximal $Q$-split torus; hence the endomorphisms of these splittings commute. Now, an appeal to the elementary Corollary 1.10 in [7] completes the proof of Proposition 6, over the field $Q$.

*Remark.* Proposition 6 can be proved for algebraic categories defined over an *arbitrary* field (of arbitrary characteristic), using the method of [6]. This approach is based on recent, very general conjugacy results due to Borel and Tits [5]. (When [6] was published, the only available reference for these conjugacy theorems was a private letter from Borel to the author.)

## 3. Formality and Weight Splittings

We recall from [9] the "dual" concepts of "formal" and "coformal" RHT's. A minimal model $M$ in category one (resp., $L$ in category three) is referred to as a *formal* minimal algebra (resp., *coformal* minimal Lie algebra), if it models its own homology. Moreover, it is proved in [9] that these dual properties are independent of arbitrary (characteristic zero) field extensions.

Explicitly, $M$ is *formal,* if and only if the canonical projection of cocycles onto cohomology, $ZM \to HM$ extends to a morphism of algebras, $M \to HM$. (Such an extension is then, of necessity, a weak equivalence.)

Similarly, $L$ is *coformal,* if and only if the canonical projection of cycles onto homology, $ZL \to HL$ extends to a morphism of Lie algebras, $L \to HL$. (Such an extension is then, of necessity, a weak equivalence.)

These dual (but distinct) properties are characterized in four different ways (each) in Propositions 3.2 and 3.3 of [9]. One purpose of this section is to supply two more characterizations for each; one in terms of weight splittings, and another in terms of automorphism groups.

If an object in an algebraic category has a differential, and a weight splitting, then the homology of this object is naturally bigraded (by degree and weight). Such a weight splitting is called a *diagonal splitting,* if the weight and degree are equal, for all non-zero components of the homology of the object.

PROPOSITION 7. *If $M$ is a minimal (associative) algebra, then the following are equivalent.*

(A) *$M$ possesses a diagonal splitting.*
(B) *Algebraic group morphism (i.e., cohomology functor)*

$$H : \mathrm{Aut}(M) \to \mathrm{Aut}(HM)$$

*is surjective.*
(C) *$M$ is formal.*

*Moreover, for diagonal splittings, the weight can* **not** *be* **less** *than the degree, for a non-zero bigraded component of $M$.*

PROPOSITION 8. *If $L$ is a connected minimal Lie algebra, then the following are equivalent.*

(A) *$L$ possesses a diagonal splitting.*
(B) *Algebraic group morphism (i.e., homology functor)*

$$H : \mathrm{Aut}(L) \to \mathrm{Aut}(HL)$$

*is surjective.*
(C) *$L$ is coformal.*

*Moreover, for diagonal splittings, the weight is a positive integer, which can* **not** *be* **greater** *than the degree, for a non-zero bigraded component of $L$.*

The proofs of these two propositions are quite similar, and Proposition 7 appears in [1] (as Lemma 6, on p. 335), together with its proof. Thus, only a sketch of the proof of Proposition 8 will be given.

(A) implies (C). Let $L(p, n)$ (resp., $ZL(p, n)$, $HL(p, n)$) denote the bigraded (degree $p$, weight $n$)-component of a weight splitting of minimal Lie algebra $L$ (resp., the cycles of $L$, the homology of $L$). Such a weight splitting is a diagonal splitting, if $HL(p, n) = 0$, unless $n = p$. Moreover, this implies that $L(p, n) = 0$, unless $0 < n < p + 1$, by an easy induction argument. Let $D$ be the direct sum of all $L(p, p)$, for $p > 0$. The projections, $L \to D$ and $D \to HL$ are then Lie algebra morphisms, and their composition is a weak equivalence. Thus, $L$ is coformal.

(C) implies (B). If $L$ is coformal, then the morphism of algebraic groups, $H : \mathrm{Aut}(L) \to \mathrm{Aut}(HL)$ is *surjective*. This follows from two facts: (1) $L$ is a cofibrant object in a closed model category, and (2) $L \to HL$ is both a fibration and a weak equivalence. Now, an appeal to axiom "CM 4" implies the above surjectivity. (See [11], Section 5.)

(B) implies (A). Consider the "grading automorphism" $a : HL \to HL$, which multiplies an element of degree $p$, by the $p$-fold power of 2. (Of course, this argument is valid, using the powers of $t$, for any rational number $t$ not equal to 1, 0, or $-1$.) By hypothesis, there is an automorphism, $f : L \to L$, which induces the grading automorphism on homology. Without loss of generality, $f$ may be chosen to be a semisimple element of $\mathrm{Aut}(L)$. An induction argument (on the maximum degree of indecomposables, $QL = L/[L, L]$) shows that $f$ can be diagonalized, with eigenvalues equal to positive integer powers of 2. Let $L(p, n)$ denote the eigenspace consisting of all degree $p$ elements of $L$, on which $f$ acts by multiplication by the $n$-fold power of 2. Now $HL(p, n) = 0$, unless $n = p$, and we have the desired diagonal splitting of $L$.

*Remark.* Observe that diagonal splittings are positive weight splittings (for both associative algebra minimal models and Lie algebra minimal models). Thus, "coformal" implies "positive weight" (Proposition 8), as does "formal" (Proposition 7).

Possession of a diagonal splitting is dual to possession of a differential with "zero perturbation" (see [9]), where "zero perturbation" simply means that the value of the differential (on a generator) can be expressed as a linear combination of quadratic terms, for some choice of algebra generators. To understand this duality, first notice that a formal (resp., coformal) minimal Lie algebra (resp., minimal associative algebra) has a weight splitting, induced by the diagonal splitting on the corresponding dual minimal associative algebra (resp., minimal Lie algebra). This induced weight splitting concentrates the minimal model's algebra generators in components with weight = degree + 1 (resp., weight = degree − 1). However, the existence of such a weight splitting is clearly equivalent to a differential with "zero perturbation". Thus, weight splittings offer an elementary proof of the equivalence of conditions (a) and (d), both in Proposition 3.2 and in Proposition 3.3 of [9]. Explicitly, weight splittings can be used to observe the following corollaries of Propositions 7 and 8, respectively.

COROLLARY 9. *A minimal Lie algebra is formal if and only if it is isomorphic to a minimal Lie algebra whose differential has "zero perturbation".*

COROLLARY 10. *A minimal associative algebra is coformal if and only if it is isomorphic to a minimal algebra whose differential has "zero perturbation".*

## REFERENCES

1. R. BODY and R. DOUGLAS, *Rational homotopy and unique factorization,* Pacific J. Math., vol. 75 (1978), pp. 331–338.

2. ———, *Tensor products of graded algebras and unique factorization,* Amer. J. Math., vol. 101 (1979), pp. 909–914.
3. ———, *Unique factorization of rational homotopy types,* Pacific J. Math., vol. 90 (1980), pp. 21–26.
4. A. Borel, *Linear algebraic groups,* Benjamin, New York, 1969.
5. A. Borel and J. Tits, *Theoremes de structure et de conjugaison pour les groupes algebriques lineaires,* C. R. Acad. Sci. Paris Ser. A-B, vol. 287 (1978), pp. A55–A57.
6. R. Douglas, *The uniqueness of coproduct decompositions,* Proceedings of the Vancouver Algebraic Topology Conference, 1977; Lecture Notes in Math., vol. 673, Springer-Verlag, New York.
7. R. Douglas and L. Renner, *Uniqueness of product and coproduct decompositions in rational homotopy theory,* Trans. Amer. Math. Soc., vol. 264 (1981), pp. 165–180.
8. P. Hilton, *Homotopy theory and duality,* Gordon and Breach, New York, 1965.
9. T. Miller and J. Neisendorfer, *Formal and coformal spaces,* Illinois J. Math., vol. 22 (1978), pp. 565–580.
10. M. Mimura and H. Toda, *On p-equivalences and p-universal spaces,* Comment. Math. Helv., vol. 46 (1971), pp. 87–97.
11. J. Neisendorfer, *Lie algebras, coalgebras, and rational homotopy theory for nilpotent spaces,* Pacific J. Math., vol. 74 (1978), pp. 429–460.
12. D. Sullivan, *Infinitesimal computations in topology,* Inst. Hautes Etudes Sci. Publ. Math., vol. 47 (1977), pp. 269–331.

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