ILLINOIS JOURNAL OF MATHEMATICS
Volume 28, Number 4, Winter 1984

FACTORIZATION OF POSITIVE MULTILINEAR MAPS

BY

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1. Introduction

Let (X, μ) be a finite measure space and let $L_0(X, \mu)$ denote the space of (equivalence classes of) all μ-measurable functions on X. E. M. Nikisin [6] and B. Maurey [5] proved several factorization theorems for linear and sub-linear operators, where a (sub-)linear operator T from a Banach space E into $L_p(X, \mu)$ $(p \geq 0)$ factors through (weak-) L_r $(r \geq 1)$, if there exists $\phi \in L_s$ for some $s \geq 0$ with $\phi > 0$ a.e., such that

$$\frac{1}{\phi} \cdot T(E) \subseteq (\text{weak-})L_r.$$

For an excellent survey of these theorems and the many applications of them we refer to J. E. Gilbert's paper [2]. In this same article Gilbert indicates that there are available versions of weak-type factorizations for maximal operators defined by multilinear operators, but it was also noticed that strong-type factorizations for multilinear operators had not yet been studied. In this paper we shall prove strong-type factorizations for positive multilinear operators. Our approach uses the positive projective tensor product of Banach lattices and we also use some of the linear operator results of Nikisin and Maurey. The results for bilinear operators are typical for the multilinear case, but we could not restrict ourselves to the bilinear case. To prove Theorem 3.2 and Theorem 3.5 for bilinear operators with values in L_r with $r > 0$, we need the result of the same theorems for trilinear operators. Therefore we consider the general multilinear case. For the same reason we shall consider tensor products of n Banach lattices. The organization of this paper is as follows. In Section 2 we develop the necessary machinery of the theory of tensor products of Banach lattices. In Section 3 we prove the factorization theorems for positive multilinear operators from $L_{p_1} \times \cdots \times L_{p_n} \to L_q$ where $q \geq 0$.

Received May 24, 1982.
[1] Research supported in part by a grant of the University of South Carolina Research and Productive Scholarship Fund.

2. Tensor product of Banach lattices

For terminology about Riesz spaces and Banach lattices, see [4] and [7]. For notations concerning the tensor products of Banach lattices we follow [1], where the theory was developed for the tensor product of two such spaces. The development of the theory for n Banach lattices is completely analogous to the case $n = 2$, so that we shall omit the proofs of the general results, but only present proofs which are relevant for our application.

Let $E_1, \dots, E_n$ and F be Archimedean Riesz spaces. An n-linear map

$$B: E_1 \times \cdots \times E_n \to F$$

is called positive if $B(x_1, \dots, x_n) \in F^+$ whenever $x_k \in E_k^+$ $(k = 1, 2, \dots, n)$; it is called a Riesz n-morphism if $B(|x_1|, \dots, |x_n|) = |B(x_1, \dots, x_n)|$ for all $x_k \in E_k$ $(k = 1, \dots, n)$. Following [1] or [8] one can construct an Archimedean Riesz space $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ and a Riesz n-morphism

$$\otimes : E_1 \times \cdots \times E_n \to E_1 \bar{\otimes} \cdots \bar{\otimes} E_n.$$

We now list only these properties of this construction which are relevant for what follows:

(a) $E_1 \otimes \cdots \otimes E_n$ is dense in $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ in the sense that for any $u \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ there exist $x_k \in E_k^+$ $(k = 1, 2, \dots, n)$ such that for all $\varepsilon > 0$ there is a $v \in E_1 \otimes \cdots \otimes E_n$ with $|u - v| \leq \varepsilon(x_1 \otimes \cdots \otimes x_n)$.

(b) If $u \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$, then there exist $x_k \in E_k^+$ $(k = 1, \dots, n)$ such that $|u| \leq x_1 \otimes \cdots \otimes x_n$.

(c) If F is a uniformly complete Archimedean Riesz space, then there is a one-to-one correspondence between positive n-linear maps $B: E_1 \times \cdots \times E_n \to F$ and positive linear maps $T: E_1 \bar{\otimes} \cdots \bar{\otimes} E_n \to F$ such that $B = T \otimes$.

In what follows we shall be mainly interested in the case that $E_k = L_{p_k}(X_k, \mu_k)$ for some $p_k \geq 1$. In that case one can identify $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ with the Riesz space generated by

$$\{f_1(x_1), \dots, f_n(x_n): f_k \in E_{p_k}, k = 1, \dots, n\}$$

in

$$L_0(X_1 \times \cdots \times X_n, \mu_1 \times \cdots \times \mu_n).$$

If $E_1, \dots, E_n$ are Banach lattices, then we can define the positive-projective norm $\| \quad \|_{|\pi|}$ on $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ by means of

$$\|u\|_{|\pi|} = \inf \left\{ \sum_{i \leq m} \prod_{k \leq n} \|x_{i,k}\| : x_{i,k} \in E_k^+ \text{ such that } |u| \leq \sum_{i \leq m} x_{i,1} \otimes \cdots \otimes x_{i,n} \right\}.$$

One can show that $\| \quad \|_{|\pi|}$ is a Riesz norm on $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$, so that we define the Banach lattice $E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$ to be the completion of $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ with respect to $\| \quad \|_{|\pi|}$. As in [1] one now proves:

(d) For any Banach lattice F there is a one-to-one normpreserving correspondence between continuous positive n-linear maps $\beta\colon E_1 \times \cdots \times E_n \to F$ and continuous positive linear maps $T\colon E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n \to F$ such that $B = T \otimes$.

The following theorem is, for $n = 2$, partly contained in [1].

THEOREM 2.1. *Let $E_1, \ldots, E_n$ be Banach lattices, Suppose that the functional ρ defined on $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ by*

$$\rho(u) = \inf \left\{ \prod_{k \leq n} \|x_k\| : x_k \in E_k^+ \ (k = 1, \ldots, n), |u| \leq x_1 \otimes \cdots \otimes x_k \right\}$$

is subadditive. Then:

(i) *$\rho(u) = \|u\|_{|\pi|}$ on $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$.*

(ii) *If $u \in E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$, then there exist $x_k \in E_k^+$ $(k = 1, \ldots, n)$ such that $|u| \leq x_1 \otimes \cdots \otimes x_n$.*

(iii) *$E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ is relatively uniform dense in $E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$; that is, if $u \in E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$, then there exist $x_k \in E_k^+$ $(k = 1, \ldots, n)$ such that for all $\varepsilon > 0$ we can find $v \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ such that*

$$|u - v| \leq \varepsilon x_1 \otimes \cdots \otimes x_k.$$

(iv) *If F is a uniformly complete Archimedean Riesz space, then there is a one-to-one correspondence between positive n-linear maps*

$$B\colon E_1 \otimes \cdots \otimes E_n \to F$$

and positive linear maps

$$T\colon E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n \to F$$

such that $B = T \otimes$.

Proof. (i) If ρ is subadditive, then ρ is a Resiz seminorm, since clearly $\rho(\alpha u) = |\alpha| \rho(|u|)$. Since $\rho(u) \geq \|u\|_{|\pi|}$ on $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$, we see that ρ is actually a Riesz norm on $E_1 \otimes \cdots \otimes E_n$. Let G denote the completion of $E_1 \otimes \cdots \otimes E_n$ with respect to ρ and let

$$B_0\colon E_1 \times \cdots \times E_n \to G$$

denote the positive n-linear map $(x_1, \ldots, x_n) \to x_1 \otimes \cdots \otimes x_n$. Then

$$\|B_0\| = \sup\,(\rho(x_1 \otimes \cdots \otimes x_n)\colon \|x_n\| \leq 1) \leq 1.$$

Hence by (d) above there exists a continuous linear map $T: E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n \to G$ of norm ≤ 1 such that $B_0 = T \otimes$, which implies that

$$\rho(u) = \rho(Tu) \leq \|u\|_{|\pi|} \quad \text{for all } u \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_n.$$

Hence $\rho(u) = \|u\|_{|\pi|}$ for all $u \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$.

(ii) Let $u \in E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$. Then there exist $u_m \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ such that

$$\|u - u_m\|_{|\pi|} < 2^{-nm-1} \quad \text{for } m = 1, 2, \ldots, .$$

Thus

$$\|u_{m+1} - u_m\| < 2^{-nm} \quad \text{for } m = 1, 2, \ldots,$$

and we conclude via (i) that there exist $x_{m,k} \in E_k^+$ $(k = 1, \ldots, n)$ such that

$$|u_{m+1} - u_m| \leq x_{m,1} \oplus \cdots \otimes x_{m,n} \text{ and } \prod_{k \leq n} \|x_{m,k}\| \leq 2^{-nm} \text{ for } m = 1, 2, \ldots, .$$

It is no loss of generality if we assume that $\|x_{m,k}\| \leq 2^{-m}$ for each $m = 1, 2, \ldots,$ and $k = 1, \ldots, n$. Then $x_k = \sum_m x_{m,k}$ exists in each E_k^+ for $k = 1, \ldots, n$ and we have

$$\begin{aligned} |u| &\leq |u_1| + \sum_{m \geq 1} |u_{m+1} - u_m| \\ &\leq |u_1| + \sum_{m \geq 1} x_{m,1} \otimes \cdots \otimes x_{m,n} \\ &\leq |u_1| + x_1 \otimes \cdots \otimes x_n \end{aligned}$$

By (b) above there exist $y_k \in E_k^+$ such that $|u_1| \leq y_1 \otimes \cdots \otimes y_n$. Hence

$$|u| \leq (x_1 + y_1) \oplus \cdots \otimes (x_n + y_n).$$

(iii) Let $u \in E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$. Then there exist $u_m \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ such that $\|u - u_m\|_{|\pi|} \leq 1/4^m$. Let $w = \sum_{m \geq 1} 2^m |u - u_m|$ in $E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$. Then by (ii) there exist $x_k \in E_k^+$ such that $|w| \leq x_1 \otimes \cdots \otimes x_n$. Hence for all m we have

$$|u - u_m| \leq 2^{-m} \quad (x_1 \otimes \cdots \otimes x_n).$$

(iv) By (c) above we can find a unique positive linear operator

$$T: E_1 \bar{\otimes} \cdots \bar{\otimes} E_n \to F$$

such that $B = T \otimes$. Let $u \in E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$. Then by (iii) we can find

$$u_m \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_n \quad \text{and} \quad x_k \in E_k^+ \quad (k = 1, \ldots, n)$$

such that

$$|u - u_m| \leq 2^{-m}(x_1 \otimes \cdots \otimes x_n) \quad \text{for all } n = 1, 2, \ldots .$$

Hence $|Tu_l - Tu_m| \leq 2^{-m+1} T(x_1 \otimes \cdots \otimes x_n)$ in F for all $l \geq m$. It follows that the relative uniform limit of $\{Tu_m\}$ exists in F. If we define Tu as this limit, then one verifies easily that Tu is well defined and extends T uniquely to a positive linear operator from $E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n \to F$.

The following theorem occurs in [1] for $n = 2$ and $E_1 = E_2 = L_2[0, 1]$.

THEOREM 2.2. *Let*

$$E_k = L_{p_k}(X_k, \mu_k) \quad (k = 1, 2, \ldots, n)$$

and assume that $\sum_{k \le n} p_k^{-1} = 1$. *Then*

$$\rho(u) = \inf (\|f_1\|_{p_1}, \ldots, \|f_n\|_{p_n}: |u| \le f_1 \otimes \cdots \otimes f_n)$$

is subadditive on $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$.

Proof. Let $u_1, u_2 \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ and $\varepsilon > 0$. Then we can find f_k and g_k in E_k^+ $(k = 1, \ldots, n)$ such that

$$|u_1| \le f_1 \otimes \cdots \otimes f_n, \quad |u_2| \le g_k \otimes \cdots \otimes g_n,$$

$$\rho(u_1) \ge \|f_1\|_{p_1}, \ldots, \|f_n\|_{p_n} - \varepsilon \quad \text{and} \quad \rho(u_2) \ge \|g_1\|_{p_1}, \ldots, \|g_n\|_{p_n} - \varepsilon.$$

Let

$$f'_k = f_k \cdot \|f_k\|_{p_k}^{-1} \cdot (\rho(u_1) + \varepsilon)^{1/p_k} \quad \text{for } k = 1, \ldots, n-1$$

and put

$$f'_n = \left\{ \prod_{k \le n-1} \|f_k\|_{p_k} (\rho(u_1) + \varepsilon)^{-1/p_k} \right\} f_n.$$

Then

$$f'_1 \otimes \cdots \otimes f'_n = f_1 \otimes \cdots \otimes f_n$$

and

$$\|f'_k\|_{p_k} \le (\rho(u_1) + \varepsilon)^{1/p_k} \quad \text{for } k = 1, \ldots, n.$$

It follows that we may assume that

$$\|f_k\|_{p_k} \le (\rho(u_1) + \varepsilon)^{1/p_k} \quad \text{for } k = 1, \ldots, n.$$

Similarly we may assume that

$$\|g_k\|_{p_k} \le (\rho(u_2) + \varepsilon)^{1/p_k} \quad \text{for } k = 1, \ldots, n.$$

Next we observe that $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ can be considered a subspace of

$$L_0(X_1 \times \cdots \times X_n, \mu_1 \times \cdots \times \mu_n),$$

so that we can apply Hölder's inequality for n factors. In case all $p_k < \infty$ we get

$$\begin{aligned} |u_1 + u_2| &\le f_1 \otimes \cdots \otimes f_n + g_1 \otimes \cdots \otimes g_n \\ &\le (f_1^{p_1} + g_1^{p_1})^{1/p_1} \otimes \cdots \otimes (f_n^{p_n} + g_n^{p_n})^{1/p_n}. \end{aligned}$$

Hence

$$\begin{aligned}\rho(u_1 + u_2) &\le \prod_{k\le n} \|(f_k^{p_k} + g_k^{p_k})^{1/p_k}\|_{p_k} \\ &= \prod_{k\le n} (\|f_k\|_{p_k}^{p_k} + \|g_k\|_{p_k}^{p_k})^{1/p_k} \\ &\le \prod_{k\le n} (\rho(u_1) + \rho(u_2) + 2\varepsilon)^{1/p_k} = \rho(u_1) + \rho(u_2) + 2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have that $\rho(u_1 + u_2) \le \rho(u_1) + \rho(u_2)$. In case one or more of the p_k's is ∞ we have to replace each $(f_k + g_k)^{1/p_k}$ by sup (f_k, g_k), but for the rest the argument remains the same.

The following theorem is now an immediate consequence of the two previous theorems.

THEOREM 2.3. *Let B be a positive n-linear map from*

$$L_{p_1}(X_1, \mu_1) \times \cdots \times L_{p_n}(X_n, \mu) \quad \text{into} \quad L_q(X, \mu)$$

where $q \ge 0$ and $\sum_k p_k^{-1} = 1$. Then there exists a unique positive linear operator

$$T: L_{p_1} \tilde{\otimes} \cdots \tilde{\otimes} L_{p_n} \to L_q$$

such that $B = T \otimes$.

Remark. If $q \ge 1$ we do not need to assume that $\sum_k p_n^{-1} = 1$ in above theorem, since B induces then a continuous linear map T from $L_{p_1} \otimes \cdots \otimes L_{p_n} \to L_q$. In case $0 \le q < 1$ the bilinear map B is jointly continuous, but does not necessarily induce a continuous linear operator from $L_{p_1} \otimes \cdots \otimes L_{p_n} \to L_q$, except when $\sum_k p_k^{-1} = 1$. In the next section we shall show that there exist a jointly continuous $B: L_2 \times L_1 \to L_{2/3}$ which does not induce a continuous linear operator from $L_2 \otimes L_1 \to L_{2/3}$. Moreover we only use above theorem for $0 \le q < 1$ in the next section.

3. Factorization of positive multilinear maps

Let (X, μ) be a σ-finite measure space. Then the following theorem is fundamental for factorization of linear maps with values in $L_0(X, \mu)$.

THEOREM 3.1. (Maurey–Nikisin, see [5]). *Let $A \subseteq L_0(X, \mu)$ be a convex set of non-negative functions bounded in measure. Then there exists $\phi > 0$ in $L_0(X, \mu)$ such that $(1/\phi) \cdot A$ is bounded in $L_1(X, \mu)$.*

We now present our first factorization result.

THEOREM 3.2. *Let* $B: L_{p_1}(X_1, \mu_1) \times \cdots \times L_{p_n}(X_n, \mu_n) \to L_0(X, \mu)$ *be a positive n-linear map and let* $r \geq 1$ *be such that* $\sum_{k \leq n} p_k^{-1} = r^{-1}$. *Then there exists* $\phi \in L_0(X, \mu)$ *with* $\phi > 0$ a.e. *such that*

$$\frac{1}{\phi} \cdot B(L_{p_1} \times \cdots \times L_{p_n}) \subseteq L_r(X, \mu).$$

Proof. Define $B_1: L_{p_1} \times \cdots \times L_{p_n} \times L_{r'} \to L_0$ by means of

$$B_1(f_1, \ldots, f_n, f_{n+1}) = f_{n+1} \cdot B(f_1, \ldots, f_n)$$

where $r^{-1} + (r')^{-1} = 1$. Then by Theorem 2.3 there exists a unique positive linear map

$$T: L_{p_1} \tilde{\otimes} \cdots \tilde{\otimes} L_{p_n} \tilde{\otimes} L_{r'} \to L_0$$

such that $T \otimes = B_1$. By the above theorem we can find $\phi \in L_0(x, \mu)$, $\phi > 0$ a.e. such that

$$\frac{1}{\phi} \cdot T(L_{p_1} \tilde{\otimes} \cdots \tilde{\otimes} L_{p_n} \tilde{\otimes} L_{r'}) \subseteq L_1(X, \mu).$$

This implies immediately that

$$\frac{1}{\phi} \cdot B(L_{p_1} \times \cdots \times L_{p_n}) \subseteq L_r.$$

We note that for $n = 1$ we have:

COROLLARY 3.3 (Nikisin [6], THEOREM 4). *If* $T: L_p(Y, \nu) \to L_0(X, \mu)$ *is a positive linear map, then there exists* $\phi \in L_0(X, \mu)$, $\phi > 0$ a.e. *such that* $(1/\phi) \cdot T(L_p) \subseteq L_p$.

We now show, that if B in Theorem 3.2 takes its values in $L_q(X, \mu)$ for some $q > 0$, then ϕ can be chosen in L_s for some s determined by r and q. The following theorem takes the place of Theorem 3.1, and does not seem to have been stated explicitly in the literature before, although our proof was partially inspired by Maurey's work.

THEOREM 3.4. *Let* $A \subseteq L_q(X, \mu)$ *be a convex set of non-negative functions such that* $\int f^q \, d\mu \leq 1$ *for all* $f \in A$. *Assume* $0 < q < 1$. *Then there exists* $\phi \geq 0$ *in* L_r *with* $\|\phi\|_r \leq 1$ *and* $r^{-1} = q^{-1} - 1$ *such that*

$$\int \frac{f}{\phi} \, d\mu \leq 1 \quad \textit{for all } f \in A.$$

Proof. Let $s = (1-q)^{-1}$ and let U_s be the positive unit ball of L_s. Then U_s is weakly compact, since $1 < s < \infty$. Define $F\colon U_s \times A \to \mathbf{R}_+ \cup \{\infty\}$ by

$$F(h, f) = \int \frac{f}{h^{1/q}}\, d\mu,$$

where we employ $0/0 = 0$ as a convention. Then for every $f \in A$, $F(h, f)$ is convex and lower semicontinuous with respect to the weak topology of L_s (see [5], p. 11). Moreover, for every $h \in U_s$, $F(h, f)$ is trivially concave on A. It follows that we can apply a slightly extended version of Ky Fan's minimax theorem (extended since we allow $+\infty$ as a value of F). Thus

$$\min_{h \in U_s} \max_{f \in A} F(h, f) = \max_{f \in A} \min_{h \in U_s} F(h, f).$$

Since $F(h, f) \leq 1$ for $h = f^{q(1-q)}$, it follows that there exists $h_0 \in U_s$ such that

$$F(h_0, f) = \int \frac{f}{h_0^{1/q}}\, d\mu \leq 1 \quad \text{for all } f \in A.$$

Put $\phi = h_0^{1/q}$ and one sees readily that $\|\phi\|_r \leq 1$.

THEOREM 3.5. *If $B\colon L_{p_1} \times \cdots \times L_{p_n} \to L_q$ $(q > 0)$ is a positive n-linear operator and $r \geq 1$ is such that $r^{-1} = \sum_k p_k^{-1}$ and $r \geq q$, then there exists $0 \leq \phi \in L_s$ with $s^{-1} = q^{-1} - r^{-1}$ such that*

$$\frac{1}{\phi} \cdot B(L_{p_1} \times \cdots \times L_{p_n}) \subseteq L_r.$$

Proof. Assume first $r = q$. Then $\phi \equiv 1$ satisfies the condition of the theorem. Assume now $r > q$ and define q_1 by $q_1^{-1} = (r')^{-1} + q^{-1}$. Define the positive n-linear map $B_1\colon L_{p_1} \times \cdots \times L_{p_n} \times L_{r'} \to L_0$ by

$$B_1(f_1, \ldots, f_n, f_{n+1}) = f_{n+1} B(f_1, \ldots, f_n).$$

We show that B_1 maps actually into L_{q_1}. Let $f_i \in L_{p_i}$ $(1 \leq i \leq n)$ and $f_{n+1} \in L_{r'}$. Then, by Hölder's inequality,

$$\int \left| B_1(f_1, \ldots, f_{n+1}) \right|^{q_1} d\mu \leq \left(\int \left| f_{n+1} \right|^{r'} d\mu \right)^{q_1/r'} \left(\int \left| B(f_1, \ldots, f_n) \right|^{q} d\mu \right)^{q_1/q} < \infty.$$

Hence B_1 maps into L_{q_1}. Applying Theorem 2.3 we find a positive linear operator

$$T\colon L_{p_1} \tilde{\otimes} \cdots \tilde{\otimes} L_{p_n} \tilde{\otimes} L_{r'} \to L_{q_1}$$

such that $B_1 = T\otimes$. Since $q_1^{-1} = (r')^{-1} + q^{-1} = 1 + (q^{-1} - r^{-1}) > 1$, we can apply Theorem 3.4 to get $0 \leq \phi \subset L_s$ with $s^{-1} = q_1^{-1} - 1 = q^{-1} - r^{-1}$ such that

$$\frac{1}{\phi} T(L_{p_1} \bar{\otimes} \cdots \tilde{\otimes} L_{p_n} \tilde{\otimes} L_{r'}) \subseteq L_1.$$

It follows as before that

$$\frac{1}{\phi} B(L_{p_1} \times \cdots \times L_{p_n}) \subseteq L_r.$$

As before we get for $n = 1$:

COROLLARY 3.6 (Maurey). *If $T\colon L_p \to L_q$ $(q > 0)$ is a positive linear operator and $p \geq q$, then there exists $0 \leq \phi \in L_s$ with $s^{-1} = q^{-1} - p^{-1}$ such that*

$$\frac{1}{\phi} \cdot T(L_p) \subseteq L_p.$$

We present some examples to indicate the scope of above theorems.

Example 1. Let $E = \{f \in L_0([0, 1]^2)\colon \text{ess sup}_s \int |f(s, t)|\, dt < \infty\}$. Define the positive linear operator $T\colon E \to L_0([0, 1]^2)$ by $(Tf)(s, t) = f(t, s)$. Assume that for some $0 < \varepsilon < 1$ there exists $X_\varepsilon \subseteq [0, 1]^2$ with $\mu(X_\varepsilon^c) \leq \varepsilon$ such that $\chi_{X_\varepsilon} \cdot T(E) \subseteq E$. Then there exists $M > 0$ such that

$$\operatorname*{ess\,sup}_s \int \chi_{X_\varepsilon}(s, t)\, |f(t, s)|\, dt \leq M \operatorname*{ess\,sup}_s \int |f(s, t)|\, dt \quad \text{for all } f \in E.$$

We apply this inequality to functions $f(s, t)$ with $f(s, t) = g(t) \in L_1([0, 1])$ to get the inequality

$$\operatorname*{ess\,sup}_s \int \chi_{X_\varepsilon}(s, t)\, |g(s)|\, dt \leq M \|g\|_1$$

for all $g \in L_1[0, 1]$. Put $h_\varepsilon(s) = \int \chi_{X_\varepsilon}(s, t)\, dt$. Then $h_\varepsilon > 0$ on a set of measure $\geq 1 - \varepsilon$ and

$$\|gh_\varepsilon\|_\infty \leq M \|g\|_1$$

for all $g \in L_1[0, 1]$, which is a contradiction. This example shows that Theorem 3.2 and Corollary 3.3 cannot be extended to arbitrary Banach function spaces.

Example 2. Let $Tf(x) = \int_0^1 |x - y|^{-1/2} f(y)\, dy$ for $f \in L_2[0, 1]$. Then T is a positive linear operator from L_2 into L_2. Suppose T factors through L_∞; i.e., suppose there exists $0 < \phi \in L_0[0, 1]$ such that $(1/\phi) \cdot T(L_2) \subseteq L_\infty$. Then there exists $M > 0$ such that $|Tf(x)| \leq M\phi(x)\|f\|_2$ a.e. This implies (see [9], theorem) that T is a Carleman operator; i.e., $y \to T(x, y) = |x - y|^{-1/2}$ is in $L_2[0, 1]$ for a.e. x, which is clearly not the case. Hence T does not factor thorugh L_∞. Define now $B: L_2 \times L_1 \to L_0$ by $B(f, g) = gTf$. Then B defines a positive bilinear operator from $L_2 \times L_1 \to L_{2/3}$. This bilinear operator extends to a positive linear operator $S: L_2 \otimes L_1 \to L_{2/3}$, which by above considerations cannot be extended to a positive linear operator from $L_2 \tilde{\otimes} L_1$. Hence we cannot drop the condition that $\sum p_k^{-1} = 1$ in Theorem 2.3 or that $\sum p_k^{-1} \leq 1$ in Theorem 3.5.

Example 3. Let (X, μ) be a probability measure space and let $\mathscr{F}$ be an ergodic family of measure-preserving transformations on X, which is closed under composition (see [10] for an explanation of these notions). Let p_1, p_2 and $r \geq 1$ such that $p_1^{-1} + p_2^{-1} = r^{-1}$. Let $B: L_{p_1}(X, \mu) \times L_{p_2}(X, \mu) \to L_0$ be a positive bilinear map. Assume B commutes simultaneously with every member of $\mathscr{F}$. Then B is a bounded map into $L_r(X, \mu)$. For the proof of this note that if $\varepsilon > 0$, then there exists $C_\varepsilon > 0$ and $A \subset X$ with $\mu(A^c) \leq \varepsilon$ such that $\int_A |B(f, g)|^r \leq C_\varepsilon$ for all f and g with $\|f\|_{p_1} \leq 1$ and $\|g\|_{p_2} \leq 1$ by Theorem 3.2. If now $w_1, \ldots, w_n \in \mathscr{F}$, then it follows that

$$\int_{w_k^{-1}(A)} |B(f, g)|^r \leq C_\varepsilon$$

for all such f and g (by the commuting property). Hence

$$\int \sum_{k=1}^{n} \lambda_k \chi_{w_k^{-1}(A)} |B(f, g)|^r \leq C_\varepsilon$$

if $\lambda_k \geq 0$ and $\sum_{k=1}^n \lambda_k = 1$. It follows now from [10] (corollary after Lemma 1) that there exists a sequence h_n of such convex combinations such that $h_n(x) \to \mu(A)$ a.e. It follows from Fatou's lemma that $\int \mu(A) |B(f, g)|^r \leq C_\varepsilon$ for all f and g with $\|f\|_{p_1} \leq 1$ and $\|g\|_{p_1} \leq 1$; i.e.,

$$\int |B(f, g)|^r \, d\mu \leq \frac{C_\varepsilon}{1 - \varepsilon}$$

for all such f and g.

We proceed by indicating the extension of the theorems of this section to positive n-linear operators defined on $E_1 \times \cdots \times E_n$, where each E_k is a Banach lattice. For the following definition and properties connected with it we refer to [3].

DEFINITION. A Banach lattice E is called p-convex if there exists a constant $M < \infty$ such that

$$\left\| \left(\sum |x_i|^p \right)^{1/p} \right\| \leq M \left(\sum \|x_i\|^p \right)^{1/p} \quad \text{if} \quad 1 \leq p < \infty,$$

or

$$\left\| \bigvee_{i=1}^{n} |x_i| \right\| \leq M \max \|x_i\| \quad \text{if } p = \infty,$$

for every choice of vectors $\{x_k: k = 1, \ldots, n\}$ in E. It is proved in [3] that a p-convex Banach lattice E can be renormed equivalently so that E, endowed with the new norm and the same order, is a p-convex Banach lattice with constant $M = 1$. Using this one proves the following theorem similarly to the theorems proved before.

THEOREM 3.7. *If E_k $(k = 1, \ldots, n)$ are p_k-convex Banach lattices and*

$$B: E_1 \times \cdots \times E_n \to L_q \quad (q \geq 0)$$

is a positive n-linear map and $r \geq 1$ such that $r^{-1} = \sum_k p_k^{-1}$ and $r \geq q$, then there exists $0 \leq \phi \in L_s$ with $s^{-1} = q^{-1} - r^{-1}$ such that

$$\frac{1}{\phi} \cdot B(E_1 \times \cdots \times E_n) \subseteq L_r.$$

We conclude by deriving an interesting consequence of Theorem 3.4 not connected with the main theme of our paper.

THEOREM 3.8. *Let $H \subseteq L_q$ $(0 < q < 1)$ be a convex set of non-negative functions which is bounded in L_q. Suppose that H is compact in L_0, then H is compact in L_q.*

Proof. By Theorem 3.4 we can find $\phi > 0$ a.e. in L_r $(r^{-1} = q^{-1} - 1)$ such that

$$\int \frac{f}{\phi} \, d\mu \leq 1 \quad \text{for all } f \in H.$$

Let $\varepsilon > 0$. Then we can write X as a disjoint union $X_1 \cup X_2$ such that

$$\left(\int_{X_1} \phi^r \, d\mu \right)^{1-q} < \varepsilon$$

and such that $\mu(X_2) < \infty$. Then we can find $\delta > 0$ such that $\mu(A) < \delta$, $A \subseteq X_2$ implies that $(\int_A \phi^r \, d\mu)^{1-q} < \varepsilon$. Let $A \subseteq X$. Then we have via Hölder's inequality

$$\begin{aligned}\int_A f^q\,d\mu &= \int_A \left(\frac{f}{\phi}\right)^q \phi^q\,d\mu \\ &\leq \left(\int\left(\frac{f}{\phi}\right)d\mu\right)^q\left(\int_A \phi^{q/(1-q)}\,d\mu\right)^{1-q} \\ &\leq \left(\int_A \phi^r\,d\mu\right)^{1-q} \quad \text{for all } f\in H.\end{aligned}$$

It follows that $\int_{X_1} f^q\,d\mu < \varepsilon$ for all $f \in H$ and that $\mu(A) < \delta$, $A \subseteq X_2$ implies that

$$\int_A f^q\,d\mu < \varepsilon$$

for all $f \in H$; i.e., $\{f^q\chi_{X_2}: f \in H\}$ is uniformly integrable. Let $f_n \in H$. Then by passing to a subsequence we may assume that $f_n(x) \to f_0(x)$ a.e. It follows from Fatou's lemma that also $\int_{X_1} f_0^q\,d\mu < \varepsilon$. By Egoroff's theorem we can find $X_0 \subseteq X_2$ with $\mu(X_2\backslash X_0) < \delta$ such that $f_n(x) \to f_0(x)$ uniformly on X_0. By the above we have $\int_{X_2\backslash X_0} f_n^q\,d\mu < \varepsilon$ for all n, so again by Fatou's lemma, $\int_{X_2\backslash X_0} f_0^q\,d\mu \leq \varepsilon$. It now follows that

$$\begin{aligned}\int |f_n - f_0|^q\,d\mu &= \int_{X_1} |f_n - f_0|^q\,d\mu + \int_{X_2\backslash X_0} |f_n - f_0|^q\,d\mu + \int_{X_0} |f_n - f_0|^q\,d\mu \\ &\leq 2^q(\varepsilon + \varepsilon) + 2^q(\varepsilon + \varepsilon) + \int_{X_0} |f_n - f_0|^q\,d\mu \\ &= 2^{q+2}\varepsilon + \int_{X_0} |f_n - f_0|^q\,d\mu.\end{aligned}$$

Since $f_n \to f$ uniformly on X_0 and since $\varepsilon > 0$ is arbitrary it follows that

$$\int |f_n - f|^q\,d\mu \to 0.$$

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