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A PROPERTY OF ATRIODIC CONTINUA

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0. Introduction

A compact metric continuum M is a *triod* provided M contains a subcontinuum H such that $M - H$ has at least three components. If M contains no triod, then it is atriodic.

In [1], T. Maćkowiak and E. D. Tymchatyn proved that each non-unicoherent subcontinuum of a compact, metric, atriodic continuum has void interior and is terminal. This paper proves this result also, but uses a different technique that displays useful properties of atriodic continua. Continua whose non-unicoherent proper subcontinua have void interior are said to have *property IUC*. Hence the class of IUC continua generalizes the classes of atriodic and hereditarily unicoherent continua.

Throughout this paper M will denote a compact metric continuum. The notation $M = A \cup B$ denotes that M is the sum of two proper subcontinua A and B unless otherwise stated.

1. Atriodic continua

We first begin with a useful lemma.

LEMMA 1.1. *If M is atriodic and $M = A \cup B$ then $M - A$ is connected.*

Proof. Suppose that $M - A$ is not connected. Hence $M - A$ has exactly two components X and Y. Now $\bar{X} \cap \bar{Y} = \emptyset$, for if not, $M = A \cup \bar{X} \cup \bar{Y}$ would contain a triod by [3, Theorem 1.8, page 443]. Also, A is irreducible about

$$(A \cap \bar{X}) \cup (A \cap \bar{Y}).$$

For if there exists a proper subcontinuum P of A containing

$$(A \cap \bar{X}) \cup (A \cap \bar{Y})$$

and H is a component of $A - P$ then $(\bar{H} \cup P) \cup (\bar{X} \cup P) \cup (\bar{Y} \cup P)$ would contain a triod.

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Let Q be a subcontinuum of B irreducible about $(A \cap \bar{X}) \cup (A \cap \bar{Y})$. If $A \nsubseteq Q$ then $A \cup Q \cup \bar{X}$ must contain a triod. Hence $A \subseteq Q \subseteq B$, a contradiction, and the lemma is proved.

Observe that if M is atriodic and $M = A \cup B$ then $M = A \cup \overline{M - A}$, where $A \cap \overline{M - A}$ is connected only in case $A \cap B$ is connected. Likewise

$$M = \overline{M - A} \cup \overline{M - (\overline{M - A})}$$

where the coherence of the intersection depends on $A \cap B$.

LEMMA 1.2. *If M is atriodic and $M = A \cup B$, where each of A and B is the closure of the complement of the other, then A and B are unicoherent.*

Proof. Suppose that $A = A_1 \cup A_2$ where $A_1 \cap A_2$ is the sum of the two mutually exclusive closed sets P and Q. Now $A_1 \cap A_2 \cap B = \emptyset$ since M is atriodic. Hence each of P and Q misses B. We will assume that A_2 intersects B.

Let U and V be open sets in M containing P and Q respectively such that $\bar{U} \cap \bar{V} = \emptyset$ and each of $\bar{U}$ and $\bar{V}$ misses B. Let D_1 and D_2 be components of $U \cap A_1$ and $V \cap A_1$ which intersect P and Q respectively. Hence no one of $\bar{D}_1 \cup A_2$, $\bar{D}_2 \cup A_2$, and $B \cup A_2$ is contained in the union of the other two, and hence their union contains a triod. Hence A, and correspondingly B, is unicoherent.

It is noted as a corollary to Lemma 1.2 that if M is atriodic and A is a proper subcontinuum of M with $A = \overline{A^0}$ then A is unicoherent, where A^0 denotes the interior of A.

THEOREM 1.3. *If M is atriodic, $M = A \cup B$, and $A \cap B$ is connected, then M is unicoherent.*

Proof. It can be assumed that $A = \overline{M - B}$, $B = \overline{M - A}$, and by Lemma 1.2 that each of A and B is unicoherent. Suppose that $M = H \cup K$ where $H \cap K$ is the sum of the disjoint continua C_1 and C_2. If $H \subseteq A$ and $K \subseteq B$ then $A = H$ and $B = K$, a contradiction since $A \cap B \neq H \cap K$. Suppose then that H intersects $M - A$ and $M - B$.

Case I. $K \subseteq B$. Then $A \subseteq H$ and each of C_1 and C_2 is in B. Now $H - A$ is connected, for if not, let E and F denote two components of $H - A$. Hence $\bar{E} \cup (A \cap B)$ and $\bar{F} \cup (A \cap B)$ are continua and

$$A \cup [\bar{E} \cup (A \cap B)] \cup [\bar{F} \cup (A \cap B)]$$

forms a triod.

But then $B = K \cup \overline{H - A}$ and $K \cap \overline{H - A} = K \cap H = C_1 \cup C_2$. Hence B is not unicoherent, a contradiction.

Case II. K intersects $M - A$ and $M - B$. We suppose that H and K differ in $M - B$. Note again that each of $H - (B \cap H)$ and $K - (B \cap K)$ is connected. Hence

$$B \cup [\overline{H - (B \cap H)} \cup (A \cap B)] \cup [\overline{K - (B \cap K)} \cup (A \cap B)]$$

contains a triod by [3]. Therefore M is unicoherent and the theorem is established.

COROLLARY 1.4. *If M is atriodic, A is a proper subcontinuum of M, and $M - A$ is not connected then M is unicoherent.*

Proof. Suppose $M - A$ is the sum of the two disjoint open sets C and D. Then $M = (C \cup A) \cup (D \cup A)$ and $(C \cup A) \cap (D \cup A) = A$ and hence M is unicoherent.

THEOREM 1.5. *If M is atriodic then each proper subcontinuum of M with interior is unicoherent.*

Proof. Let H be a proper subcontinuum of M with interior. If $M - H$ is not connected then, by Corollary 1.4, M is unicoherent, and by [2] the theorem is proved.

Suppose then that $M - H$ is connected. Now $H^0 = M - (\overline{M - H})$ is connected and hence $\overline{H^0}$ is unicoherent. Also $K = M - \overline{H^0}$ is connected since

$$M = \overline{H^0} \cup \overline{M - H}.$$

If the boundary of H, bdy H, is connected then bdy $H = H \cap \overline{M - H}$ and by Theorem 1.3 M is unicoherent, which completes the proof. Likewise if bdy $\overline{H^0}$ is connected M is unicoherent and therefore so is H.

Hence we may suppose that bdy H is the sum of the disjoint continua B_1 and B_2, and bdy $\overline{H^0}$ is the sum of the disjoint continua D_1 and D_2 where $D_1 \subseteq B_1$ and $D_2 \subseteq B_2$.

If $D_1 = B_1$ and $D_2 = B_2$ then $H = \overline{H^0}$ and hence H is unicoherent. So suppose that D_1 is a proper subcontinuum of B_1. But then

$$H = (\overline{H^0} \cup B_2) \cup B_1 \quad \text{and} \quad (\overline{H^0} \cup B_2) \cap B_1 = D_1$$

and again H is unicoherent by Theorem 1.3.

2. IUC continua

The definition of property IUC is motivated by Theorem 1.5.

DEFINITION. The continuum M is said to have property IUC (M is an IUC continuum) provided each proper subcontinuum of M with interior is unicoherent. If M has property IUC hereditarily, M is said to be an HIUC continuum.

Clearly the class of IUC continua contains the class of hereditarily unicoherent continua (HUC), and by Theorem 1.5 also contains all atriodic continua. The next example easily shows that it is proper containment.

Example 1. Let K denote the sum of a circle and an arc whose intersection is an endpoint of the arc. Let M be the sum of K and a half-ray which limits on K. Hence M has property IUC and is neither atriodic nor hereditarily unicoherent.

Theorem 1.3 did not depend on the atriodicity of M, but rather the IUC property of M, as is now shown.

THEOREM 2.1. *If M has property* IUC *and $M = A \cup B$ where $A \cap B$ is connected, then M is unicoherent.*

Proof. Let $C = A \cap B$ and suppose that $M = A_1 \cup B_1$ where $A_1 \cap B_1$ is the sum of two disjoint closed sets C_1 and C_2. Let A' be minimal with respect to $M = A' \cup B$. Note that since M has property IUC, A' is unique and $A - B \subseteq A'$. Also $A' \cap B = A' \cap C$ and hence is a continuum. Let B' be the unique subcontinuum of B minimal with respect to $M = A' \cup B'$. Now $A' \cap B' = C'$ is a continuum and hence, without loss of generality, we suppose that $A' = A$, $B' = B$, and $C' = C$.

It will now be shown that each of A_1 and B_1 intersects both A and B. Suppose rather that $A_1 \subseteq M - B$. Hence B_1 intersects both A and B. Let X_1 be a component of $B_1 \cap (M - B)$ from C_1 to bdy $(M - B)$ and X_2 be a component of $B_1 \cap (M - B)$ from C_2 to bdy $(M - B)$ and X_2 be a component of $B_1 \cap (M - B)$ from C_2 to bdy $(M - B)$. Now $D = \bar{X}_1 \cup C \cup \bar{X}_2$ is a continuum. But $A_1 \cup D$ is a proper subcontinuum of M with interior and $A_1 \cap D$ is not connected. Hence A_1 intersects both A and B.

Also each component of C_1 and C_2 intersects C. For suppose the component D of C_1 misses C. Hence we may suppose that $D \subseteq M - A$. Let X_1 be a component of $A_1 - A$ from D to the boundary of $A_1 - A$ and let X_2 be a component of $B_1 - B$ from D to the boundary of $B_1 - B$. Hence one of

$$A \cup \bar{X}_1 \cup D \cup \bar{X}_2 \quad \text{or} \quad C \cup \bar{X}_1 \cup D \cup \bar{X}_2$$

is a proper subcontinuum of M with interior which is not unicoherent, a contradiction.

Now if $A^0 \subseteq A_1$ and $B^0 \subseteq B_1$ then, by the minimality of A and B, $A \subseteq A_1$ and $B \subseteq B_1$. But $C = A \cap B \subseteq A_1 \cap B_1$ and since each component of $A_1 \cap B_1$ intersects C we have that $A_1 \cap B_1$ is connected. So we may suppose that $A^0 \nsubseteq A_1$.

If $B^0 \subseteq B_1$ then $B \subseteq B_1$, but $A_1 \cup B$ is then a proper subcontinuum of M with interior, and $A_1 \cap B$ is the union of the disjoint sets $C_1 \cap B$ and $C_2 \cap B$. Hence B_1 does not contain B^0.

Now C does not lie wholly in either A_1 or B_1, and each of $A_1 \cap C$ and $B_1 \cap C$ is connected since M has property IUC. But $A_1 \cup (B_1 \cap C)$ is a proper subcontinuum of M with interior and

$$A_1 \cap (B_1 \cap C) = (A_1 \cap B_1) \cap C = (C_1 \cap C) \cup (C_2 \cap C)$$

and hence is not connected, a contradiction. Hence M is unicoherent.

COROLLARY 2.2. *If M has property* IUC, *M is not unicoherent, and A is a proper subcontinuum of M then $M - A$ is connected.*

Proof. If $M - A$ is the union of two disjoint open sets X and Y then each of $A \cup X$ and $A \cup Y$ is a continuum and $M = (A \cup X) \cup (A \cup Y)$ where the intersection is A. Hence by Theorem 2.1, M is unicoherent, a contradiction.

DEFINITION. The subcontinuum H of M is *terminal* provided that if K is a subcontinuum of M which intersects H then $K \subseteq H$ or $H \subseteq K$.

The following theorem and its proof was noted by Professor W. T. Ingram, and completes the proof of the theorem of Mackowiak and Tymchatyn.

THEOREM 2.3. *If M is an H* IUC *continuum then each non-unicoherent subcontinuum of M is terminal.*

Proof. If H is a non-unicoherent subcontinuum of M, K is a subcontinuum of M intersecting H, and $K \nsubseteq H$ then $H \subseteq K$. For otherwise, H would be a non-unicoherent subcontinuum with interior with respect to $H \cup K$, a contradiction.

Example 1 shows that Theorem 2.3 is not true for IUC continua in general.

In regards to the structure of IUC continua we make the following observation.

COROLLARY 2.4. *If M has property* IUC *and is not* unicoherent, *then M is the sum of two irreducible subcontinua with non-connected intersection.*

Proof. If $M = A \cup B$ where $A \cap B$ is not connected then

$$M = \overline{M - A} \cup \overline{M - (\overline{M - A})}.$$

3. ICT continua

DEFINITION. The continuum M has *property* ICT provided each triod of M has void interior.

Hence property ICT seems to be a more natural generalization of atriodic continua. In fact we show that property ICT is stronger than property IUC.

Theorem 3.1. *Property* ICT *implies* IUC.

Proof. Suppose M has property ICT and K is a proper subcontinuum of M with interior which is not unicoherent. Hence $K = K_1 \cup K_2$ where $K_1 \cap K_2$ is the sum of two continua C_1 and C_2. Since K has interior we may assume that K_2 has interior with respect to M. We now show that K_2 is irreducible about $C_1 \cup C_2$.

Suppose A is a proper subcontinuum of K_2 containing $C_1 \cup C_2$. Since K_2 is not a triod $K_2 - A$ has one or two components. Hence either A or a component of $K_2 - A$ has interior with respect to M. Let $\mathscr{U}$ and $\mathscr{V}$ be open in M containing C_1 and C_2 respectively such that $\overline{\mathscr{U}} \cap \overline{\mathscr{V}}$ is void. Let D_1 and D_2 be components of $\mathscr{U} \cap K_1$ and $\mathscr{V} \cap K_1$ intersecting C_1 and C_2 respectively. Let X be a component of $K_2 - A$, choosing X to have interior in case A has void interior. Then $A \cup \bar{D}_1 \cup \bar{D}_2 \cup \bar{X}$ is a triod with interior, a contradiction.

Now $K_2 - (C_1 \cup C_2)$ is connected, so define $\hat{K}_2 = \overline{K_2 - (C_1 \cup C_2)}$. Hence

$$K = K_1 \cup \hat{K}_2$$

and $K_1 \cap \hat{K}_2$ is the sum of the two continua $C_1 \cap \hat{K}_2$ and $C_2 \cap \hat{K}_2$ with void interior with respect to K. Hence we may assume that $K_2 = \hat{K}_2$. If K_1 has interior we may duplicate the above and redefine K_1 correspondingly.

Consider now $M - K$. Since M is not a triod $M - K$ has one or two components. If $M - K$ is connected then bdy $(\overline{M - K})$ has only one or two components, and if $M - K$ has two components B_1 and B_2 then necessarily bdy $(\bar{B}_1)$ and bdy $(\bar{B}_2)$ are continua. Hence we consider three cases:

(I) $\overline{M - K}$ and bdy $(\overline{M - K})$ are continua.
(II) $\overline{M - K}$ is a continuum and bdy $(\overline{M - K})$ has two components.
(III) $M - K$ has two components B_1 and B_2.

Suppose (I). If bdy $(\overline{M - K})$ misses $C_1 \cup C_2$ then we may assume

$$\text{bdy } (\overline{M - K}) \subseteq K_1.$$

Let $\mathscr{U}$ and $\mathscr{V}$ be open sets containing C_1 and C_2 respectively such that $\overline{\mathscr{U}}$ misses $\overline{\mathscr{V}}$. Let D_1 and D_2 be components of $\mathscr{U} \cap K_2$ and $\mathscr{V} \cap K_2$ intersecting C_1 and C_2 respectively. Hence

$$\overline{M - K} \cup K_1 \cup \bar{D} \cup \bar{D}_2$$

is a triod with interior.

Suppose then that bdy $(\overline{M - K})$ intersects $C_1 \cup C_2$ and K_2 has interior. Now $K_1 \cup$ bdy $(M - K)$ is a proper subcontinuum of K. But

$$E = K - [K_1 \cup \text{bdy } (\overline{M - K})]$$

is connected since M has property ICT. Now the boundary of $\bar{E}$ with respect to K, $\mathrm{bdy}_K(\bar{E})$, has only one or two components. If $\mathrm{bdy}_K(\bar{E})$ is a continuum then so is

$$Q = [\mathrm{bdy}\,(\overline{M - K}) \cup K_1] \cap K_2.$$

But Q is a proper subcontinuum of K_2 containing C_1 and C_2, contradicting the irreducibility of K_2. Also, if $\mathrm{bdy}_K(\bar{E})$ has two components, one can easily construct a triod using $K_1 \cup \mathrm{bdy}\,(\overline{M - K})$ as the core and $\overline{M - K}$ as one leg with interior.

Suppose (II). Let $\mathrm{bdy}\,(\overline{M - K}) = D_1 \cup D_2$ where each of D_1 and D_2 is a continuum. There are three subcases to consider.

(1) $D_1 \cup D_2$ misses $C_1 \cup C_2$. Without loss of generality we may assume $D_1 \subseteq K_1$. One can boundary bump from each of C_1 and C_2 into K_2 missing D_2 and construct a triod with the core K_1, and $\overline{M - K}$ as one leg. Boundary bumping refers to the technique used in the proof of Lemma 1.2.

(2) D_1 intersects $C_1 \cup C_2$ and D_2 misses $C_1 \cup C_2$. We may suppose that $D_2 \subseteq K_2$. Since K is not a triod $K - (D_1 \cup K_1)$ has one or two components. If there are two components one can construct a triod with interior using each component as a leg of the triod.

Hence $K - (D_1 \cup K_1)$ is connected. Now $\mathrm{bdy}_K [K - (D_1 \cup K_1)]$ is not connected, for if so, $K_2 \cap (D_1 \cup K_1)$ is a proper subcontinuum of K_2 containing $C_1 \cup C_2$. So let $\mathrm{bdy}_K [K - (D_1 \cup K_1)]$ be the sum of the two disjoint closed sets B_1 and B_2. Once again, boundary bumping from B_1 and B_2 into $K - (D_1 \cup K_1)$ missing D_2 helps us to build a triod with interior.

(3) Each of D_1 and D_2 intersects $C_1 \cup C_2$. Now

$$K - (D_1 \cup K_1 \cup D_2) = E$$

is connected since M has property ICT. But now an argument analogous to (2) applies.

Case (III) is analogous to Case (I) and hence the proof is omitted.

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