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MODULES DETERMINED BY THEIR COMPOSITION FACTORS

BY

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Introduction

Throughout this paper Λ denotes an artin algebra over a commutative artin ring R; i.e., Λ is an R-algebra which is finitely generated as an R-module. All Λ-modules are assumed to be finitely generated Our main concern is to develop criteria for when two indecomposable modules are isomorphic, at least for some special types of algebras Λ.

Given a module M, we denote by $[M]$ the image of M in the Grothendieck group of Λ. Thus $[M_1] = [M_2]$ if and only if they have the same composition factors including multiplicities. Now it is classical that Λ is semisimple if and only if $[M_1] = [M_2]$ implies that M_1 and M_2 are isomorphic, where M_1 and M_2 are arbitrary Λ-modules. So while the composition factors do not determine arbitrary modules over nonsemisimple artin algebras, it is still a sensible question to ask which artin algebras have the property that their indecomposable modules are determined up to isomorphism by their composition factors. While we do not give a definitive answer to this question, we do extend considerably the class of artin algebras previously known to have this property.

By a short chain we mean a sequence of nonzero morphisms $A \to B \to D\operatorname{Tr} A$ where A and B are in $\operatorname{ind}\Lambda$, the category whose objects are the indecomposable Λ-modules. We show in Section 1 that if Λ has no short chains, then indecomposable Λ-modules are determined up to isomorphism by their composition factors. In the case Λ is of finite representation type, then Λ has no short chains if its AR-quiver has no oriented cycles, where by an oriented cycle we mean a sequence of nonzero morphisms between indecomposable modules $A_0 \to A_1 \to \cdots \to A_n$, not all of which are isomorphisms, with $A_0 \cong A_n$. Therefore this result contains earlier results by Happel-Ringel [10] and Bautista [4]. It is worth noting that the class of algebras with no short chains is much more extensive than the class of algebras having no oriented cycles, as follows easily from the theory of coverings [8].

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Algebras with no short chains have another interesting and somewhat surprising criterion when two indecomposable modules A and B are isomorphic. Namely, let $P_1 \to P_0 \to A \to 0$ and $Q_1 \to Q_0 \to B \to 0$ be minimal projective presentations. Then A is isomorphic to B if P_i is isomorphic to Q_i for $i = 0, 1$. It would be interesting to know which artin algebras satisfy this criterion for indecomposable modules to be isomorphic.

These results follow rather easily from two general facts which seem to be of interest in themselves. The first is the fact that two arbitrary modules A and B are isomorphic if either of the following conditions is satisfied:

(a) $\langle X, A \rangle = \langle X, B \rangle$ for all X in $\operatorname{ind} \Lambda$,

(b) $\langle A, X \rangle = \langle B, X \rangle$ for all X in $\operatorname{ind} \Lambda$,

where $\langle U, V \rangle$ denotes the length as an R-module of $\operatorname{Hom}_\Lambda(U, V)$ [2] (see also [6]).

The second general fact is the following. Let A and B be arbitrary Λ-modules and $P_1 \to P_0 \to A \to 0$ a minimal projective presentation of A. Then

$$\langle A, B \rangle - \langle B, D\operatorname{Tr} A \rangle = \langle P_0, B \rangle - \langle P_1, B \rangle.$$

In fact, the notion of short chains as well as the results cited above were suggested by this formula. The last section of the paper is devoted to giving some applications of this formula in somewhat different directions as an indication of other uses it may have. Amongst other things we show that if Λ is a symmetric algebra, then $|P(A, B)| = |P(B, A)|$ for all Λ-modules A and B where $|P(A, B)|$ denotes the length over R of the R-submodule of $\operatorname{Hom}_\Lambda(A, B)$ consisting of the morphisms factoring through projective modules. Benson and Parker showed this result for group algebras in [6].

We would like to thank Sverre Smalø for helpful discussions, especially on the material in Section 2.

1. Algebras with no short chains

In this section we prove that if A and B are indecomposable Λ-modules which are not the middle of any short chain, then they are isomorphic if $[A] = [B]$. Actually we deduce this as a consequence of more general results.

We say that a subcategory $\mathscr{C}$ of $\operatorname{ind} \Lambda$ is contravariantly determined if two modules A and B in $\operatorname{add} \mathscr{C}$ are isomorphic whenever $\langle X, A \rangle = \langle X, B \rangle$ for all X in $\mathscr{C}$, where $\operatorname{add} \mathscr{C}$ consists of all finite sums of modules in $\mathscr{C}$. Now it is well known that $[A] = [B]$ if and only if $\langle P, A \rangle = \langle P, B \rangle$ for all projective P in $\operatorname{ind} \Lambda$, if and only if $\langle A, I \rangle = \langle B, I \rangle$ for all injective I in $\operatorname{ind} \Lambda$. Therefore if we assume that a subcategory $\mathscr{C}$ of $\operatorname{ind} \Lambda$ is contravariantly determined, then showing that A and B in $\mathscr{C}$ are isomorphic whenever $[A] = [B]$, is the same thing as showing that $\langle P, A \rangle = \langle P, B \rangle$ for all projectives P in $\operatorname{ind} \Lambda$ implies that $\langle X, A\rangle = \langle X, B\rangle$ for all X in $\mathscr{C}$. Dually, we say that $\mathscr{C}$ is covariantly determined if A and B in add $\mathscr{C}$ are isomorphic whenever $\langle A, X\rangle = \langle B, X\rangle$ for all X in $\mathscr{C}$.

In view of the results of this paper it is of interest to know which subcategories $\mathscr{C}$ of ind Λ are contravariantly or covariantly determined. This question will be studied in another paper. As stated in the introduction this has been proved for $\mathscr{C}=$ ind Λ. The proofs known carry over to show that $\mathscr{C}$ is contravariantly and covariantly determined if it is closed under irreducible morphisms. We include a proof for the sake of completeness. To avoid using functors as in the proof of Auslander, we use the language of almost split sequences as in [6].

Let $\mathscr{C}$ be a subcategory of ind Λ. We denote by $K_0(\mathscr{C}, 0)$ the free abelian group generated by the isomorphism classes of modules in $\mathscr{C}$. We define the bilinear map $\langle\ ,\ \rangle\colon K_0(\mathscr{C},0)\times K_0(\mathscr{C},0)\to Z$ by

$$\left\langle \sum n_i C_i, \sum m_j C_j\right\rangle = \sum n_i m_j \langle C_i, C_j\rangle.$$

Associated with each object $A = \coprod n_i C_i$ in add $\mathscr{C}$ is the element $\Sigma n_i C_i$ in $K_0(\mathscr{C},0)$, which we will often denote more simply by A. The following result gives a useful way of showing that $\mathscr{C}$ is contravariantly determined.

PROPOSITION 1.1. *Let $\mathscr{C}$ be a subcategory of* ind Λ.

(a) *If for each C in $\mathscr{C}$ there is a $\hat{C}$ in $K_0(\mathscr{C},0)$ such that*

(i) $\langle \hat{C}, C\rangle \neq 0$ *and*

(ii) $\langle \hat{C}, X\rangle = 0$ *for X in $\mathscr{C}$ not isomorphic to C,*

then $\mathscr{C}$ is contravariantly determined.

(b) *If for each C in $\mathscr{C}$ there is a $\hat{C}$ in $K_0(\mathscr{C},0)$ such that*

(i) $\langle C, \hat{C}\rangle \neq 0$ *and*

(ii) $\langle X, \hat{C}\rangle = 0$ *for X in $\mathscr{C}$ not isomorphic to C,*

then $\mathscr{C}$ is covariantly determined.

Proof. (a) Let $A = \coprod n_i C_i$ be in add $\mathscr{C}$. Then $n_i = \langle \hat{C}_i, A\rangle / \langle \hat{C}_i, C_i\rangle$ for all i. Therefore A is completely determined by the numbers $\langle C, A\rangle$ for all C in $\mathscr{C}$, which shows that $\mathscr{C}$ is contravariantly determined.

(b) This is analogous to the proof of (a).

We now apply this to show the following.

COROLLARY 1.2. *Let $\mathscr{C}$ be a subcategory of* ind Λ *which is closed under irreducible morphisms. Then $\mathscr{C}$ is contravariantly and covariantly determined.*

Proof. Suppose A in $\mathscr{C}$ is not injective and let $0\to A\to B\to C\to 0$ be an almost split sequence. Then B and C are in $\mathscr{C}$ since $\mathscr{C}$ is closed under irreducible morphisms. Define $\hat{A} = B - A - C$ in $K_0(\mathscr{C},0)$. If A in $\mathscr{C}$ is injective, then $A/\operatorname{soc}A$ is in $\mathscr{C}$ since $A \to A/\operatorname{soc}A$ is irreducible. Define $\hat{A} = A - A/\operatorname{soc}A$ in $K_0(\mathscr{C},0)$. We claim that for each A in $\mathscr{C}$ we have

(a) $\langle \hat{A}, A\rangle \neq 0$ and

(b) $\langle \hat{A}, X\rangle = 0$ for X in $\mathscr{C}$ not isomorphic to A.

For if A is not injective, the almost split sequence $0 \to A \to B \to C \to 0$ gives rise to the exact sequence

$$0 \to \operatorname{Hom}_\Lambda(C, X) \to \operatorname{Hom}_\Lambda(B, X) \to \operatorname{Hom}_\Lambda(A, X)$$

for each X in $\mathscr{C}$, and by the definition of almost split sequences

$$0 \to \operatorname{Hom}_\Lambda(C, X) \to \operatorname{Hom}_\Lambda(B, X) \to \operatorname{Hom}_\Lambda(A, X) \to 0$$

is exact if and only if $X \not\simeq A$. And if A is injective,

$$\operatorname{Hom}_\Lambda(A/\operatorname{soc}A, X) \to \operatorname{Hom}_\Lambda(A, X)$$

is a monomorphism for X in $\mathscr{C}$, which is an isomorphism if and only if $X \not\simeq A$. Hence $\mathscr{C}$ is contravariantly determined by Proposition 1.1. It is proved similarly that $\mathscr{C}$ is covariantly determined.

We now point out a condition on a subcategory $\mathscr{C}$ of ind Λ which is useful in showing that if $[A] = [B]$ for A, B in $\mathscr{C}$, then $\langle X, A\rangle = \langle X, B\rangle$ for every X in $\mathscr{C}$. Suppose we have a map $f\colon \mathscr{C} \times \mathscr{C} \to \mathscr{N}$ (nonnegative integers) such that for each X in $\mathscr{C}$ we have that $g(A) = \langle X, A\rangle - f(X, A)$ depends only on the composition factors of A and let $\mathscr{C}'$ be the subcategory of $\mathscr{C}$ consisting of the modules C such that $\langle X, C\rangle \cdot f(X, C) = 0$ for all X in $\mathscr{C}$. Then we have the following.

LEMMA 1.3. *Let A and B be in $\mathscr{C}'$ such that $[A] = [B]$. Then*

(a) *$\langle X, A\rangle = \langle X, B\rangle$ for all X in $\mathscr{C}$,*

(b) *$A \simeq B$ if $\mathscr{C}$ is contravariantly determined.*

Proof. (a) Since $[A] = [B]$ we have

$$\langle X, A\rangle - f(X, A) = \langle X, B\rangle - f(X, B) \quad \text{for each } X \text{ in } \mathscr{C}.$$

If $\langle X, A\rangle \neq 0$, then $f(X, A) = 0$. Then $\langle X, B\rangle - f(X, B) > 0$, so that $\langle X, B\rangle \neq 0$ since $f(X, B) \geq 0$. Therefore $f(X, B) = 0$, which shows that $\langle X, A\rangle = \langle X, B\rangle$ in this case. If $\langle X, A\rangle = 0$, then $\langle X, B\rangle - f(X, B) \leq 0$ so that $\langle X, B\rangle = 0$.

(b) Trivial.

In the case Λ is hereditary, a natural f: $\operatorname{mod} \Lambda \times \operatorname{mod} \Lambda \to \mathcal{N}$ suggests itself, where $\operatorname{mod} \Lambda$ in the category of finitely generated Λ-modules. Consider $f(X, A) = |\operatorname{Ext}_\Lambda^1(X, A)|$ where $|\operatorname{Ext}_\Lambda^1(X, A)|$ means the length of $\operatorname{Ext}_\Lambda^1(X, A)$ as an R-module. Then if $0 \to P_1 \to P_0 \to X \to 0$ is a projective resolution of X, we have that

$$\langle X, A\rangle - |\operatorname{Ext}^1(X, A)| = \langle P_0, A\rangle - \langle P_1, A\rangle$$

which depends only on the composition factors of A. Since Λ is hereditary, $\operatorname{Ext}_\Lambda^1(X, \ )$ is right exact and so we have that

$$\operatorname{Ext}_\Lambda^1(X, A) \xrightarrow{\sim} \operatorname{Ext}_\Lambda^1(X, \Lambda) \otimes_\Lambda A.$$

Therefore

$$\begin{aligned} |\operatorname{Ext}_\Lambda^1(X, A)| &= |\operatorname{Ext}_\Lambda^1(X, \Lambda) \otimes_\Lambda A| \\ &= |D(\operatorname{Ext}_\Lambda^1(X, \Lambda) \otimes_\Lambda A)| = \langle A, D \operatorname{Ext}_\Lambda^1(X, \Lambda)\rangle \\ &= \langle A, D \operatorname{Tr} X\rangle. \end{aligned}$$

So we have

$$\langle X, A\rangle - \langle A, D \operatorname{Tr} X\rangle = \langle P_0, A\rangle - \langle P_1, A\rangle.$$

Remarkably, this formula is true for arbitrary artin algebras, as we now show.

THEOREM 1.4. (a) *Let X and A be arbitrary modules over an artin algebra Λ. Suppose $P_1 \to P_0 \to X \to 0$ is a minimal projective presentation. Then we have*

$$\langle X, A\rangle - \langle A, D \operatorname{Tr} X\rangle = \langle P_0, A\rangle - \langle P_1, A\rangle.$$

(b) *Let X and A be in* $\operatorname{mod} \Lambda$ *and let $0 \to X \to I_0 \to I_1$ be a minimal injective copresentation. Then we have*

$$\langle A, X\rangle - \langle \operatorname{Tr} DX, A\rangle = \langle A, I_0 - I_1\rangle.$$

Proof. (a) Denoting $\operatorname{Hom}_\Lambda(P_i, \Lambda)$ by P_i^*, the exact sequence

$$P_0^* \to P_1^* \to \operatorname{Tr} X \to 0$$

gives rise to the exact sequence

$$P_0^* \otimes_\Lambda A \to P_1^* \otimes_\Lambda A \to \operatorname{Tr} X \otimes_\Lambda A \to 0.$$

Using the functorial isomorphisms $P_i^* \otimes_\Lambda A \to \mathrm{Hom}_\Lambda(P_i, A)$, we get the exact sequence

$$0 \to \mathrm{Hom}_\Lambda(X, A) \to (P_0, A) \to (P_1, A) \to \mathrm{Tr}\, X \otimes_\Lambda A \to 0.$$

Using the fact that $D(\mathrm{Tr}\, X \otimes_\Lambda A) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(A, D\,\mathrm{Tr}\, X)$ as R-modules, we have

$$|\mathrm{Tr}\, X \otimes_\Lambda A| = \langle A, D\,\mathrm{Tr}\, X\rangle$$

which gives (a).

(b) Follows by duality.

This result suggests the following definitions.

Let $\mathscr{C}$ be a subcategory of ind Λ. A chain $A \to B \to D\,\mathrm{Tr}\, A$ of nonzero morphisms is said to be a left short chain in $\mathscr{C}$ if A and B are in $\mathscr{C}$, a right short chain in $\mathscr{C}$ if B and $D\,\mathrm{Tr}\, A$ are in $\mathscr{C}$, and a short chain in $\mathscr{C}$ if it is both a left and a right short chain in $\mathscr{C}$. For $\mathscr{C} =$ ind Λ, or more generally if $\mathscr{C}$ is closed under irreducible maps, then all these definitions coincide.

Clearly B in $\mathscr{C}$ is not the middle of any left short chain in $\mathscr{C}$ if and only if $\langle X, B\rangle\langle B, D\,\mathrm{Tr}\, X\rangle = 0$ for all X in $\mathscr{C}$, and not in the middle of any right short chain if $\langle \mathrm{Tr}\, DX, B\rangle\langle B, X\rangle = 0$ for all X in $\mathscr{C}$. Therefore, combining our previous comments, we have one of our main results.

THEOREM 1.5. (a) *Suppose a subcategory $\mathscr{C}$ of* ind Λ *is contravariantly determined (for instance, $\mathscr{C}$ is closed under irreducible morphisms), and assume that A, B in $\mathscr{C}$ are not the middle of any left short chain in $\mathscr{C}$. If $[A] = [B]$, then $A \simeq B$.*

(b) *Let $\mathscr{C}$ be a covariantly determined subcategory of* ind Λ *and assume that A, B in $\mathscr{C}$ are not the middle of any right short chain in $\mathscr{C}$. If $[A] = [B]$, then $A \simeq B$.*

As a special case of this theorem we get the following result of Happel [9] which generalizes an earlier result of Happel-Ringel [10].

COROLLARY 1.6. *Let $\mathscr{C}$ be a subcategory of* ind Λ *which is a preprojective component. If $[A] = [B]$ for A and B in $\mathscr{C}$, then $A \simeq B$.*

Proof. Since a preprojective component is closed under irreducible morphisms, it suffices to show by Theorem 1.5 that there are no short chains in $\mathscr{C}$. One of the consequences of $\mathscr{C}$ being a preprojective component is that for each X in $\mathscr{C}$ there are only a finite number of Y in ind Λ such that $\mathrm{Hom}_\Lambda(Y, X) \neq 0$ (see [10]). Hence there is a nonzero morphism $Y \to X$ in $\mathscr{C}$ which is not an isomorphism if and only if there is a finite chain of irreducible morphisms from Y to X with nonzero composition. Therefore the fact that there are no oriented cycles of irreducible morphisms in $\mathscr{C}$ is equivalent to there being no oriented cycles of nonzero morphisms in $\mathscr{C}$, where not all the morphisms are isomorphisms. But this implies there are no short chains in $\mathscr{C}$. For suppose there is a short chain $A \to B \to D\operatorname{Tr} A$ in $\mathscr{C}$. Then the almost split sequence

$$0 \to D\operatorname{Tr} A \to E \to A \to 0$$

gives rise to a sequence of irreducible morphisms $D\operatorname{Tr} A \to E' \to A$ in $\mathscr{C}$ which in turn gives rise to an oriented cycle of nonzero morphisms

$$A \to B \to D\operatorname{Tr} A \to E' \to A$$

in $\mathscr{C}$, not all of which are isomorphisms, which is a contradiction. Hence $\mathscr{C}$ has no short chains, which is what we wanted to show.

As another consequence of Theorem 1.5 we have the following.

COROLLARY 1.7. *Suppose* ind Λ *has no short chains. Then A and B in* ind Λ *are isomorphic if* $[A] = [B]$.

In view of the above, it would seem that algebras having no short chains should be of considerable interest. They are probably of finite representation type since they have the property that their indecomposable modules are determined by their composition factors. For this reason we concentrate on algebras of finite representation type in discussing algebras with no short chains. We now show that this is a rather extensive class of algebras.

As our first example we have the following.

PROPOSITION 1.8. *Suppose Λ is of finite representation type and has no oriented cycles of irreducible morphisms. Then Λ has no short chains.*

Proof. This follows from the proof of Corollary 1.6.

The algebras of finite representation type having no oriented cycles do not by any means exhaust the class of algebras of finite representation type having no short chains, as can be seen from the following result.

PROPOSITION 1.9. *Suppose Λ is an algebra of finite type over an algebraically closed field k.*

(a) *There are finite coverings of $k(\Gamma_\Lambda)$ such that the associated algebras have no short chains, where $k(\Gamma_\Lambda)$ denotes the mesh category of the AR-quiver.*

(b) *If Λ has no short chains, then every Λ' where $k(\Gamma_{\Lambda'})$ is a finite covering of $k(\Gamma_\Lambda)$ has no short chains.*

Proof. (a) Follows from the proof of Corollary 5.3. in [8].
(b) Trivial consequence of the definition of coverings.

While we will return later to discuss other properties of algebras having no short chains, we end this section with the following.

PROPOSITION 1.10. *If Λ is an algebra of finite type having no short chains, then Λ is Schurian.*

Proof. Assume to the contrary that Λ is not Schurian, i.e., there is some A in $\operatorname{ind}\Lambda$ having a nonzero endmorphism $f\colon A \to A$ which is not an isomorphism. Then there is an oriented cycle of irreducible morphisms

$$A = A_0 \xrightarrow{f_1} A_1 \to \cdots \to A_{n-1} \xrightarrow{f_n} A_n = A$$

with nonzero composition.
By the theorem of Bautista-Smalø [5], there is some B such that $B \cong A_i$ and $D\operatorname{Tr} B \cong A_j$ for some i and j. But then we have the short chain $B \to A \to D\operatorname{Tr} B$, which is a contradiction.

2. Modules not on cycles

Our purpose in this section is to exploit the following observation.

Let $P_1, \dots, P_t$ be nonisomorphic indecomposable projective Λ-modules and suppose $\Gamma = (\operatorname{End} P)^{\mathrm{op}}$, where $P = \coprod_{i=1}^{t} P_i$, has the property that indecomposable Γ-modules are determined by their composition factors. The $\operatorname{Hom}_\Lambda(P, P_i)$ are the indecomposable projective Γ-modules, and the natural morphism $\operatorname{Hom}_\Lambda(Y, X) \to \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(P, Y), \operatorname{Hom}_\Lambda(P, X))$ is an isomorphism when $Y = P$ and hence also when $Y = P_i$ for each i. Suppose $\mathscr{C}$ is a subcategory of $\operatorname{ind}\Lambda$ such that the functor $(P, \ )\colon \mathscr{C} \to \operatorname{mod}\Lambda$ given by

$$C \to \operatorname{Hom}_\Lambda(P, C)$$

is fully faithful. Then two objects C and C' in $\mathscr{C}$ are isomorphic if $\langle P_i, C\rangle = \langle P_i, C'\rangle$ for $i = 1, \dots, t$, in particular if $[C] = [C']$. For we have by the above that

$$\langle P, C\rangle = \langle \operatorname{Hom}_\Lambda(P, P), \operatorname{Hom}_\Lambda(P, C)\rangle$$

and

$$\langle P_i, C\rangle = \langle \operatorname{Hom}_\Lambda(P, P_i), \operatorname{Hom}_\Lambda(P, C')\rangle.$$

Hence $\langle P_i, C\rangle = \langle P_i, C'\rangle$ for $i = 1,\ldots,t$ implies $[(P,C)] = [(P,C')]$ in $\operatorname{mod}\Gamma$. Since the indecomposable Γ-modules are determined by their composition factors, we have $(P,C) \simeq (P,C')$, and by the assumption on $(P,\ ): \mathscr{C} \to \operatorname{mod}\Gamma$, we have $C \simeq C'$.

We now describe a way of finding families of indecomposable projective modules $P_1,\ldots,P_t$ such that $\Gamma = \operatorname{End}(\coprod P_i)^{\mathrm{op}}$ has the property that indecomposable Γ-modules are determined by their composition factors in the case Λ is of finite representation type. We begin by recalling some general facts.

Let Λ be an arbitrary artin algebra, $P_1,\ldots,P_t$ some family of nonisomorphic indecomposable projective modules and $\Gamma = (\operatorname{End} P)^{\mathrm{op}}$ where $P = \coprod_{i=1}^{t} P_i$. We now describe three subcategories $\mathscr{C}_0, \mathscr{C}_1, \mathscr{C}_2$ of $\operatorname{ind}\Lambda$ such that the functor $(P,\ )\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ given by $C \to (P,C) = \operatorname{Hom}_\Lambda(P,C)$ when restricted to the subcategories $\mathscr{C}_0, \mathscr{C}_1, \mathscr{C}_2$ induce equivalences with $\operatorname{ind}\Gamma$ (see [1]).

(a) A is in $\mathscr{C}_0$ if and only if in its minimal projective presentation $Q_1 \to Q_0 \to A \to 0$ the simple summands of $Q_1/\mathbf{r}Q_1$ and $Q_0/\mathbf{r}Q_0$ are amongst the simples $P_1/\mathbf{r}P_1,\ldots,P_t/\mathbf{r}P_t$, where $\mathbf{r}$ is the radical of Λ.

(b) A is in $\mathscr{C}_1$ if and only if the simple summands of $A/\mathbf{r}A$ and $\operatorname{soc} A$ are amongst the simples $P_1/\mathbf{r}P_1,\ldots,P_t/\mathbf{r}P_t$.

(c) A is in $\mathscr{C}_2$ if and only if in its minimal injective copresentation $0 \to A \to I_0 \to I_1$ the simple summands of $\operatorname{soc} I_0$ and $\operatorname{soc} I_1$ are amongst the simples $P_1/\mathbf{r}P_1,\ldots,P_t/\mathbf{r}P_t$.

Our aim now is to prove the main result of this section.

THEOREM 2.1. *Let Λ be an artin algebra of finite representation type. Suppose C is in $\operatorname{ind}\Lambda$ and $P_1,\ldots,P_t$ are a complete set of nonisomorphic indecomposable projective Λ-modules such that $\langle P_i, C\rangle \neq 0$. Further suppose that C, which lies in $\mathscr{C}_1$ for $P_1,\ldots,P_t$, does not lie on any oriented cycle of nonzero nonisomorphic morphisms in $\mathscr{C}_1$. Then $\Gamma = (\operatorname{End}\coprod_{i=1}^{t} P_i)^{\mathrm{op}}$ has the property that indecomposable Γ-modules are determined by their composition factors.*

Proof. Clearly C is in $\mathscr{C}_1$. Since $(P,\ )\colon \mathscr{C}_1 \to \operatorname{ind}\Gamma$ is an equivalence, the fact that C does not lie on any oriented cycle of nonzero nonisomorphic morphisms in $\mathscr{C}_1$ implies that (P,C) does not lie on any oriented cycle of nonzero nonisomorphic morphisms in $\operatorname{ind}\Gamma$. Hence Γ is of finite representation type with a sincere module (P,C) (i.e., an indecomposable Γ-module with all simple Γ-modules as composition factors), not lying on any oriented cycle of irreducible morphisms in $\operatorname{ind}\Gamma$.

We want to show that if Y is on an oriented cycle of irreducible morphisms in $\operatorname{ind}\Gamma$, then $D\operatorname{Tr} Y$ is not on the cycle. Assume there is a cycle containing Y and $D\operatorname{Tr} Y$ and let $\mathbf{C}$ be a maximal such cycle, in the sense that there is no cycle with more nonisomorphic modules in it. We claim that $\mathbf{C}$ must contain a projective and an injective module.

Suppose $\mathbf{C}$ does not contain a projective module. Then $D\operatorname{Tr}\mathbf{C}$, with its obvious meaning, is a cycle of irreducible morphisms which contains $D\operatorname{Tr}Y$ and so the join of $\mathbf{C}$ and $D\operatorname{Tr}\mathbf{C}$ is a cycle containing Y and $D\operatorname{Tr}Y$. Hence the modules in $D\operatorname{Tr}\mathbf{C}$ are amongst the modules in $\mathbf{C}$. Therefore the modules in $\mathbf{C}$ are $D\operatorname{Tr}$-periodic. Since Γ is an indecomposable algebra of finite representation type, there must be some indecomposable T not in $\mathbf{C}$ such that there is an irreducible morphism $T \to S$ or $S \to T$ with S in $\mathbf{C}$. Otherwise the modules in $\mathbf{C}$ would be closed with respect to irreducible morphisms and would be all of $\operatorname{ind}\Gamma$, which is impossible since all the modules in $\mathbf{C}$ are $D\operatorname{Tr}$-periodic. Suppose T is not in $\mathbf{C}$ and there is an irreducible morphism $T \to S$ with S in $\mathbf{C}$. (The case $S \to T$ is handled similarly.) Then $D\operatorname{Tr}S \to T \to S$, when joined to $\mathbf{C}$, would give an oriented cycle containing Y, $D\operatorname{Tr}Y$ and T, contradicting the maximality of $\mathbf{C}$. Therefore not all the modules in $\mathbf{C}$ are $D\operatorname{Tr}$-periodic which means that $\mathbf{C}$ contains a projective module. Similarly, $\mathbf{C}$ contains an injective module.

Let Q be a projective module in $\mathbf{C}$ and I an injective module in $\mathbf{C}$. Since (P, C) is a sincere module and Γ is of finite type, we know there is a chain of irreducible morphisms $Q \to \cdots \to (P, C) \to \cdots \to I$. Therefore (P, C) lies on a cycle since Q and I lie on the cycle $\mathbf{C}$, which is a contradiction. Therefore no oriented cycle of irreducible morphisms in $\operatorname{ind}\Gamma$ contains both Y and $D\operatorname{Tr}Y$ for any Y in $\operatorname{ind}\Gamma$.

We now indicate two different ways of completing the proof of the theorem. If there were a short chain $A \to B \to D\operatorname{Tr}A$ in $\operatorname{ind}\Gamma$, then there would be an oriented cycle of irreducible morphisms containing A and $D\operatorname{Tr}A$ since Γ is of finite representation type. Therefore there are no short chains in $\operatorname{ind}\Gamma$, which shows by Corollary 1.7 that the indecomposable Γ-modules are determined by their composition factors.

Alternatively, Bautista-Smalø [5] have shown that for any oriented cycle of irreducible morphisms there must be a Y such that Y and $D\operatorname{Tr}Y$ are in the cycle. This shows that there are no oriented cycles of irreducible morphisms in $\operatorname{ind}\Gamma$ and so by the result of Happel-Ringel [10] the indecomposable Γ-modules are determined by their composition factors.

As an immediate consequence of our discussion so far we have the following result.

COROLLARY 2.2. *Let Λ be an artin algebra of finite representation type, C an indecomposable Λ-module and $P_1, \ldots, P_t$ a complete set of nonisomorphic indecomposable projective Λ-modules such that $\langle P_i, C\rangle \neq 0$. Suppose C is not on any oriented cycle of nonzero nonisomorphic morphisms in the subcategory $\mathscr{C}_1$ of* $\operatorname{ind}\Lambda$ *given by $P_1, \ldots, P_t$. Then two indecomposable Λ-modules A and B with $[A] = [B]$ are isomorphic if they both are in one of the subcategories $\mathscr{C}_0$, $\mathscr{C}_1$, $\mathscr{C}_2$ of* $\operatorname{ind}\Lambda$ *given by the projectives $P_1, \ldots, P_t$.*

As a special case of this result we have the following.

COROLLARY 2.3. *Let Λ be an artin algebra of finite representation type. Suppose C in* $\operatorname{ind}\Lambda$ *does not lie on any oriented cycle of irreducible morphisms. If A and B are in* $\operatorname{ind}\Lambda$ *with* $[A]=[B]$ *such that the simples occuring as composition factors for* $\operatorname{soc}A$, $A/\mathbf{r}A$, $\operatorname{soc}B$, $B/\mathbf{r}B$ *are amongst those occuring for C, then $A \cong B$.*

Proof. Let $P_1,\dots,P_t$ be the projective covers of the simples occuring as composition factors of C. The hypothesis means that A and B are in the subcategory $\mathscr{C}_1$ of $\operatorname{ind}\Lambda$ given by $P_1,\dots,P_t$. Hence the result follows from Corollary 2.2.

3. Another isomorphism criterion

Let Λ be an artin algebra and $\mathscr{C}$ a subcategory of $\operatorname{ind}\Lambda$. In Section 1 we obtained some information about modules in $\mathscr{C}$ which are not the middle of left short chains in $\mathscr{C}$ when $\mathscr{C}$ is contravariantly determined. This section is devoted to studying modules in $\mathscr{C}$ which are not the beginning of left short chains in $\mathscr{C}$ when $\mathscr{C}$ is covariantly determined. The results generally follow directly from the formulas given in Theorem 1.4, in much the same manner as the proofs in section one followed directly from these formulas.

THEOREM 3.1. *Let Λ be an artin algebra and $\mathscr{C}$ a subcategory of* $\operatorname{ind}\Lambda$. *Assume A and B in $\mathscr{C}$ are not the start of any left short chain in $\mathscr{C}$ and their minimal projective presentations*

$$P_1(A)\to P_0(A)\to A\to 0 \quad \textit{and} \quad P_1(B)\to P_0(B)\to B\to 0$$

have the property $P_0(A)-P_1(A)=P_0(B)-P_1(B)$ in $K_0(\mathscr{C},0)$. Then we have:
(a) $\langle A,X\rangle=\langle B,X\rangle$ *and* $\langle X,D\operatorname{Tr}A\rangle=\langle X,D\operatorname{Tr}B\rangle$ *for all X in $\mathscr{C}$.*
(b) *If $\mathscr{C}$ is covariantly determined, then $A\simeq B$.*
(c) *If $\mathscr{C}$ is contravariantly determined and $D\operatorname{Tr}C$ is in $\mathscr{C}$ for each nonprojective C in $\mathscr{C}$, then $A\simeq B$.*

Proof. (a) By Theorem 1.4, we have

$$\langle A,X\rangle-\langle X,D\operatorname{Tr}A\rangle=\langle P_0(A)-P_1(A),X\rangle=\langle B,X\rangle-\langle X,D\operatorname{Tr}B\rangle$$

for all X in $\mathscr{C}$. The fact that

$$\langle A,X\rangle=\langle B,X\rangle \quad \text{and} \quad \langle X,D\operatorname{Tr}A\rangle=\langle X,D\operatorname{Tr}B\rangle$$

for all X in $\mathscr{C}$ now follows from the fact that

$$\langle A, X\rangle \cdot \langle X, D\operatorname{Tr} A\rangle = 0 = \langle B, X\rangle \cdot \langle X, D\operatorname{Tr} B\rangle$$

since A and B are not the start of any left short chain in $\mathscr{C}$.

(b) Obvious consequence of (a).

(c) Suppose A is not projective. Then $D\operatorname{Tr} A$ is in $\mathscr{C}$ and so by (a),

$$\langle D\operatorname{Tr} A, D\operatorname{Tr} B\rangle \neq 0$$

which means that B is not projective. Since $\mathscr{C}$ is contravariantly determined, it follows from (a) that $D\operatorname{Tr} A \simeq D\operatorname{Tr} B$ and so $A \simeq B$. If A is projective, then B is also projective by the previous argument and so $A = P_0(A) - P_1(A) = P_0(B) - P_1(B) = B$.

We now state the dual version of Theorem 3.1.

THEOREM 3.2. *Let Λ be an artin algebra and $\mathscr{C}$ a subcategory of* $\operatorname{ind}\Lambda$. *Assume A and B in $\mathscr{C}$ are not the ends of any right short chain in $\mathscr{C}$ and their minimal injective copresentations*

$$0 \to A \to I_0(A) \to I_1(A) \quad \textit{and} \quad 0 \to B \to I_0(B) - I_1(B)$$

have the property that $I_0(A) - I_1(A) = I_0(B) - I(B)$ in $K_0(\mathscr{C}, 0)$. Then we have:

(a) *$\langle X, A\rangle = \langle X, B\rangle$ and $\langle \operatorname{Tr} DA, X\rangle = \langle \operatorname{Tr} DB, X\rangle$ for all X in $\mathscr{C}$.*

(b) *If $\mathscr{C}$ is contravariantly determined, then $A \cong B$.*

(c) *If $\mathscr{C}$ is covariantly determined and* $\operatorname{Tr} DC$ *is in $\mathscr{C}$ for each noninjective C in $\mathscr{C}$, then $A \cong B$.*

Combining results from Section 1 with the above we obtain the following.

THEOREM 3.3. *Let Λ be an artin algebra and $\mathscr{C}$ a subcategory of* $\operatorname{ind}\Lambda$ *having no left or right short chains. Suppose in addition $\mathscr{C}$ is both covariantly and contravariantly determined. Let A and B be in $\mathscr{C}$.*

(a) *The following are equivalent.*

(i) $A \cong B$.

(ii) $[A] = [B]$.

(iii) $P_0(A) - P_1(A) = P_0(B) - P_1(B)$.

(iv) $I_0(A) - I_1(A) = I_0(B) - I_1(B)$.

(b) *If $\langle A, B\rangle \neq 0$, then we have the reciprocity law*

$$\langle P_0(A) - P_1(A), B\rangle = \langle A, B\rangle = \langle A, I_0(B) - I_1(B)\rangle.$$

Proof. (a) Already proven.

(b) This is a special case of the following slightly more general obvious consequence of the formulas in Theorem 1.4.

PROPOSITION 3.4. *Let* Λ *be an artin algebra and* A *and* B *in* $\operatorname{ind}\Lambda$ *such that* $\langle A, B\rangle \neq 0$. *If* $\langle D\operatorname{Tr}B, A\rangle = 0 = \langle B, D\operatorname{Tr}A\rangle$, *then*

$$\langle A, I_0(B) - I_1(B)\rangle = \langle A, B\rangle = \langle P_0(A) - P_1(A), B\rangle.$$

It is worth noting that our previous results imply that the conclusion of Theorem 3.3 is also valid under the following circumstances.

THEOREM 3.5. *Let* Λ *be an artin algebra and* $\mathscr{C}$ *a subcategory of* $\operatorname{ind}\Lambda$ *having the property that* $D\operatorname{Tr}C$ *is in* $\mathscr{C}$ *for each nonprojective* C *in* $\mathscr{C}$ *and* $\operatorname{Tr}DC$ *is in* $\mathscr{C}$ *for each noninjective* C *in* $\mathscr{C}$. *If in addition* $\mathscr{C}$ *has no short chains, then* A *and* B *in* $\mathscr{C}$ *satisfy* (a) *and* (b) *of Theorem* 3.3 *provided* $\mathscr{C}$ *is either contravariantly or covariantly determined.*

The rest of this section is devoted to studying some properties of modules which are not the start of left short chains in a subcategory of $\operatorname{ind}\Lambda$.

PROPOSITION 3.6. *Let* Λ *be an artin algebra,* $\mathscr{C}$ *a subcategory of* $\operatorname{ind}\Lambda$ *and* A *in* $\mathscr{C}$ *not the start of any left short chain in* $\mathscr{C}$. *Then we have the following where* $P_1(A) \to P_0(A) \to A \to 0$ *is a minimal projective presentation.*

(a) $\langle A, X\rangle$ *and* $[X, D\operatorname{Tr}A\rangle$ *depend only on* $[X]$ *for* X *in* $\mathscr{C}$.

(b) *If* P *is an indecomposable projective module such that* $P/\mathbf{r}P$ *is in* $\mathscr{C}$, *then* $P_0(A)$ *and* $P_1(A)$ *do not have* P *as a common summand. Hence, if* $\mathscr{C}$ *contains all the simple modules,* $P_0(A)$ *and* $P_1(A)$ *have no common summands.*

(c) *If* S *is a simple module whose injective envelope is in* $\mathscr{C}$, *then* S *is not a common composition factor of* Ω^2A *and* A. *Hence if* $\mathscr{C}$ *contains all injective modules, then* Ω^2A *and* A *have no composition factors in common.*

Proof. (a) Let X and Y be in $\mathscr{C}$ with $[X] = [Y]$. Then

$$\begin{aligned}\langle A, X\rangle - \langle X, D\operatorname{Tr}A\rangle &= \langle P_0(A) - P_1(A), X\rangle = \langle P_0(A) - P_1(A), Y\rangle \\ &= \langle A, Y\rangle - \langle Y, D\operatorname{Tr}A\rangle.\end{aligned}$$

It then follows that $\langle A, X\rangle = \langle A, Y\rangle$ and $\langle X, D\operatorname{Tr}A\rangle = \langle Y, D\operatorname{Tr}A\rangle$ since A is not the start of any left short chain.

(b) Let P be an indecomposable projective module such that the simple module $S = P/\mathbf{r}P$ is in $\mathscr{C}$. Since A is not the start of a left short chain in $\mathscr{C}$, we have that $\langle A, S\rangle \cdot \langle S, D\operatorname{Tr}A\rangle = 0$. Therefore $A/\mathbf{r}A$ and $\operatorname{soc}D\operatorname{Tr}A$ do not have S as a common summand. Since $\operatorname{soc}D\operatorname{Tr}A = \Omega A/\mathbf{r}\Omega A$, it follows that P is not a common summand of $P_0(A)$ and $P_1(A)$.

(c) This is an easy consequence of the following general fact.

PROPOSITION 3.7. *Let A be an arbitrary module over an artin algebra Λ and I an injective Λ-module. Then*

(a) $\langle \Omega^2 A, I\rangle = \langle I, D\operatorname{Tr} A\rangle$ *and*

(b) $\operatorname{pd} A \leq 1$ *if and only if* $\langle I, D\operatorname{Tr} A\rangle = 0$

for all indecomposable injective modules I.

Proof. (a) Let

$$0 \to \Omega^2 A \to P_1(A) \to P_0(A) \to A \to 0$$

be exact with $P_1(A) \to P_0(A) \to A \to 0$ a minimal projective presentation. Then

$$\begin{aligned} 0 \to \operatorname{Hom}_\Lambda(A, I) \to \operatorname{Hom}_\Lambda(P_0(A), I) \to \operatorname{Hom}_\Lambda(P_1(A), I) \\ \to \operatorname{Hom}_\Lambda(\Omega^2 A, I) \to 0 \end{aligned}$$

is exact since I is injective. Therefore

$$\langle A, I\rangle - \langle \Omega^2 A, I\rangle = \langle P_0(A) - P_1(A), I\rangle = \langle A, I\rangle - \langle I, D\operatorname{Tr} A\rangle.$$

Hence $\langle \Omega^2 A, I\rangle = \langle I, D\operatorname{Tr} A\rangle$.

(b) Trivial consequence of (a).

Returning to the proof of Proposition 3.6(c), we have that if I is an indecomposable injective module in $\mathscr{C}$ and $\langle A, I\rangle \neq 0$, then $\langle \Omega^2 A, I\rangle = \langle I, D\operatorname{Tr} A\rangle = 0$ since A is not the start of a left short chain in $\mathscr{C}$. Therefore $\operatorname{soc} I$ is not a common composition factor of A and $\Omega^2 A$.

For the sake of completeness we end this section by giving the duals of Propositions 3.6 and 3.7 in reverse order.

PROPOSITION 3.8. *Let B be an arbitrary module over an artin algebra Λ and let P be a projective Λ-module. Then*

(a) $\langle P, \Omega^{-2} B\rangle = \langle \operatorname{Tr} DB, P\rangle$ *and*

(b) $\operatorname{inj\,dim} B \leq 1$ *if and only if* $\langle \operatorname{Tr} DB, \Lambda\rangle = 0$.

The dual of Proposition 3.6 is the following.

PROPOSITION 3.9. *Let Λ be an artin algebra, $\mathscr{C}$ a subcategory of* $\operatorname{ind}\Lambda$ *and B in $\mathscr{C}$ not the end of any right short chain in $\mathscr{C}$. Then we have the following where $0 \to B \to I_0(B) \to I_1(B)$ is a minimal injective copresentation.*

(a) $\langle X, B\rangle$ *and* $\langle \operatorname{Tr} DB, X\rangle$ *depend only on* $[X]$ *for X in $\mathscr{C}$.*

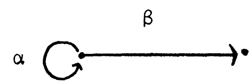
(b) *If I is an indecomposable injective module such that* $\operatorname{soc} I$ *is in $\mathscr{C}$, then $I_0(B)$ and $I_1(B)$ do not have I as a common summand. Hence if $\mathscr{C}$ contains all indecomposable injectives, $I_0(B)$ and $I_1(B)$ have no common summands.*

(c) *If S is a simple module whose projective cover is in $\mathscr{C}$, then S is not a common composition factor of B and $\Omega^{-2}B$. Hence if $\mathscr{C}$ contains all indecomposable projective modules, then B and $\Omega^{-2}B$ have no composition factors in common.*

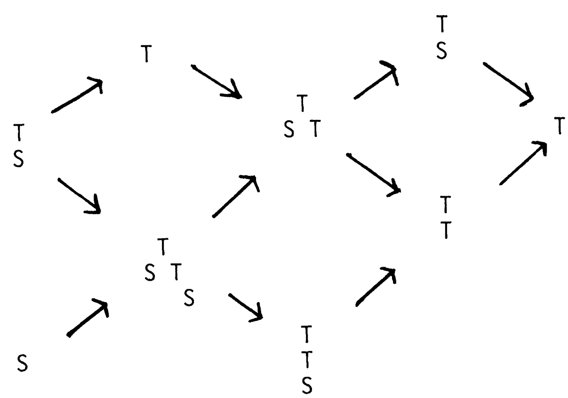
4. Examples

There are some natural ways to try to generalize our results. We give counterexamples to some of the things that do not work, and pose some questions. We first give two examples, which we use to give our counterexamples.

Example 4.1. Let Λ be given by the following quiver with relations



$\alpha^2 = 0$, over an algebraically closed field k. Denoting the simple projective Λ-module by S and the other simple Λ-module by T, we have the following AR-quiver.



The indecomposables which are not the start of short chains are

$$\begin{array}{cccccc} & & & T & T & \\ & & T & T & S\,T & \\ S, & T, & S, & & S, & \end{array}$$

and the only indecomposable not the middle of a short chain is S.

Example 4.2. Let Λ be an indecomposable selfinjective Nakayama algebra, with indecomposable projective modules $P_1, \ldots, P_n$ and of Loewy length t. Let A be an indecomposable Λ-module.

(a) A is not the start of a short chain if and only if $l(A) \le n - t$, where $l(A)$ denotes the length of A.

(b) A is not the middle of a short chain if and only if $l(A) \leq n - t + 1$.

(c) If A is not projective, $\operatorname{Hom}_\Lambda(A, D\operatorname{Tr} A) = 0$ if and only if $l(A) \leq n - 1$.

(d) If A is not projective,

$$\underline{\operatorname{Hom}}_\Lambda(A, D\operatorname{Tr} A) = \operatorname{Hom}_\Lambda(A, D\operatorname{Tr} A)/P(A, D\operatorname{Tr} A) = 0$$

if and only if $l(A) \leq n - 1$ or $l(A) \geq t - n + 1$, where $P(A, D\operatorname{Tr} A)$ denotes the morphisms from A to $D\operatorname{Tr} A$ factoring through projective modules.

(e) A is uniquely determined by its composition factors if and only if $n = 1$ or $l(A)$ is not a multiple of n.

(f) $\operatorname{End}(A)$ is a division ring if and only if $l(A) \leq n$.

(g) A is determined by $P_0(A) - P_1(A)$ if and only if $l(A) \geq t - n$.

We have seen that $\operatorname{ind} \Lambda$ having no short chains is sufficient for $[A] = [B]$ to imply $A \simeq B$ when A and B are in $\operatorname{ind} \Lambda$, and Λ being Schurian is a more general condition which is not sufficient as seen from Example 4.2(e), (f).

In Section 3 we saw that if an algebra of finite type had no short chains, then A in $\operatorname{ind} \Lambda$ is uniquely determined by $P_0(A) - P_1(A)$. This last property is not sufficient for A in $\operatorname{ind} \Lambda$ to be determined by $[A]$, nor does the converse hold, as seen from Example 4.2(e), (g).

If Λ has no short chains, we also have $\langle A, D\operatorname{Tr} A\rangle = 0$ for all A in $\operatorname{ind} \Lambda$ and $\langle \underline{A, D\operatorname{Tr} A}\rangle = 0$ for all A in $\operatorname{ind} \Lambda$ where $\langle \underline{A, D\operatorname{Tr} A}\rangle$ is the R-length of $\underline{\operatorname{Hom}}_\Lambda(A, D\operatorname{Tr} A)$. But neither of these conditions is sufficient, as seen by Example 4.2(c), (d).

We next discuss possible "local" conditions. We do not know if it is sufficient for $[A] = [B]$ to imply $A \simeq B$ that only A is assumed not to be the middle of a short chain.

The property of A not being the start of a short chain neither implies nor is implied by A not being the middle of a short chain. This is seen from the above examples. We do not know if not being the start of a short chain is a sufficient condition for modules to be determined by their composition factors. But if only one of the modules is not the start of a short chain, it is not sufficient, as we see by choosing

$$A = \begin{matrix} T \\ T \\ S \end{matrix} \quad \text{and} \quad B = \begin{matrix} & T & \\ S & & T \end{matrix}$$

in Example 4.1.

In Section 3 we saw that if A in $\operatorname{ind} \Lambda$ is not the start of a short chain in $\operatorname{ind} \Lambda$, then $\langle A, X\rangle$ is determined by $[X]$. It would be interesting to know if when both A and B in $\operatorname{ind} \Lambda$ have this latter property, $[A] = [B]$ implies $A \simeq B$.

$\langle \underline{A}, D\operatorname{Tr} A\rangle = 0$ is not a sufficient condition when A is indecomposable nonprojective, as seen from Example 4.2(d). But we do not know if $\langle A, D\operatorname{Tr} A\rangle = 0$ is a sufficient local condition if A is indecomposable nonprojective. Also we do not know if $\langle A, D\operatorname{Tr} A\rangle = 0$ for all indecomposable nonprojective A implies that the indecomposable nonprojective modules are determined by their composition factors.

It would also be interesting to know if any of the above properties, along with the property of not lying on a cycle of irreducible maps, is preserved for indecomposables with composition factors amongst those for the given one.

5. A symmetry for symmetric algebras

Let Λ be an artin algebra over the commutative artin ring R. Given two Λ-modules A and B we denote by $I(A, B)$ the R-submodule of $\operatorname{Hom}(A, B)$ consisting of those morphisms factoring through injective R-modules. Our aim in this section is to show how Theorem 1.4 can be applied to study $|P(A, B)|$ and $|I(A, B)|$, the R-lengths of $P(A, B)$ and $I(A, B)$ respectively. Amongst other things we show that $|P(A, B)| = |P(B, A)|$ for all A and B if Λ is a symmetric algebra, generalizing a result of Benson and Parker for group rings [6]. We also obtain information on $\langle A, B\rangle - \langle B, A\rangle$ for symmetric algebras.

To prove our results in this section we also make essential use of the isomorphism $\operatorname{Ext}^1_\Lambda(A, DB) \simeq D\underline{\operatorname{Hom}}_\Lambda(\operatorname{Tr} B, A)$ from [3, Prop. 2.2 and page 253] for A in $\operatorname{mod}\Lambda$ and B in $\operatorname{mod}\Lambda^{\mathrm{op}}$. Here $\underline{\operatorname{Hom}}_\Lambda(X, Y)$ denotes $\operatorname{Hom}_\Lambda(X, Y)/P(X, Y)$. This isomorphism, which in addition to being a group isomorphism is an $\underline{\operatorname{End}}(A)^{\mathrm{op}} - \underline{\operatorname{End}}(B)^{\mathrm{op}}$ bimodule isomorphism, has been useful in the representation theory of artin algebras, in particular it was used to prove the existence theorem for almost split sequences over artin algebras in [3], and also in the work of Bongartz on tilted algebras [7, page 34].

PROPOSITION 5.1. *Let A and B be modules over the artin algebra Λ. Then we have*:

(a) $|I(A, B)| = \langle P - \Omega\operatorname{Tr} DB, A\rangle$ *where* $P = P(\Omega\operatorname{Tr} DB)$, *a projective cover for* $\Omega\operatorname{Tr} DB$, *and where* ΩX *denotes the first syzygy module for* X.

(b) $|P(A, B)| = \langle B, I - \Omega^{-1}D\operatorname{Tr} A\rangle$ *where* $I = I(\Omega^{-1}D\operatorname{Tr} A)$, *an injective envelope for* $\Omega^{-1}D\operatorname{Tr} A$.

(c) $|P(A, B)| = \langle A, P(B) - \Omega B\rangle$.

(d) $|I(A, B)| = \langle I(A) - \Omega^{-1}A, B\rangle$.

Proof. (a) By Theorem 1.4 we have

$$\langle \operatorname{Tr} DB, A\rangle = \langle A, B\rangle + \langle P_0 - P_1, A\rangle$$

where $P_1 \to P_0 \to \operatorname{Tr} DB \to 0$ is a minimal projective presentation of $\operatorname{Tr} DB$. From the exact sequence $0 \to \Omega\operatorname{Tr} DB \to P_0 \to \operatorname{Tr} DB \to 0$ we obtain the exact sequence

$$0 \to \operatorname{Hom}_\Lambda(\operatorname{Tr} DB, A) \to \operatorname{Hom}_\Lambda(P_0, A) \to \operatorname{Hom}_\Lambda(\Omega \operatorname{Tr} DB, A)$$
$$\to \operatorname{Ext}^1_\Lambda(\operatorname{Tr} DB, A) \to 0.$$

Using the fact that

$$D \operatorname{Ext}^1_\Lambda(\operatorname{Tr} DB, A) \cong \underline{\operatorname{Hom}}_\Lambda(\operatorname{Tr} DA, \operatorname{Tr} DB) \cong \overline{\operatorname{Hom}}_\Lambda(A, B)$$

where $\overline{\operatorname{Hom}}_\Lambda(X, Y) = \operatorname{Hom}_\Lambda(X, Y)/I(X, Y)$, from the last exact sequence we get

$$\langle \operatorname{Tr} DB, A \rangle = \langle P_0, A \rangle + |\overline{\operatorname{Hom}}_\Lambda(A, B)| - \langle \Omega \operatorname{Tr} DB, A \rangle.$$

Comparing these two expressions for $\langle \operatorname{Tr} DB, A \rangle$ we obtain

$$\langle A, B \rangle - |\overline{\operatorname{Hom}}_\Lambda(A, B)| = \langle P_1 - \Omega \operatorname{Tr} DB, A \rangle$$

from which our desired result follows since

$$|I(A, B)| = \langle A, B \rangle - |\overline{\operatorname{Hom}}_\Lambda(A, B)|.$$

(b) Follows from (a) by duality.

(c) From the exact sequence $0 \to \Omega B \to P(B) \to B \to 0$ we obtain the exact sequence

$$0 \to \operatorname{Hom}_\Lambda(A, \Omega B) \to \operatorname{Hom}_\Lambda(A, P(B)) \to \operatorname{Hom}_\Lambda(A, B).$$

Since $P(A, B) = \operatorname{Im}(\operatorname{Hom}_\Lambda(A, P(B)) \to \operatorname{Hom}_\Lambda(A, B))$, the sequence

$$0 \to \operatorname{Hom}_\Lambda(A, \Omega B) \to \operatorname{Hom}_\Lambda(A, P(B)) \to P(A, B) \to 0$$

is exact, from which (c) follows trivially.

(d) Dual of (c).

The next two propositions give information about when two Λ-modules A and B satisfy $|P(X, A)| = |P(X, B)|$ or $|I(X, A)| = |I(X, B)|$ for all indecomposable X. They generalize a result from [6] on group algebras, and some of the arguments are similar to those in [6].

PROPOSITION 5.2. *Let A and B be modules over an artin algebra Λ where the radical has no nonzero submodule which is projective.*

(a) *$|P(X, A)| = |P(X, B)|$ for all indecomposable X if and only if $(\Omega A)_{\mathscr{P}} \simeq (\Omega B)_{\mathscr{P}}$, where $C_{\mathscr{P}}$ denotes the sum of the nonprojective indecomposable summands of C.*

(b) *Suppose, in addition,* Λ *is selfinjective and* A *and* B *have no nontrivial projective summands. Then* $A \simeq B$ *if*

$$|I(X, A)| = |P(X, A)| = |P(X, B)| = |I(X, B)|$$

for all indecomposable Λ*-modules* X.

Proof. (a) By Proposition 5.1(c) we have that $|P(X, A)| = |P(X, B)|$ for all X is equivalent to $\langle X, P(A) - \Omega A\rangle = \langle X, P(B) - \Omega B\rangle$ for all X, which is equivalent to $P(A) - \Omega A = P(B) - \Omega B$ in $K_0(\operatorname{ind}\Lambda, 0)$ since $\operatorname{ind}\Lambda$ is contravariantly determined. It is easy to see that this last equality is equivalent to $(\Omega A)_{\mathscr{P}} = (\Omega B)_{\mathscr{P}}$.

(b) This is a special case of (a). For if Λ is selfinjective, then $\Omega A = (\Omega A)_{\mathscr{P}}$ and $\Omega B = (\Omega B)_{\mathscr{P}}$ and $\Omega A \simeq \Omega B$ implies $A \simeq B$ if A and B have no nontrivial projective summands.

PROPOSITION 5.3. *Let* A *and* B *be modules over an artin algebra* Λ *where the radical has no nonzero submodule which is projective. Then* $|I(X, A)| = |I(X, B)|$ *for all indecomposable modules* X *if and only if* $(\Omega \operatorname{Tr} DA)_{\mathscr{P}} \simeq (\Omega \operatorname{Tr} DB)_{\mathscr{P}}$.

Proof. By Proposition 5.1(a) we have that

$$|I(X, A)| = |I(X, B)| \quad \text{for all } X$$

is equivalent to

$$\langle P(\Omega \operatorname{Tr} DA) - \Omega \operatorname{Tr} DA, X\rangle = \langle P(\Omega \operatorname{Tr} DB) - \Omega \operatorname{Tr} DB, X\rangle \quad \text{for all } X,$$

which is equivalent to

$$P(\Omega \operatorname{Tr} DA) - \Omega \operatorname{Tr} DA = P(\Omega \operatorname{Tr} DB) - \Omega \operatorname{Tr} DB \quad \text{in } K_0(\operatorname{ind}\Lambda, 0)$$

since $\operatorname{ind}\Lambda$ is covariantly determined. It is now easy to see that this last equality is equivalent to $(\Omega \operatorname{Tr} DA)_{\mathscr{P}} \cong (\Omega \operatorname{Tr} DB)_{\mathscr{P}}$.

We now apply Proposition 5.1 to show that $|P(A, B)| = |P(B, A)|$ for all indecomposable A and B when Λ is symmetric, generalizing a result from [6] on group algebras. We deduce this as a special case of a result for selfinjective algebras. Denote here by N the equivalence $(D(\ \ ), \Lambda)$ from $\operatorname{mod}\Lambda$ to $\operatorname{mod}\Lambda$. If C is indecomposable nonprojective, we see that $NC \simeq \operatorname{Tr} D\Omega^2 C$, by considering the following exact sequences:

$$0 \to \Omega^2 C \to P_1 \to P_0 \to C \to 0,$$

$$0 \to D(C) \to D(P_0) \to D(P_1) \to D(\Omega^2 C) \to 0,$$

$$0 \to (D\Omega^2 C)^* \to D(P_1)^* \to D(P_0)^* \to D(C)^* \to 0.$$

N is isomorphic to the identity functor if and only if Λ is a symmetric algebra. With this notation we have the following.

THEOREM 5.4. *Let Λ be a selfinjective algebra and A and B in* $\operatorname{mod}\Lambda$. *Then* $|P(B, A)| = |P(NA, B)|$.

Proof. By Proposition 5.1(b), if A has no nonzero injective summands then $|P(\operatorname{Tr} D\Omega^2 A, B)|$ equals $\langle B, I(\Omega A) - \Omega A\rangle$, which is equal to $\langle B, P(A) - \Omega A\rangle$, and hence to $|P(B, A)|$ by Proposition 5.1(c). If Q is indecomposable projective, then $\operatorname{soc} Q \simeq DQ^*/\mathbf{r}DQ^* = NQ/\mathbf{r}(NQ)$. Hence $\langle B, Q\rangle = \langle NQ, B\rangle$ since both sides measure the multiplicity of composition factors. This finishes the proof for all A, B since $\langle B, Q\rangle = |P(B, Q)|$ and $\langle NQ, B\rangle = |P(NQ, B)|$.

THEOREM 5.5. *If Λ is a symmetric algebra, then $|P(B, A)| = |P(A, B)|$ for all A, B in* $\operatorname{mod}\Lambda$.

This result can be viewed as a generalization of the fact that the Cartan matrix of a symmetric algebra is symmetric. For this generalized matrix to be symmetric, it is clearly necessary that Λ is weakly symmetric, which we see by choosing A to be simple and B to be indecomposable projective.

There is another natural generalization of the Cartan matrix, by using $\langle A, B\rangle$ for A and B indecomposable. This matrix will rarely be symmetric. In fact, we have the following.

PROPOSITION 5.6. *Let Λ be an indecomposable artin algebra. Then $\langle A, B\rangle = \langle B, A\rangle$ for all indecomposable A and B if and only if Λ is a local Nakayama algebra.*

Proof. If $\langle A, S\rangle = \langle S, A\rangle$ for all indecomposable A, S with S simple, then we have already seen that Λ is weakly symmetric, and we have $\operatorname{soc} A \simeq A/\mathbf{r}A$. In particular, this holds if $A/\mathbf{r}A$ is simple, so that each indecomposable projective is uniserial and has only one simple in its composition series. This shows that Λ is local Nakayama, and the converse is obvious.

We now use our main theorem to describe $\langle A, B\rangle - \langle B, A\rangle$ for modules over symmetric algebras. This result is based on the following specialization of our formula $D(\operatorname{Ext}^1_\Lambda(X, A)) \simeq \underline{\operatorname{Hom}}_\Lambda(\operatorname{Tr} DA, X)$ [3] to selfinjective algebras. We recall that $\langle \underline{X}, \underline{Y}\rangle$ is the R-length of $\underline{\operatorname{Hom}}_\Lambda(X, Y)$.

LEMMA 5.7. *Let Λ be a selfinjective algebra. Then $\langle \underline{B}, \underline{A}\rangle = \langle \underline{A}, \underline{D\operatorname{Tr}\Omega^{-1}B}\rangle$ for all Λ-modules A and B. In particular $\langle \underline{B}, \underline{A}\rangle = \langle \underline{A}, \underline{\Omega B}\rangle$ if Λ is symmetric.*

Proof. From the exact sequence $0 \to B \to I(B) \to \Omega^{-1}B \to 0$ we get the exact sequence

$$0 \to \operatorname{Hom}_\Lambda(\Omega^{-1}B, A) \to \operatorname{Hom}_\Lambda(I(B), A) \to \operatorname{Hom}_\Lambda(B, A) \to \operatorname{Ext}^1_\Lambda(\Omega^{-1}B, A) \to 0$$

from which it follows that $\underline{\mathrm{Hom}}_\Lambda(B, A) \cong \mathrm{Ext}^1_\Lambda(\Omega^{-1}B, A)$. Since

$$D\big(\mathrm{Ext}^1_\Lambda(\Omega^{-1}B, A)\big) \simeq \underline{\mathrm{Hom}}_\Lambda(\mathrm{Tr}\,DA, \Omega^{-1}B) \simeq \underline{\mathrm{Hom}}_\Lambda(A, D\,\mathrm{Tr}\,\Omega^{-1}B),$$

we have $\langle \underline{B}, \underline{A} \rangle = \langle \underline{A}, \underline{D\,\mathrm{Tr}\,\Omega^{-1}B} \rangle$.

Assume now that Λ is symmetric. Then $D\,\mathrm{Tr} = \Omega^2$, so $\langle \underline{B}, \underline{A} \rangle = \langle \underline{A}, \underline{\Omega B} \rangle$.

As a consequence of Lemma 5.7, we have the following result.

PROPOSITION 5.8. *Let A and B be modules over a symmetric algebra Λ. Then* $\langle A, B \rangle - \langle B, A \rangle = \langle \underline{B}, \underline{\Omega A} - \underline{A} \rangle$.

Proof. Now $\langle A, B \rangle = |P(A, B)| + \langle \underline{A}, \underline{B} \rangle = |P(B, A)| + \langle \underline{B}, \underline{\Omega A} \rangle$ by Theorem 5.5 and Lemma 5.7. Since $\langle B, A \rangle = |P(B, A)| + \langle \underline{B}, \underline{A} \rangle$, we have

$$\langle \underline{A}, \underline{B} \rangle - \langle B, A \rangle = \langle \underline{B}, \underline{\Omega A} - \underline{A} \rangle.$$

We next investigate what it means to have $\langle \underline{X}, \underline{A} \rangle = \langle \underline{X}, \underline{\Omega A} \rangle$ for all X for an A in $\mathrm{mod}\,\Lambda$.

LEMMA 5.9. *Let Λ be a selfinjective algebra where the projective modules are determined by their composition factors, and let A be in* $\mathrm{mod}\,\Lambda$ *having no nonzero projective summand. Then $\langle \underline{X}, \underline{A} \rangle = \langle \underline{X}, \underline{\Omega A} \rangle$ for all X if and only if $\Omega^2 A \simeq A$.*

Proof. Let A be in $\mathrm{mod}\,\Lambda$ with no nonzero projective summand. Assume that $\langle \underline{X}, \underline{A} \rangle = \langle \underline{X}, \underline{\Omega A} \rangle$ for all X in $\mathrm{mod}\,\Lambda$. The exact sequence $0 \to \Omega A \to P(A) \to A \to 0$ gives rise to exact sequences

$$\begin{aligned} 0 \to \mathrm{Hom}_\Lambda(X, \Omega A) &\to \mathrm{Hom}_\Lambda(X, P(A)) \\ &\to \mathrm{Hom}_\Lambda(X, A) \to \underline{\mathrm{Hom}}_\Lambda(X, A) \to 0 \end{aligned}$$

for X in $\mathrm{mod}\,\Lambda$. And $0 \to \Omega^2 A \to P(\Omega^1 A) \to \Omega A \to 0$ gives exact sequences

$$\begin{aligned} 0 \to \mathrm{Hom}_\Lambda(X, \Omega^2 A) &\to \mathrm{Hom}_\Lambda(X, P(\Omega A)) \\ &\to \mathrm{Hom}_\Lambda(X, \Omega A) \to \underline{\mathrm{Hom}}_\Lambda(X, \Omega A) \to 0. \end{aligned}$$

This show that $\langle X, \Omega^2 A \amalg \Omega A \amalg P(A) \rangle = \langle X, \Omega A \amalg A \amalg P(\Omega A) \rangle$ for all X in $\mathrm{mod}\,\Lambda$. Hence $\Omega^2 A \simeq A$ by Corollary 1.2.

Assume conversely that $\Omega^2 A \simeq A$, and consider the exact sequence $0 \to A \to P_1 \to P_0 \to A \to 0$, where $P_0 = P(A)$ and $P_1 = P(\Omega A)$. Then $[P_0] = [P_1]$, so that by our assumption $P_1 \simeq P_0$. Then $\langle \underline{X}, \underline{A} \rangle = \langle \underline{X}, \underline{\Omega A} \rangle$ follows from the above exact sequences.

We have the following consequence of Proposition 5.8 and Lemma 5.9.

COROLLARY 5.10. *Let Λ be a symmetric algebra where the projective modules are determined by their composition factors, for example a group algebra. Then the following are equivalent for A in* mod Λ *with no nonzero projective summand.*

(a) $\langle X, A\rangle = \langle A, X\rangle$ *for all X in* mod Λ.

(b) $\Omega^2 A \simeq A$.

(c) *There are nonprojective indecomposable Λ-modules C_i and positive integers $n_i, i = 1, \ldots, t$, where $\Omega^{2n_i}C_i \simeq C_i$ such that A is isomorphic to*

$$\coprod_{i=1}^{t}\left(C_i \coprod \Omega^2 C_i \coprod \cdots \coprod \Omega^{2n_i-2}C_i\right).$$

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