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ON THE COHOMOLOGY OF THE LIE ALGEBRA OF FORMAL VECTOR FIELDS PRESERVING A FLAG

BY

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1. Let

$$\mathscr{A}_{n,r} = \left\{ \sum_{i=1}^{r} f_i(x_1, \ldots, x_r) \frac{\partial}{\partial x_i} + \sum_{i=r+1}^{n} f_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_1} \,\middle|\, f_i\text{-formal power series in the variables concerned.} \right\}$$

and

$$\mathscr{A}_r = \left\{ \sum_{i=1}^{r} f_i(x_1, \ldots, x_r) \frac{\partial}{\partial x_i} \,\middle|\, f_i\text{-formal power series in } x_1, \ldots, x_r \right\}$$

The cohomology groups of $\mathscr{A}_r$ were studied by Gelfand and Fuks [4]. In this paper we prove that $\mathscr{A}_{n,r}$ is r-connected: $H^i(\mathscr{A}_{n,r}\mathbf{R}) = 0$ for $0 < i \leq r$.

In this context Professor A. Haefliger asked the author whether

$$H^i(\mathscr{A}_{n,r}, \mathbf{R}) \simeq H^i(\mathscr{A}_r, \mathbf{R}) \quad \text{for } i \leq 2n \text{ (canonically).}$$

Here we prove this isomorphism for $i \leq n - r$ only (Theorem 3.6). The method of this paper is not powerful enough to answer Haefliger's question for $i > n - r$.

The method of proof is essentially that employed by M. Jacques Vey [10] in proving a vanishing theorem for the cohomology of the formal Poisson algebra.

We describe below how the cohomology groups of $\mathscr{A}_{n,r}$ ($\mathscr{A}_r$) are related to the characteristic classes of a flag of foliations (a foliation). For more details see [3] and [1].

Let M^m be a smooth manifold of dimension m. A flag of smooth foliations of codimensions r, n ($r \leq n$) is a pair of foliations $\mathscr{F}_r$, $\mathscr{F}_n$ on M of codimensions r, n respectively such that the leaves of $\mathscr{F}_n$ are contained in the leaves of $\mathscr{F}_r$. Let ν_r be the normal bundle of $\mathscr{F}_r$ and let

$$\nu_{n-r} = \frac{\text{normal bundle of } \mathscr{F}_n}{\text{normal bundle of } \mathscr{F}_r}.$$

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Let $E(\mathscr{F}_n, \mathscr{F}_r)$ be the principal $Gl(r) \times Gl(n-r)$ bundle associated to $\nu_r \oplus \nu_{n-r}$; let $E(\mathscr{F}_r)$ be the principal $Gl(r)$ bundle associated to ν_r. The inclusion

$$i: \nu_r \rightarrow \nu_r \oplus \nu_{n-r}$$

and the projection

$$\pi: \nu_r \oplus \nu_{n-r} \rightarrow \nu_r$$

induce

$$i': E(\mathscr{F}_r) \rightarrow E(\mathscr{F}_n, \mathscr{F}_r) \quad \text{and} \quad \pi': E(\mathscr{F}_n, \mathscr{F}_r) \rightarrow E(\mathscr{F}_r)$$

such that $\pi' \circ i' = \mathrm{id}_{E(\mathscr{F}_r)}$. Hence i', π' induce on the cohomology level maps i^* and π^* satisfying

$$\begin{array}{ccc} H^*(E(\mathscr{F}_n, \mathscr{F}_r), \mathbf{R}) & \xrightarrow{i^*} & H^*(E(\mathscr{F}_r), \mathbf{R}) \\ & \searrow^{\pi^*} & \downarrow \mathrm{id} \\ & & H^*(E(\mathscr{F}_r), \mathbf{R}). \end{array}$$

Similarly there is a canonical inclusion $i_1: \mathscr{A}_r \rightarrow \mathscr{A}_{n,r}$ and a projection $\pi_1: \mathscr{A}_{n,r} \rightarrow \mathscr{A}_r$ such that $\pi_1 \circ i_1 = \mathrm{id}_{\mathscr{A}_r}$. Hence on the cohomology level, we have $\pi_1^*: H^*(\mathscr{A}_r, \mathbf{R}) \rightarrow H^*(\mathscr{A}_{n,r}, \mathbf{R})$ which is injective and $i_1^*: H^*(\mathscr{A}_{n,r}, \mathbf{R}) \rightarrow H^*(\mathscr{A}_r, \mathbf{R})$ which is surjective.

Given a smooth foliation $\mathscr{F}_r$ of codimension r on M^m, there is a homomorphism

$$j_{\mathscr{F}_r}: H^*(\mathscr{A}_r, \mathbf{R}) \rightarrow H^*(E(\mathscr{F}_r), \mathbf{R})$$

whose image depends on the integrable homotopy class of $\mathscr{F}_r$. The elements of the image of $j_{\mathscr{F}_r}$ are called characteristic classes of $\mathscr{F}_r$. For this reason one may view $H^*(\mathscr{A}_r, \mathbf{R})$ as universal characteristic classes of codimension r foliations.

In a similar way, given a smooth flag on M^m, one can construct a homomorphism

$$j_{(\mathscr{F}_n, \mathscr{F}_r)}: H^*(\mathscr{A}_{n,r}, \mathbf{R}) \rightarrow H^*(E(\mathscr{F}_n, \mathscr{F}_r), \mathbf{R}).$$

Given a flag $(\mathscr{F}_n, \mathscr{F}_r)$ of foliations, we have the following commutative diagram:

$$\begin{array}{ccc} H^k(\mathscr{A}_{n,r}, \mathbf{R}) & \xrightarrow{i_1^*} & H^k(\mathscr{A}_{r,}, \mathbf{R}) \\ \downarrow j_{(\mathscr{F}_n, \mathscr{F}_r)} & & \downarrow j_{\mathscr{F}_r} \\ H^k(E(\mathscr{F}_n, \mathscr{F}_r), \mathbf{R}) & \xrightarrow{i} & H^k(E(\mathscr{F}_r), \mathbf{R}). \end{array}$$

The i_1^* associates to a characteristic class of a flag of foliations, the corresponding characteristic class of the bigger foliation. Thus the elements of the kernel of i_1^* are precisely those additional characteristic classes one gets by subfoliating a codimension r foliation. The geometric implication of the canonical isomorphism $H^k(A_{n,r}, \mathbf{R}) \simeq H^k(A_r, \mathbf{R})$ $(k \leq n - r)$ is that these additional characteristic classes can appear only in $H^k(\mathscr{A}_{n,r}, \mathbf{R})$, $k > n - r$.

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2. Let S denote the polynomial algebra in n indeterminates $x_1, \ldots, x_n$ over $\mathbf{R}$. Let $\Lambda = \bigoplus_{p=0}^{n} \Lambda^p$ denote the exterior algebra with generators $e_1, \ldots, e_n$ over $\mathbf{R}$ with deg $e_i = 1$. Let M be a module over S. We have the following complex:

$$C^s(M)\colon \cdots \to \Lambda^p \otimes M \xrightarrow{d} \Lambda^{p-1} \otimes M \to \cdots \to M \to 0 \tag{2.1}$$

where d is defined by

$$d((e_{i_1} \wedge \cdots \wedge e_{i_p}) \otimes m) = \sum (-1)^{t+1} (e_{i_1} \wedge \cdots \wedge \hat{e}_{i_t} \wedge \cdots \wedge e_{i_p}) \otimes x_{i_t} m.$$

We have $d^2 = 0$ and the homology of this complex is denoted by $H^S(M)$.

Consider the map $\varepsilon\colon S \to \mathbf{R}$ defined by $\varepsilon(x_i) = 0$. Then $\mathbf{R}$ becomes a S module and we consider the complex

$$\to \Lambda^p \otimes S \to \Lambda^{p-1} \otimes S \to \cdots \to S \xrightarrow{\varepsilon} \mathbf{R} \to 0. \tag{2.2}$$

It is a free S-module resolution for $\mathbf{R}$ and is known as Koszul resolution. For more details on this see [7].

3. Let L be a topological Lie algebra and L^* its topological dual: $L^* = \{f\colon L \to \mathbf{R} \mid f \text{ is continuous and linear}\}$. Let $T \subset L$ be a finite dimensional abelian subalgebra. Then T acts on L and L^* via adjoint representation and this extends to an action of the universal enveloping algebra of T which is $S(T)$, the symmetric algebra of T.

PROPOSITION 3.1. *Let L, T be as above. Assume that L^* is a free $S(T)$ module. Then we have*

$$H^i(L, \mathbf{R}) = 0 \quad \textit{for } 0 < i \leq \dim T.$$

Proof. We need the following lemma.

LEMMA 3.2. *Under the assumptions of the proposition, $C^i(L, \mathbf{R}) = \Lambda^i(L^*)$ is a projective $S(T)$ module.*

Let $C = \bigoplus_{i>0} \Lambda^i(L^*)$. Consider the double complex

$$A = (\Lambda T \oplus_{\mathbf{R}} S(T)) \oplus_{S(T)} C, \quad A^{p,q} = (\Lambda^{-p} T \oplus_{\mathbf{R}} S(T)) \oplus_{S(T)} C^q$$

There are two differentials d' and d'' on A:

$$d'\colon A^{p,q} \to A^{p+1,q}, \quad d''\colon A^{p,q} \to A^{p,q+1};$$

d' is induced by the boundary operator of the Koszul resolution and d'' by the coboundary operator of the complex C. One can check that $d'\,d'' + d''\,d' = 0$. $H(A)$ denotes the cohomology of A with respect to the total differential $d = d' + d''$.

This double complex is zero outside the strip $-n \le i \le 0$, $j \ge 0$ where $n = \dim T$. Hence the associated spectral sequences converge.

Consider the first filtration of A, given by $F^pA = \bigoplus_{i \ge p} A^{i,q}$. The E_0 term of the corresponding spectral sequence is given by

$$E_0^p = \frac{F^pA}{F^{p+1}A} \simeq A^{p*} = \bigoplus_q A^{p,q};$$

the differential d_0 on E_0 is the differential d'' on A. Hence

$$E_1^p = H(E_0^p) = H_{d''}(A^{p*}).$$

Consider

$$A^{p,q} = (\Lambda^{-p} T \otimes_{\mathbf{R}} S(T) \otimes_{S(T)} C^q) \simeq \Lambda^{-p} T \otimes_{\mathbf{R}} C^q.$$

As the differential in the complex C does not involve any action of L on T (and hence on $\Lambda^{-p}T$) we have

$$H_{d''}(A^{p*}) = H(\Lambda^{-p} T \otimes C) = \Lambda^{-p} T \otimes H(C).$$

The differential d_1 on E_1 is that on the Koszul complex, and as the action of $S(T)$ on $H(C)$ is trivial we have

$$E_2 = \Lambda T \otimes_{\mathbf{R}} H(C)$$

That is, $E_2^{p,q} = \Lambda^{-p} T \otimes H^q(C)$.

Similarly considering the second filtration, the E_0 term of the associated spectral sequence is given by

$$E_0^q = (\Lambda T \otimes_{\mathbf{R}} S(T)) \otimes_{S(T)} C^q.$$

The differential d_0 on E_0 is induced by that of the Koszul complex $\Lambda T \otimes S(T)$. As the Koszul complex is a free $S(T)$ module resolution for $\mathbf{R}$ and C^q is a projective $S(T)$ module we have

$$E_1^q = \mathbf{R} \otimes_{S(T)} C^q.$$

This spectral sequence collapses and $E_2 = H(\mathbf{R} \otimes_{S(T)} C)$. Therefore for the first spectral sequence,

$$E_2^{p,q} \to H^{p+q}(\mathbf{R} \otimes_{S(T)} C).$$

Let r be the first integer such that $H^r(C) \neq 0$. Then

$$E_2^{-n,r} = H^r(C) \neq 0 \quad \text{and} \quad E_2^{-n-i,r+i} = E_2^{-n+i,r-1} = 0.$$

This implies that $E_2^{p,q} \simeq E_\alpha^{p,q}$ whenever $p + q = -n + r$. Hence

$$E_2^{-n,r} = H^r(C) \simeq H^{-n+r}(\mathbf{R} \otimes_{S(T)} C).$$

But $H^i(\mathbf{R} \otimes C) = 0$ for $i \leq 0$. Therefore $-n + r > 0$. This proves the proposition.

Proof of Lemma 3.2. As L^* is $S(T)$ free, we can write $L^* = S(T) \otimes V$. As $\Lambda^k(L^*)$ is the direct summand of $\otimes^k (L^*)$, it is enough to prove that $\otimes^k (L^*)$ is $S(T)$ free. We know that $\otimes^k (L^*)$ is $\otimes^k S(T)$ free because $\otimes^k (L^*) = \otimes^k (S(T)) \otimes \otimes^k V$. Therefore it is sufficient to prove that $\otimes^k (S(T))$ is $S(T)$ free.

$S(T)$ acts on $\otimes^k (L^*)$ through the map α: $S(T) \to \otimes^k S(T)$ where

$$x_i \to x_i \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes x_i \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes x_i,$$

$$S(T) = \mathbf{R}[t_1, \ldots, t_n] \quad \text{and} \quad \otimes^k S(T) = \mathbf{R}[t_j^i],\ i \leq n, j \leq k,$$

and the action α sends $t_i \to \sum t_i^j$. Using new indeterminates s_j^i we have the isomorphism ϕ: $\mathbf{R}[s_j^i] \to \mathbf{R}[t_j^i]$ where $s_i^1 \to \sum t_i^j$ and $s_i^j \to t_i^j$, $j > 1$. $\mathbf{R}[s_j^i]$ is a free $\mathbf{R}[t_i]$ module through the action $\phi^{-1} \circ \alpha$ because $\phi^{-1} \circ \alpha(t_i) = s_i^j$. Therefore $\mathbf{R}[t_j^i]$ is a free $\mathbf{R}[t_i]$ module.

We next prove that the assumption of Proposition 3.1 is satisfied for $L = a_{n,r}$ and $T = \{\sum_{i=1}^r \alpha_i\, \partial/\partial x_i \,|\, a_i \in \mathbf{R}\}$.

LEMMA 3.3. *Let* $L = \mathscr{A}_{n,r}$ *and* $T = \{\sum_{i=1}^r \alpha_i\, \partial/\partial x_i \,|\, \alpha_i \in \mathbf{R}\}$. *Then* L^* *is free over* $S(T)$.

Proof. For $i \leq r$ let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a multiindex and for $r + 1 \leq i \leq n$ let $\beta = (\beta_1, \ldots, \beta_n)$ be a multiindex.

Define ∂_α^i, $\partial_\beta^j \in L^*$ by

$$\partial_\alpha^i\left(\sum f_i \frac{\partial}{\partial x_1}\right) = \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f_i}{\partial^{\alpha_1}, \ldots, \partial^{\alpha_r}}\bigg|_0,$$

$$\partial_\beta^j\left(\sum f_i \frac{\partial}{\partial x_i}\right) = \frac{1}{\beta!} \frac{\partial^{|\beta|} f_j}{\partial^{\beta_1}, \ldots, \partial^{\beta_n}}\bigg|_0$$

It is known [2] that $\{\partial_\alpha^i, \partial_\beta^j \,|\, 1 \leq i \leq r, r + 1 \leq j \leq n\}$ generates L^*.

If θ denotes the adjoint representation of L on L^* then

$$\theta\left(\frac{\partial}{\partial x_k}\right)\partial^i_\alpha = -(\alpha_k + 1)\,\partial_{(\alpha_1,\ldots,\alpha_k+1,\ldots,\alpha_r)} \quad \text{if } k \le r$$
$$= 0 \quad \text{if } k > r,$$

and

$$\theta\left(\frac{\partial}{\partial x_k}\right)\partial^j_\beta = -(\beta_k + 1)\,\partial^j_{(\beta_1,\ldots,\beta_k+1,\ldots,\beta_n)}.$$

The following algebraic fact is well known (for example, see [8]).

Let M be a graded module over $\mathbf{R}[x_1, \ldots, x_n]$. The following are equivalent:

(i) M is free over $\mathbf{R}[x_1, \ldots, x_n]$.
(ii) x_i is a nonzero divisor of

$$\frac{M}{(x_1, \ldots, x_{i-1})M}$$

for $1 \le i \le n$.

By virtue of this fact it suffices to prove that if $k \le r$ and

$$\partial\left(\frac{\partial}{\partial x_k}\right)\partial^i_\alpha = -(\alpha_k + 1)\,\partial_{(\alpha_1,\ldots,\alpha_k+1,\ldots,\alpha_r)} \in \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right)L^*$$

then

$$\partial^i_\alpha \in \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{k-1}}\right)L^*.$$

This implies that $\alpha_i > 0$ for $i \le k-1$; say $\alpha_1 > 0$. Then

$$\partial^i_\alpha = \frac{-1}{\alpha_1}\,\theta\left(\frac{\partial}{\partial x_1}\right)\partial_{(\alpha_1-1,\ldots,\alpha_r)}.$$

Hence

$$\partial^i_\alpha \in \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{k-1}}\right)L^*$$

Now we use Proposition 3.1 to get:

THEOREM 3.4. *$H^i(\mathscr{A}_{n,r}, \mathbf{R}) = 0$ for $i \le r$.*

Consider

$$\Pi = \left\{\sum_{r+1}^{n} f_i(x_1, \ldots, x_n)\,\frac{\partial}{\partial x_i}\right\} \subset \mathscr{A}_{n,r}$$

and

$$T = \left\{ \sum_{i=r+1}^{n} \alpha_i \partial/\partial x_i \mid \alpha_i \in \mathbf{R} \right\}.$$

then T is abelian and Π^* is generated by

$$\{\partial^i_\beta \mid r + 1 \leq i \leq n \quad \text{and} \quad \beta = (\beta_1, \ldots, \beta_n)\}$$

As above, one can show that Π^* is $S(T)$ free. Therefore we have the following result.

THEOREM 3.5. *$H^i(\Pi, \mathbf{R}) = 0$ for $0 < i \leq n - r$.*
From this we deduce the required theorem:

THEOREM 3.6. *$H^i(\mathscr{A}_{n,r}, \mathbf{R}) \simeq H^i(\mathscr{A}_r, \mathbf{R})$ for $i \leq n - r$.*

Proof. Consider the Hochschild-Serre spectral sequence [6] for $\mathscr{A}_{n,r}$ relative to the ideal Π. The E_2 term is given by

$$E_2^{p,q} = H^q(\mathscr{A}_r, H^p(\Pi, \mathbf{R})).$$

Thus $E_2^{0,q} = H^q(\mathscr{A}_r, \mathbf{R})$ and $E_2^{p,q} = 0$ for $0 < q \leq n - r$.

As the Hochschild-Serre spectral sequence converges to $H^*(\mathscr{A}_{n,r}, \mathbf{R})$, we have

$$H^i(\mathscr{A}_{n,r}, \mathbf{R}) \simeq H^i(\mathscr{A}_r, \mathbf{R}), \quad i \leq n - r, \qquad \text{Q.E.D.}$$

Remark. Let $r \leq n$. Let $\Gamma_{n,r}$ be the topological groupoid of germs of local diffeomorphisms of $\mathbf{R}^n$ of the form

$$f(x, y) = (g(x), h(x, y))$$

where $(x, y) = (x_1, \ldots, x_r, y_1, \ldots, y_{n-r}) \in \mathbf{R}^r \times \mathbf{R}^{n-r}$, g is a local diffeomorphism of $\mathbf{R}^r$ and h is a smooth map from an open set of $\mathbf{R}^n$ to $\mathbf{R}^{n-r}$. Let Γ_r be the topological groupoid of germs of local diffeomorphisms of $\mathbf{R}^r$. Let $B\Gamma_{n,r}$, $B\Gamma_r$ denote the Haefliger's classifying spaces for $\Gamma_{n,r}$, Γ_r respectively [5]; $B\Gamma_{n,r}$, $B\Gamma_r$ classify $\Gamma_{n,r}$ foliations (flags of foliations) and codimension r foliations, respectively.

There is a canonical morphism from $\Gamma_{n,r}$ to $\Gamma_r \times Gl_{n-r}$ given by

$$f = (g, h) \rightarrow (g, d_y(h))$$

This induces a map π on the classifying space level:

$$\pi \colon B\Gamma_{n,r} \rightarrow B\Gamma_r \times BGl_{n-r}$$

The author has proved in his thesis [9] that π is n-connected.

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