ILLINOIS JOURNAL OF MATHEMATICS
Volume 29, Number 3, Fall 1985

WASHNITZER'S CONJECTURE AND THE COHOMOLOGY OF A VARIETY WITH A SINGLE ISOLATED SINGULARITY

BY

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Introduction

Let X be an irreducible quasi-projective variety defined over $\mathbf{C}$, the field of complex numbers, and let $H^*(X, \mathbf{C})$ denote the singular cohomology.

One has a morphism of sites π: $X_{\text{Class}} \to X_{\text{Zar}}$, hence a Leray spectral sequence

$$H^p_{\text{Zar}}(X, R^q_{\pi_*}\mathbf{C}) \to H^{p+q}(X, \mathbf{C}),$$

which yields a decreasing filtration in $H^{p+q}(X, \mathbf{C})$. Washnitzer conjectured that if X is non-singular the filtration above coincides with the filtration by "coniveau". Recall that this filtration, also called the arithmetic filtration, is given by

$$N^pH^m = \cup \operatorname{Ker}\{H^m(X) \to H^m(X - Z)\colon Z \text{ is Zariski closed and cod } Z \geq p\}.$$

Bloch and Ogus proved Washnitzer's conjecture in [2].

We extend their results to the case of a variety with at most a single isolated singularity.

We fix a distinguished closed point x_0 on X and assume that $X - \{x_0\}$ is non-singular. In this case we say that X is almost non-singular [3].

We define $N^{+0} = H^m$ and for $p \geq 1$

$$N^{+p}H^m = \cup \operatorname{Ker}\{H^m(X) \to H^m(X - Z)\colon Z \text{ is Zariski closed, cod } Z \geq p \text{ and } x_0 \notin Z\}.$$

Our result is that this arithmetic filtration coincides with the Leray filtration induced by the morphism of sites π described above. More precisely the

Received February 8, 1983
Supported in part by grants from the Italian Ministry of Public Education and the C.N.R.

arithmetic filtration N^+ is the filtration of a natural spectral sequence which we show to coincide from E_2^{pq} on with the Leray one.

It is known that the Leray spectral sequence coincides with the second spectral sequence of hyper-cohomology associated to the algebraic de-Rham cohomology described in [5]. There (p. 8), Hartshorne proposes the problem of understanding the related filtration for a variety with arbitrary singularities. Our result provides therefore an answer to Hartshorne's question in the particular case of almost non-singular varieties.

1. Arithmetic $E_2^{pq} = H^p(X, \mathscr{H}^q)$

In this section we use singular cohomology with either integral or complex coefficients. Following [2] we let $\mathscr{H}^m$ be the sheaf in the Zariski topology associated to the presheaf $H^m(U)$.

Let X^i be the set of points (i.e. irreducible cycles) of codimension i on X and let

$$X^{+i} = \left\{ x \in X^i \colon x_0 \notin \overline{\{x\}} \right\}.$$

Let Z^0 be the set of all Zariski closed subsets of X, $Z^+ = \{W \in Z^0\colon x_0 \notin W\}$; note that there is a filtration

$$Z^0 \supset Z^+ = Z^{+1} \supset Z^{+2} \supset \cdots \quad \text{where } Z^{+i} = \{W \in Z^+\colon \operatorname{cod} Z \geq i\}.$$

Let $G^m(*, W)$ be the presheaf in the Zariski topology defined by

$$G^m(U, W) = H^m(U \cap (X - W))$$

and let

$$G^m(U) = \lim G^m(U, W), \quad W \in Z^+. \tag{1.1}$$

We denote by $\mathscr{G}^m$ the sheaf associated to the presheaf G^m. Similarly we set

$$F^m(U, W) = H^m(U, U \cap (X - W)) = H^m_{W \cap U}(U),$$

and write

$$F^m(U) = \lim F^m(U, W), \quad W \in Z^+, \tag{1.2}$$

$$\mathscr{F}^m = \text{the sheaf associated to } F^m.$$

From the long exact sequence of cohomology for the couple $(U, U \cap (X - W))$, taking direct limits, one has an exact sequence

$$\ldots F^m(U) \to H^m(U) \to G^m(U) \to F^{m+1}(U) \ldots . \tag{1.3}$$

The associated exact sequence of sheaves is

$$\ldots \mathscr{F}^m \to \mathscr{H}^m \to \mathscr{G}^m \to \mathscr{F}^{m+1} \ldots . \tag{1.4}$$

(1.5) LEMMA. $0 \to \mathscr{H}^m \to \mathscr{G}^m \to \mathscr{F}^{m+1} \to 0$ *is exact.*

Proof. We prove that $\mathscr{F}^m \to \mathscr{H}^m$ is the zero map. Over the smooth open set $V = X - \{x_0\}$ the map factors as

$$\mathscr{F}^m_{/V} \to \mathscr{H}^m_{V, Z^1} \xrightarrow{a} \mathscr{H}^m_V,$$

where $\mathscr{H}^m_{V, Z^1}$ is the sheaf on V associated to the presheaf of the cohomology groups supported on subvarieties of cod ≥ 1. The map a is zero by [2, 4.2.3]. To end the proof it suffices to show that the stalk $\mathscr{F}^m_{x_0} = 0$. If $y \in \mathscr{F}^m_{x_0}$ then $y = \text{image}\, y'$, $y' \in F^m(U, W)$ for some U, W with $x_0 \in U$, $W \in Z^+$. Let $V = U - (W \cap U)$; then $y' \to 0$ in $F^m(V, W) = 0$, hence $y = 0$.

If A is an abelian group and $x \in X$ let $i_x A$ denote the constant sheaf A on $\overline{\{x\}}$, extended by zero to all of X. Let $H^m(x) = \lim H^m(V)$ V open $\subseteq \overline{\{x\}}$.

(1.6) PROPOSITION. (Gersten resolution). *There is an exact complex*

$$0 \to \mathscr{F}^{m+1} \to \coprod_{x \in X^{+1}} i_x H^{m-1}(x) \to \coprod_{x \in X^{+2}} i_x H^{m-2}(x) \to \cdots$$

Proof. First we build the complex. Set

$$F^m_{Z^{+p}}(X) = \lim H^m_W(X), \quad W \in Z^{+p}, \ p \geq 1. \tag{1.7}$$

In particular, $F^m_{Z^{+1}}(X) = F^m(X)$; see [1] and [2]. As in [1, (4.15)], there are long exact sequences

$$\ldots F^m_{Z^{+p+1}} \to F^m_{Z^{+p}} \to \coprod_{x \in X^{+p}} H^{m-2p}(x) \to F^{m+1}_{Z^{+p+1}} \to F^{m+1}_{Z^{+p}} \ldots \tag{1.8}$$

hence a spectral sequence

$$E_1^{pq} = \coprod_{x \in X^{+p+1}} H^{q-p-2}(x) \Rightarrow F^{p+q}(X). \tag{1.9}$$

The Gersten complex we look for is the sheafified form of the complex

$$F^{m+1}(X) \to E_1^{0,m+1} \to E_1^{1,m+1} \to \cdots .$$

As in [2, (4.2.2)], the Gersten complex is exact if the natural map of sheaves $\mathscr{F}^{m+1}_{Z^{+p+1}} \to \mathscr{F}^{m+1}_{Z^{+p}}$ is the zero map, $p \geq 1$. The same argument as the one given in [2, p. 191] applies provided we prove the following claim: Given $W' \in Z^{+p+1}$, $p \geq 1$, $x \in W'$, there exists a $W \in Z^{+p}$ containing W' and an affine neighborhood U of x in X such that the map $W' \cap U \to W \cap U$ is locally homologically effaceable at x. Now the same proof for the claim [2] works if we use a finite morphism f (notations as in [2]) having the properties stated in Lemma (2.9) of [3].

(1.10) Corollary

$$0 \to \mathscr{H}^m \to \mathscr{G}^m \to \coprod_{x \in X^{+1}} i_x H^{m-1}(x) \to \coprod_{x \in X^{+2}} i_x H^{m-2}(x) \to \cdots$$

is a resolution of $\mathscr{H}^m$.

(1.11) Lemma. $H^0(X, \mathscr{G}^m) = G^m(X)$.

Proof. Let R be the local ring O_{X, x_0} and let i: $\operatorname{Sp} R \to X$ be the natural map. Set $\mathscr{L}^m = i^{-1}\mathscr{H}^m$. Then $\mathscr{G}^m = i_* \mathscr{L}^m$. Now $H^0(X, \mathscr{G}^m) = H^0(\operatorname{Sp} R, \mathscr{L}^m) = \mathscr{L}^m_{x_0} = G^m(X)$; cf. [3, (3.8)].

(1.12) Proposition. $\mathscr{G}^m$ *is acyclic.*

Proof. See (3.14) below.

Recall the exact sequence (1.3) and the sequences (1.8). The exact couple technique yields a spectral sequence

$$(1.13) \qquad E_1^{0q} = G^q(X), \qquad E_1^{pq} = \coprod_{x \in X^{+p}} H^{q-p}(x), \quad p > 0, \text{ with } E_1^{pq} \Rightarrow H^{p+q}(X).$$

We call this spectral sequence the arithmetic spectral sequence and remark that the filtration it induces on $H^m(X)$ is the filtration N^{+p}, which we have defined in the introduction.

(1.14) Theorem

$$\text{Arithmetic } E_2^{pq} = H^p(X, \mathscr{H}^q).$$

Proof. Since $\mathcal{G}^q$ is acyclic and since the other terms in the resolution (1.10) are flabby, the cohomology group $H^p(X, \mathcal{H}^q)$ is just the cohomology of the complex of global sections of the resolution. This last complex is exactly the spectral complex $E_1^{0q} \to E_1^{1q} \to \cdots$ of (1.14).

2. Arithmetic filtration = Leray filtration

In this section we use singular cohomology with complex coefficients.

We have seen above that Arithmetic $E_2^{pq} = H^p(X, \mathcal{H}^q)$. Now $R^q\pi_*\mathbf{C} = \mathcal{H}^q$ by definition, then Arithmetic $E_2^{pq} \simeq$ Leray E_2^{pq}. In order to prove that the two spectral sequences coincide from E_2^{pq} terms on we need to produce a map of spectral sequences which induces the given isomorphism.

Following [2] we indicate how to produce the required map using the algebraic de Rham cohomology $H_{DR}^m(X)$. Recall [5] that $H_{DR}^m(X)$ is defined in the following way.

(2.1) Let X be embedded as a closed subscheme in Y, where Y is non-singular.

Let $\hat{Y}$ be the formal completion of Y along X and let $\hat{\Omega}^\cdot$ be the completion of $\Omega^\cdot$, the complex of sheaves of regular differential forms on Y.

Then $H_{DR}^m(X) =$ hypercohomology $\mathbf{H}^m(\hat{Y}, \hat{\Omega}^\cdot)$ of the complex $\hat{\Omega}^\cdot$ on the formal scheme $\hat{Y}$. Since topologically $\hat{Y} = X$, $H_{DR}^m(X)$ is the hypercohomology of a certain complex of abelian sheaves on X. Note that $H_{DR}^m(X) \simeq H^m(X, \mathbf{C})$ [5].

(2.2) LEMMA. *From E_2^{pq} on the Leray spectral sequence coincides with the second spectral sequence of hypercohomology associated to the D-R complex $\hat{\Omega}^\cdot$.*

Proof. The same argument as for the smooth case [2, (6.4)] applies if one uses the formal analytic Poincaré lemma of [5, (IV, 2.1)].

We consider now a modified form of the Cousin complex associated to an abelian sheaf $\mathcal{F}$ on X; cf. [4, (IV, 2)]. Given the filtration $Z^0 \supset Z^{+1} \supset \cdots$ of Section 1, there are long exact sequences

$$(2.3) \qquad \ldots \mathcal{H}^i_{Z^{+p+1}}(\mathcal{F}) \to \mathcal{H}^i_{Z^{+p}} \to \mathcal{H}^i_{Z^{+p}/Z^{+p+1}} \to \mathcal{H}^{i+1}_{Z^{+p+1}} \ldots$$

and a spectral sequence

$$(2.4) \qquad \mathcal{E}_1^{pq} = \mathcal{H}^{p+q}_{Z^{+p}/Z^{+p+1}}(\mathcal{F}) \Rightarrow \mathcal{H}^n(\mathcal{F}).$$

The "Cousin" complex we need is the complex $\mathcal{F} \to \mathcal{E}_1^{00} \to \mathcal{E}_1^{10} \to \cdots$ namely

$$(2.5) \qquad 0 \to \mathcal{F} \to \mathcal{H}^0_{Z^0/Z^{+1}}(\mathcal{F}) \to \mathcal{H}^1_{Z^{+1}/Z^{+2}}(\mathcal{F}) \to \cdots$$

(2.6) *Remark.* If $n \geq 1$,

$$\mathscr{H}^n_{Z^{+n}/Z^{+n+1}}(\mathscr{F}) = \coprod_{x \in X^{+n}} i_x H^n_x(\mathscr{F}),$$

where $H^n_x(\mathscr{F})$ is the n-th local cohomology group with support at x. $\mathscr{H}^0_{Z^0/Z^{+1}}(\mathscr{F}) = i_* i^{-1}\mathscr{F}$ where i: $X_{x_0} \to X$ is the embedding of the local scheme at x_0.

(2.7) THEOREM. *If $\mathscr{F}$ is locally free, of finite rank, either as a sheaf on the scheme X or on the formal scheme $\hat{Y}$* (cf. (2.1)) *then* (i) *sequence* (2.5) *is a resolution of $\mathscr{F}$ and* (ii) *$\mathscr{H}^0_{Z^0/Z^{+1}}(\mathscr{F})$ is acyclic.*

(2.8) COROLLARY. *Under the hypotheses of* (2.7) *sequence* (2.5) *is an acyclic resolution of $\mathscr{F}$.*

To prove (2.7) we need the following:

(2.9) LEMMA. *If $\mathscr{F}$ is as* (2.7) *then $\mathscr{H}^i_{Z^{+p}/Z^{+p+1}}(\mathscr{F}) = 0$, $i \neq p$.*

Proof. If $p = 0$, by (2.3) it suffices to show $\mathscr{H}^i_{Z^{+1}}(\mathscr{F}) = 0$, $i > 1$. Now

$$\mathscr{H}^i_{Z^{+1}}(\mathscr{F}) = \lim_{Z \in Z^+} H^i_Z(\mathscr{F})$$

by [4, IV, var. 5, motif D]. Also $H^i_Z(\mathscr{F}) = R^{i-1}j_*(\mathscr{F}_{/X-Z})$, $i > 1$, j: $X - Z \to X$ [4, var. 3, motif B].

If $X - Z$ is affine then $R^{i-1}j_*(\mathscr{G}) = 0$, $i > 1$, for any coherent sheaf $\mathscr{G}$, because the cohomology on affine schemes is 0 (cf. E.G.A. II ∮ 4, 4.1.7, for the formal case). Since the set of Z's such that $X - Z$ is affine is a cofinal family in Z^{+1}, then $\mathscr{H}^i_{Z^{+1}}(\mathscr{F}) = 0$, $i > 1$.

If $p > 0$,

$$\mathscr{H}^i_{Z^{+p}/Z^{+p+1}}(\mathscr{F}) = \coprod_{x \in X^{+p}} H^i_x(\mathscr{F}),$$

by (2.6). When $\mathscr{F}$ is locally free on the scheme X then $(+)$ $H^i_x(\mathscr{F}) = 0$, $i \neq p$, because X is Cohen-Macaulay at the point x of codimension p [4, (IV, 2.6)]. We do not know of a reference for $(+)$ in the case of the formal scheme $\hat{Y}$, hence we sketch a proof of it.

In order to compute local cohomology at x we restrict the formal sheaf $\mathscr{F}$ to the local space X_x. Setting $U = X_x - \{x\}$, $(+)$ is equivalent to:

(1) $H^0(X_x, \mathscr{F}) \to H^0(U, \mathscr{F})$ is surjective,

and

(2) $H^i(U, \mathscr{F}) = 0$, $i > 0$, $i \neq p + 1$.

In any case $H^m(U, \mathscr{F}) = 0$ if $m > p - 1$, because U has combinatorial dimension $(p - 1)$. Since $\mathscr{F}$ is locally free and we work locally it suffices to prove (1) and (2) with $\mathscr{F} = \mathscr{O}^{\wedge}$, the completed structure sheaf of $Y^{\wedge}$. Let $\mathscr{I}$ be the ideal of X in Y; there are exact sequences

$$0 \to \mathscr{N}_n \to \mathscr{O}_n \to \mathscr{O}_{n-1} \to 0$$

where $\mathscr{O}_n$ is the structure sheaf of the subscheme of Y with ideal $\mathscr{I}^n$. Since X and Y are non-singular at x, then $(\mathscr{N}_n)_x$ is free. By induction on n one has

(1) $H^0(X_x, \mathscr{O}_n) \to H^0(U, \mathscr{O}_n)$ is surjective,

and

(2) $H^i(U, \mathscr{O}_n) = 0$, $i < (p - 1)$.

The same properties hold for the sheaf $\mathscr{O}^{\wedge}$, because of [5, (I.4.5)]. This completes the proof of $(+)$, hence of (2.9).

According to [5, IV, 1, Coda, Motif G], the spectral sequence (2.4) converges. The exactness of (2.5) follows then from the lemma; recall that $\mathscr{H}^n(\mathscr{F}) = 0$, $n > 0$. The acyclicity of $\mathscr{H}^0_{Z^0/Z^{+1}}(\mathscr{F})$ is proved below in (3.16).

(2.10) THEOREM. *The Leray spectral sequence and the arithmetic spectral sequence coincide.*

Proof. As in [2, (6.4)], using the Cousin complex introduced above in (2.3), instead of Hartshorne's.

3. Some homological algebra

This section is independent of the preceding ones.

We establish a sufficient condition for acyclicity of a sheaf $\mathscr{F}^0$ which we have used before in (1.12) and (2.8). This condition may be used to provide another proof for ∮ 4 of [3].

(3.1) Let x_0 be a distinguished closed point on X and let A be a sheaf of abelian groups on X. We start with an exact sequence

$$(3.1.1) \qquad 0 \to \mathscr{A} \to \mathscr{F}^0 \to \mathscr{F}^1 \to \cdots \to \mathscr{F}^n \xrightarrow{d_n} \mathscr{F}^{n+1} \to \cdots$$

and make the following hypotheses: (1) if $i > 0$ then $\mathscr{F}^i$ is flabby and $0 = \mathscr{F}^i_{x_0}$, the stalk at x_0; (2) there is a complex

$$(3.1.2) \qquad 0 \to \mathscr{A} \to \mathscr{E}^0 \to \mathscr{E}^1 \to \cdots \to \mathscr{F}^n \xrightarrow{d_n} \mathscr{F}^{n+1} \to \cdots$$

which is exact on the open set $X - \{x_0\}$; (3) $\mathscr{E}^i$ is flabby, $i \geq 0$; (4)

$\mathscr{E}^i = \mathscr{F}^i \oplus \mathscr{G}^i$, $i \geq 1$; (5) $H^0(X, \mathscr{G}^i) = \mathscr{G}^i_{x_0}$, the stalk at x_0; (6) there is a morphism of complexes (3.1.1) → (3.1.2) which induces the identity on $\mathscr{A}$ and which is the natural inclusion on $\mathscr{F}^i \to \mathscr{E}^i$, $i \geq 1$.

(3.2) PROPOSITION. *$\mathscr{F}^0$ is acyclic.*

(3.3) There is a spectral sequence $E_1^{pq} = H^q(X, \mathscr{F}^p) \Rightarrow E^{pq} = H^{p+q}(X, \mathscr{A})$.

The proposition amounts to (+) $E_1^{0q} = 0$, $q \geq 1$. From hypothesis (1), $E_1^{pq} = 0$, $p \geq 1$, $q \geq 1$; therefore $E_1^{0q} = E_2^{0q}$, $q \geq 1$.

We shall see that $E_2^{p0} \xrightarrow{\sim} E_\infty^p$, $p > 1$, and also that $E_2^{10} \to E_\infty^1$ is surjective. Property (+) follows immediately.

In the following we adopt the convention that an italic letter represents the global sections group of the corresponding sheaf, e.g., $F = H^0(X, \mathscr{F})$. Also we write $H^i(\mathscr{F})$ instead of $H^i(X, \mathscr{F})$, $i > 0$. We recall that E_2^{p0} is the *p-th* cohomology group of

$$(3.4) \qquad 0 \to F^0 \to F^1 \to \cdots \to F^n \to F^{n+1} \to \cdots$$

which is the complex of global sections associated with (3.1.1).

Let

$$(3.5) \qquad \mathscr{B}^{+j} = \text{Image: } \mathscr{F}^{j-1} \to \mathscr{F}^j, \qquad \mathscr{B}^j = \text{Image: } \mathscr{E}^{j-1} \to \mathscr{E}^j$$

$$\mathscr{Z}^{+j} = \text{Ker: } \mathscr{F}^j \to \mathscr{F}^{j+1}, \qquad \mathscr{Z}^j = \text{Ker: } \mathscr{E}^j \to \mathscr{E}^{j+1}$$

By our hypotheses $\mathscr{A} = \mathscr{Z}^{+0}$, $\mathscr{B}^{+j} = \mathscr{Z}^{+j}$, $j \geq 1$; then $A = Z^{+0}$, $B^{+j} = Z^{+j}$, $j \geq 1$.

(3.6) (a) $E_2^{p0} = H^{p-1}(\mathscr{B}^{+1})$, $p \geq 2$.

(b) $E_2^{10} = B^{+1}/\text{Image}\, F^0$.

Proof. (a) $0 \to \mathscr{B}^{+1} \to \mathscr{F}^1 \to \mathscr{F}^2 \to \cdots$ is exact.

(b) $B^{+1} = Z^{+1} = \text{Ker: } F^1 \to F^2$.

By hypothesis, (3.1.2) there is an exact sequence

$$(3.7) \qquad 0 \to \mathscr{B}^j \to \mathscr{Z}^j \to \mathscr{R}^j \to 0, \quad j \geq 1,$$

with $\mathscr{R}^j$ a skyscraper sheaf, supported at x_0. Using (3.7) and the exact sequences $0 \to \mathscr{Z}^j \to \mathscr{E}^j \to \mathscr{B}^{j+1} \to 0$ one has, by simple chase,

$$(3.8) \quad H^i(\mathscr{A}) = H^{i-1}(\mathscr{B}^1) = H^{i-1}(\mathscr{Z}^1) = \cdots = H^1(\mathscr{B}^{i-1}), \quad i \geq 2,$$

and, similarly,

$$(3.9) \qquad H^{i-1}(\mathscr{B}^{+1}) = H^1(\mathscr{B}^{+(i-1)}).$$

(3.10) LEMMA. (a) *The natural map* $H^1(\mathscr{B}^{+(i-1)}) \to H^1(\mathscr{B}^{i-1})$ *is an isomorphism*, $i \geq 2$.

(b) $B^{+1} \to H^1(\mathscr{A})$ *is surjective.*

The proposition follows because by (3.6), (3.9) and (3.8) the lemma is equivalent to (a) $E_2^{i0} \xrightarrow{\sim} E_\infty^i$, $i > 1$ and (b) $E_2^{10} \to E_\infty^1$ is surjective.

Proof of (3.10)(*a*). In the following we omit the index $(i-1)$ when there is no confusion.

There is a diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{Z}^{+} & \to & \mathscr{F} & \to & \mathscr{B}^{+i} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{B} & \to & \mathscr{E} & \to & \mathscr{Q} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{Z} & \to & \mathscr{E} & \to & \mathscr{B}^{i} & \to & 0 \end{array} \tag{3.11}$$

where $\mathscr{Q}$ is defined by exactness. The associated diagram of global sections is

$$\begin{array}{ccccccccccc} 0 & \to & Z^{+} & \to & F & \xrightarrow{d^+} & B^{+i} & \to & C^{+} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow r & & \downarrow g & & \\ 0 & \to & B & \to & E & \xrightarrow{h} & Q & \to & T & \to & 0 \\ & & \downarrow & & \downarrow = & & \downarrow s & & \downarrow f & & \\ 0 & \to & Z & \to & E & \xrightarrow{d} & B^{i} & \to & C & \to & 0 \end{array} \tag{3.12}$$

where C^+, T, C are defined to be the cokernels of d^+, h, d. Since $\mathscr{F}$ and $\mathscr{E}$ are acyclic, $C^+ = H^1(\mathscr{Z}^{+i-1})$ and $T = H^1(\mathscr{B}^{i-1})$. One has to prove that g is an isomorphism.

(3.13) LEMMA. *Let* $j \geq 0$. *If* $z \in Z^j$ *is a global section whose restriction at the stalk at* x_0 *is* 0 (*i.e.*, $z_{x_0} = 0$) *then* $z \in B^{+j}$.

Proof. Consider the diagram

$$\begin{array}{ccccc} \mathscr{B}^{+j} & = & \mathscr{Z}^{+j} & \to & \mathscr{F}^{j} \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{B}^{j} & \to & \mathscr{Z}^{j} & \to & \mathscr{E}^{j} \end{array}$$

where all maps are injective. Associated to it there is the corresponding diagram of global sections, which we omit, and the maps are still injective.

By hypothesis, $z_{x_0} = 0$, then $z \in F^j$ by (3.1.1), (3.4), (3.5), hence $z \in Z^{+j}$, because $d^+z = dz = 0$.

Proof of the surjectivity of g. Let $t \in T$; we shall find a representative $q' \in Q$ for t such that $q' = r(b^+)$, some $b^+ \in B^{+i}$. Let q be some representative of t in Q. Since $\mathscr{E} \to \mathscr{Q}$ is surjective and $\mathscr{E}$ is flabby, there is a global section $e \in E$ such that in the stalk at x_0, $h(e)_{x_0} = q_{x_0}$. We take $q' = q - h(e)$; note $q'_{x_0} = 0$. Then $b = s(q')$ has image zero in the stalk at x_0, so $b \in B^{+i}$ (3.13).

We claim that $r(b) = q'$. The sequence (3.7) induces an exact sequence $0 \to \mathscr{R} \to \mathscr{Q} \to \mathscr{B}^i \to 0$, by chase on (3.11). Since $\mathscr{R}$ is acyclic the corresponding sequence of global sections

$$0 \to R \to Q \xrightarrow{s} B^i \to 0$$

is exact. To prove the claim we note that (i) $sr(b) = s(q')$, i.e. $(r(b) - q') \in R$ and (ii) in the stalk at x_0, $0 = (r(b) - q')_{x_0}$.

Therefore $0 = r(b) - q'$, because $\mathscr{R}$ is skyscraper, supported at x_0.

Proof of the injectivity of g. Let $a \in \operatorname{Ker} g$ be represented by $b^+ \in B^{+i}$; we shall show $h^+ \in \operatorname{Image} d^+\colon F \to B^{+i}$. We have $r(b^+) = h(e)$, some $e \in E$, because $g(a) = 0$. Evaluating at the stalk at x_0, $b^+_{x_0} = 0 = r(b^+)_{x_0} = h(e)_{x_0}$.

By exactness of the middle row of (3.11), in the stalk at x_0 there is $\beta \in \mathscr{B}_{x_0}$ such that $\beta = e_{x_0}$. Now $d\colon \mathscr{E}^{i-2} \to \mathscr{B}$ is surjective and $\mathscr{E}^{i-2}$ is flasque, because $i > 1$; hence there is a global section $w \in E^{i-2}$ with $(dw)_{x_0} = \beta = e_{x_0}$. Since $(e - dw)_{x_0} = 0$ in the stalk $\mathscr{E}_{x_0}$ then $(e - dw) \in F$, by (3.1), (3.4), (3.5). We claim that $b^+ = d^+(e - dw)$. It suffices to show

$$srd^+(e - dw) = sr(b^+),$$

because sr is an inclusion. Now

$$srd^+(e - dw) = d(e - dw) = d(e) = sh(e) = sr(b^+).$$

Proof of (3.10)(*b*). The proof given for (a) applies, using the diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathscr{A} & \to & \mathscr{F}^0 & \to & \mathscr{B}^{+1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{B}^0 & \to & \mathscr{E}^0 & \to & \mathscr{Q} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{Z}^0 & \to & \mathscr{E}^0 & \to & \mathscr{B}^1 & \to & 0
\end{array}$$

which is the analogue of (3.11) with $i = 1$, where $\mathscr{B}^0 = \operatorname{Image} \mathscr{A} \to \mathscr{E}^0$.

We apply the proposition to the sheaf $\mathscr{G}^m$ of Section 1.

(3.14) COROLLARY. *$\mathscr{G}^m$ is acyclic.*

Proof. The hypotheses in (3.1) are satisfied, taking the exact complex (3.1.1) to be the complex (1.10) and (3.1.2) to be the complex

$$0 \to \mathcal{H}^m \to \coprod_{x \in X^0} i_x H^m(x) \to \coprod_{x \in X^1} i_x H^{m-1}(x) \to \cdots$$

which is exact on $X - \{x_0\}$ by [2].

Similarly for the sheaf $\mathcal{H}^0_{Z^0/Z^{+1}}(\mathcal{F})$ of Section 2:

(3.15) COROLLARY. *$\mathcal{H}^0_{Z^0/Z^{+1}}(\mathcal{F})$ is acyclic.*

Proof. The hypotheses (3.1) are satisfied taking (3.1.1) to be the resolution (2.6) and the complex (3.1.2) to be

$$0 \to \mathcal{F} \to \coprod_{x \in X^0} i_x H^0_x(\mathcal{F}) \to \coprod_{x \in X^1} i_x H^1_x(\mathcal{F}) \to \cdots$$

which is exact on $X - \{x_0\}$ by [4, (IV.2)].

4. Final remarks

(4.1) In the following we use singular cohomology with integer coefficients.

Let $B^i(Y)$ be the group of cycles of codim i mod algebraic equivalence on Y, a smooth variety; by [2, (7.4)], $B^i(Y) = H^i(Y, \mathcal{H}^i)$. In particular $H^1(Y, \mathcal{H}^1)$ $=$ Image: $\operatorname{Pic} Y \to H^2(Y)$.

PROPOSITION . *If X is almost non-singular,* $H^1(Y, \mathcal{H}^1) =$ Image: $\operatorname{Pic} X \to H^2(X)$.

Proof. Since X is irreducible, $\mathcal{H}^0_X$ is the constant sheaf $\mathbf{Z}$ in the Zariski topology; therefore $H^i(X, \mathcal{H}^0) = 0$, $i > 0$. From the Leray spectral sequence we have exact sequence

$$0 \to H^1(X, \mathcal{H}^1) \to H^2(X) \to H^0(X, \mathcal{H}^2) \to H^2(X, \mathcal{H}^1).$$

Although not needed later we note that $H^2(X, \mathcal{H}^1) = 0$ by (1.10). Further, from the description in (1.10) and (1.15) we see that $H^1(X, \mathcal{H}^1)$ is generated by the classes of the irreducible divisors which do not contain x_0. The proposition follows because $\operatorname{Pic} X$ is generated by such divisors.

Question. Let X be a variety with arbitrary singularities. Has $H^1(X, \mathcal{H}^1)$ any reasonable geometric interpretation? Our motivation is that for K-theory $H^1(X, \mathcal{K}^1) = \operatorname{Pic} X$.

(4.2) Let (X, x_0) be almost non-singular, let $Z^p(X, x_0)$ be the free abelian group with set of generators X^{+p}, let $R(X, x_0)$ be the relations of algebraic equivalence which avoid x_0 (cf. [3] for the definition in the case of rational equivalence). We have

$$H^p(X, \mathscr{H}^p) = Z^p(X, x_0)/R(X, x_0), \quad p > 1,$$

by the same argument as for [2, (7)], using our (1.10).

It does not appear that $H^p(X, \mathscr{H}^p)$, $p > 1$, is a reasonable candidate for an extension to the almost non-singular case of the notion of $B^m(X)$; see (4.1). Indeed one expects such a group to be countably generated at most, because this is the case when the variety is smooth. On the other hand some computation we have show that if X is $\mathbf{P}^3$ blown up along a rational curve with one single node, so that X is almost non-singular, then $H^2(X, \mathscr{H}^2)$ is not countably generated. We sketch the example. Let Y be the rational curve with a single node in $\mathbf{P}^3$. By the same arguments as in my paper "Grothendieck's K theory and the cubic threefold with an ordinary double point" one has an isomorphism $CH^2(X) \simeq \operatorname{Pic} Y \oplus CH^2(\mathbf{P}^3)$, where $CH^2(X)$ denotes the group of codimension 2 cycles on X which avoid the singular point x_0 modulo the relations of rational equivalence which avoid x_0 [3]. In the isomorphism, $\operatorname{Pic} Y$ corresponds to the subgroup of $CH^2(X)$ generated by the classes of lines in the exceptional divisor which avoid x_0. We recall that $\operatorname{Pic} Y = \mathbf{C}^* \oplus \mathbf{Z}$, and let $A^2(X)$ be the subgroup of $CH^2(X)$ which is isomorphic to $\mathbf{C}^*$. Let f: $CH^2(X) \to H^2(X, \mathscr{H}^2)$ be the natural map. We shall prove that the restriction of f to $A^2(X)$ is injective; from this it follows that $H^2(X, \mathscr{H}^2)$ is not countably generated as an abelian group. Let Z_1 and Z_2 be effective 1-cycles on X for which

$$\operatorname{class}(Z_1 - Z_2) \in \operatorname{Ker}(f) \cap A^2(X).$$

By our Bloch-Ogus type result one can produce a complete and smooth parameter curve T and a cycle W in $T \times X$, such that for every point $t \in T$, W_t is a 1-cycle on X which avoids x_0, and there are t_1 and t_2 with $W_{t_i} = Z_i + R$, $i = 1, 2$. The correspondence g: $T \to A^2(X)$ given by $g(t) = \operatorname{class}(W_t - Z_1 - R)$ maps a complete curve to $\mathbf{C}^*$; hence it is constant.

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