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AUTOMORPHISMS OF METABELIAN GROUPS WITH TRIVIAL CENTER

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1. Introduction

Let $F(n)$ denote the free group of rank n and let $B(n) = F(n)/F(n)''$, the free metabelian group of rank n. The automorphism group $\mathrm{Aut}(B(n))$ has been independently and jointly investigated by Bachmuth and Mochizuki in a series of papers dating from 1965 to 1987. In [1], the outer automorphism group $\mathrm{Out}(B(2))$ is shown to be isomorphic to $GL_2(\mathbb{Z})$. When $n = 3$, $\mathrm{Aut}(B(n))$ has been shown to be infinitely generated in [2]. For $n \geq 4$, they showed [3] that

$$\mathrm{Aut}(F(n)) \to \mathrm{Aut}(B(n)) \to 1;$$

i.e., every automorphism of $B(n)$ is induced by an automorphism of $F(n)$ and hence, $\mathrm{Aut}(B(n))$ is finitely generated. This is carried out using the faithful Magnus representation of $IA(B(n))$ as a subgroup of $GL_n(\mathbb{Z}[F(n)/F(n)'])$ ($IA(G)$ is the normal subgroup of $\mathrm{Aut}(G)$ consisting of automorphisms of G which induce the identity on the quotient G/G'), and ideas and methods influenced by matrices and matrix groups over integral Laurent polynomial rings.

Instead of considering the automorphism group of a given metabelian group, we propose to approach the problem from the opposite direction, namely:

> Which groups can be realized as the automorphism groups of metabelian groups?

That is, for which groups H does there exist a metabelian group G such that $\mathrm{Aut}\, G$ is isomorphic to H?

The case when G is a torsion free, nilpotent group of class 2, hence metabelian with non-trivial center, has been considered by Dugas and Göbel. In [8], they adapt Zalesskii's matrix construction of a torsion free, nilpotent group of rank 3 and class 2 with no outer automorphisms, to show that any group H can be realized as

$$\mathrm{Aut}(G)/\mathrm{Stab}(G) \cong H$$

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for some torsion free, nilpotent G of class 2, where

$$\mathrm{Stab}(G) = \{\alpha \in \mathrm{Aut}(G) : \alpha \text{ induces the identity on } Z(G) \text{ and } G/Z(G)\}.$$

Notice that in this setting, $\mathrm{Inn}(G) \subseteq \mathrm{Stab}(G)$ and $\mathrm{Stab}(G)$ is abelian. Using the Baer-Lazard theorem, which provides a correspondence between nilpotent groups of class 2 and alternating bilinear maps, they arrive at a similar result [9] with the added information that $\mathrm{Stab}(G)/\mathrm{Inn}(G)$ is isomorphic to a direct sum of $|G|$-copies of the cyclic group $\mathbb{Z}_2$ of order 2.

In this paper, we consider metabelian groups with trivial center. If a group G has trivial center, then G automatically embeds as $\mathrm{Inn}\, G$ in $\mathrm{Aut}\, G$. The quotient $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is usually called $\mathrm{Out}(G)$, the outer automorphism group of G. Hence we ask:

> For which groups H does there exist a metabelian group G with trivial center such that $\mathrm{Out}\, G = H$?

A partial answer is supplied below by our main results.

A group is said to be *complete* if its center and outer automorphism group are both trivial. In [11], Gagen and Robinson classified all finite, metabelian and complete groups. Using homological methods, Robinson considered infinite soluble and complete groups in [20]. Here, we consider, the infinite metabelian case and provide infinitely many non-isomorphic complete, metabelian groups in the following theorem.

THEOREM. *Let B be a free metabelian group of rank λ with $3 \leq \lambda < 2^{\aleph_0}$. Then there exists a torsion free, complete, metabelian group G embedding B, with G containing an abelian and characteristic subgroup A of cardinality $2^{\aleph_0}$ such that $G/A \cong B/B'$.*

It is interesting to note that this non-abelian result is obtained by mainly applying methods from abelian group theory, properties of group rings and the Magnus representation of a free metabelian group.

A group is said to be a *unique product group* (UP group) if, given any two non-empty finite subsets A and B of G, there exists at least one element x of G that has a unique representation in the form $x = ab$ with $a \in A$ and $b \in B$. Free groups and, more generally, right ordered groups are examples of UP groups. It is a well-known fact that no group can have its automorphism group be cyclic of odd order > 1 or infinite cyclic (see Robinson [19], [21] and Pettet [16], [17], [18] for other examples of non-automorphism groups). In contrast, we see from our second main result that every abelian group is isomorphic to the automorphism group of some metabelian group modulo its inner automorphisms.

THEOREM. *Every abelian group and every UP group can be realized as the outer automorphism group of some metabelian group with trivial center.*

The construction is founded on the endomorphism rings of torsion free abelian groups, the extraction of the multiplicative units of a ring and the semi-direct product of two abelian groups.

2. Representation of free metabelian groups

For convenience, we include two well-known results concerning free soluble groups and matrix representation of groups, which are due, respectively, to Smélkin and Magnus.

The following is a special case of a lemma due to Magnus [14], which is referred to in the literature as Magnus representation and has proved to be a useful tool in various contexts.

Suppose F is a non-cyclic free group with basis $\{x_i : i \in I\}$. Let

$$\{s_i = x_i F' : i \in I\} \text{ and } \{a_i = x_i F'' : i \in I\}$$

be generators of F/F' and F/F'' respectively. Let $\oplus_{i\in I}\mathbb{Z}[F/F']t_i$ be a free $\mathbb{Z}[F/F']$-module of rank $|I|$. The set of matrices

$$\begin{bmatrix} F/F' & 0 \\ \oplus_{i\in I}\mathbb{Z}[F/F']t_i & 1 \end{bmatrix}$$
$$= \left\{ \begin{pmatrix} g & 0 \\ \sum_{i\in I} r_i t_i & 1 \end{pmatrix} : g \in F/F', \sum_{i\in I} r_i t_i \in \oplus_{i\in I}\mathbb{Z}[F/F']t_i \right\}$$

forms a group under formal matrix multiplication.

LEMMA 2.1 [14]. *The map*

$$a_j \to \begin{bmatrix} s_j & 0 \\ t_j & 1 \end{bmatrix}$$

extends to an injective homomorphism

$$\psi\colon F/F'' \to \begin{bmatrix} F/F' & 0 \\ \oplus_{i\in I}\mathbb{Z}[F/F']t_i & 1 \end{bmatrix}.$$

If B is a metabelian group, then $\bar{B} = B/B'$ acts on B' via conjugation. Hence there exists a homomorphism $\varphi\colon \bar{B} \to \mathrm{Aut}(B')$. This extends to a ring homomorphism

$$\varphi\colon \mathbb{Z}[\bar{B}] \to \mathrm{End}(B'),$$

and so B' can be viewed as a $\mathbb{Z}[\bar{B}]$-module. In the case when B is free metabelian, the Magnus representation enables us to see that each nonzero element of $\varphi(\mathbb{Z}[\bar{B}]) \subset \mathrm{End}(B')$ is a monomorphism. We express this in terms of modules:

COROLLARY 2.2. *Let $B = F/F''$. Then B' is a torsion free $\mathbb{Z}[\bar{B}]$-module.*

Proof. Using the Magnus representation, we identify B with $\psi(B)$ and notice that B' embeds in

$$C = \begin{bmatrix} 1 & 0 \\ \oplus_{i \in I} \mathbb{Z}[F/F']t_i & 1 \end{bmatrix}.$$

If $a = \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} \in B$ and $Z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \in B'$, then $Z^a = \begin{pmatrix} 1 & 0 \\ zx & 1 \end{pmatrix}$. Moreover, if

$$b = \overline{\sum n_j \begin{pmatrix} u_j & 0 \\ v_j & 1 \end{pmatrix}} \in \mathbb{Z}[\bar{B}],$$

then $Z^b = \begin{pmatrix} 1 & 0 \\ z \cdot \sum n_j u_j & 1 \end{pmatrix}$, where $n_j \in \mathbb{Z}$, $u_j \in F/F'$ and

$$^- : B \to B/B'$$

is the canonical epimorphism. Let $z = \sum_{i \in I} b_i t_i$, where $b_i \in \mathbb{Z}[F/F']$. Since $\mathbb{Z}[F/F']$ is an integral domain (see [12]),

$$\left(\sum b_i t_i\right) \cdot \left(\sum n_j u_j\right) = 0 \text{ iff } b_i = 0 \text{ or } \sum n_j u_j = 0 \text{ for each } i.$$

Hence B' is a torsion free $\mathbb{Z}[\bar{B}]$-module. □

From now on, we identify B' with a subgroup of $\oplus_{i \in I} \mathbb{Z}[\bar{B}]t_i$. Suppose S is a ring and G is a group. Let $I(S, G)$ denote the *augmentation ideal of* $S[G]$, which consists of all $x = \sum n_g g \in S[G]$ such that $\sum n_g = 0$. Let $\sum_{i \in I} b_i t_i$ correspond to an element of B'. A characterization of the b_i's is given in [1], but for our purposes it is enough to know that each b_i is an element of $I(\mathbb{Z}, \bar{B})$. Since the action of $b \in \mathbb{Z}[\bar{B}]$ on $z = \sum b_i t_i \in B'$ is defined by $z^b = \sum b b_i t_i$ and the commutator equality

$$[uv, w] = [u, w]^v [v, w]$$

holds for elements of any group, it suffices to verify that the free generators $\psi(a_i)$ satisfy

$$[\psi(a_i), \psi(a_j)] = (s_j - 1)t_i + (1 - s_i)t_j.$$

Hence B' is a subgroup of $\oplus_{i \in I} I(\mathbb{Z}, \bar{B})t_i$.

Let $F^{(d)}$ denote the d-th derived subgroup of the non-cyclic free group F.

THEOREM 2.3 (Smélkin [22]). *Let $G = F/F^{(d)}$ be a free soluble group and $\alpha \in \operatorname{Aut}(G)$ which is the identity when restricted to $F^{(d-1)}/F^{(d)}$. Then $\alpha \in \operatorname{Inn} G$. In fact, α is conjugation by an element of $F^{(d-1)}/F^{(d)}$.*

The following corollary follows easily.

COROLLARY 2.4. *Suppose $B = F/F''$, where F is a free group and B embeds in a group G. Let α, $\beta \in \operatorname{Aut}(B)$ such that $\alpha|_{B'} = \beta|_{B'}$ Then*

(i) *$\alpha \in \beta \cdot \operatorname{Inn} B$ and*
(ii) *α extends to G if and only if β extends to G.*

3. Extensions of free metabelian groups

Recall that an abelian group A is p-reduced if $\bigcap_{n\in\omega} p^n A = 0$, for some prime number p. If an abelian group A is p-reduced and torsion free, we denote its completion relative to the p-adic topology by $\widehat{A}$. Note that conjugation by $b \in B$ extends uniquely to $\widehat{B'}$. Suppose B is free metabelian, $B' \leq H \leq \widehat{B'}$ and H is B-invariant, i.e., closed under conjugation by elements of B. The set $G = H \cdot B$ of elements of the form $h \cdot b$ forms a group under the operation

$$(h \cdot b)(g \cdot c) = hg^{b^{-1}} \cdot bc \text{ where } h, g \in H, \; b, c \in B.$$

Note however that representation of elements of G in the form $h \cdot b$ is not unique; i.e., G is not a semi-direct product.

LEMMA 3.1. *Suppose $G = H \cdot B$, where B is free metabelian, $B' \leq H < \widehat{B'}$ and H is B-invariant. If A is a normal, abelian subgroup of G then $A \leq H$. Hence H is the largest normal, abelian subgroup of G and so is characteristic in G.*

Proof. We first observe using Corollary 2.2 , that if $b \in B$ and $x \in B'$ with $x^b = x$, then either $b \in B'$ or $x = 1$; i.e., conjugation by elements $b \in B \setminus B'$ does not leave non-trivial elements of B' fixed. By the continuity of homomorphisms on H and the B-invariance of H, it follows that conjugation by $b \in B \setminus B'$ does not leave non-trivial elements of H fixed.

Suppose there exists $x \in A$ such that $x = h \cdot b$, $h \in H$ and $b \in B \setminus B'$. Since A is normal, abelian in G, $x^c \in A$ and $x^c x^{-1} = x^{-1} x^c \in A$ for all $c \in B$; i.e., $[c, x^{-1}] = [x, c]$. Hence $[c, x]^{x^{-1}} = [c, x]$. Since $[c, x] \in H$ and $h \in H$, $[c, x]^{b^{-1}} = [c, x]$. Taking $c = b$, we get $[b, h]^b = [b, h]$. This implies $[b, h] = 1$, i.e., $h^b = h$. Since $b \in B \setminus B'$, it follows that $h = 1$. So $x = b \in (B \setminus B') \cap A$ and $b^c \in A$ for all $c \in B$. Since $b^c b^{-1} \in A \cap B'$, $b \cdot b^c b^{-1} = b^c b^{-1} \cdot b = b^c$. This means $(b^c)^{b^{-1}} = b^c$, and so, by our first observation, $b^c = 1$ for all $c \in B$. But this occurs only if $b = 1$, thus giving us a contradiction. □

Throughout this section and the next, we adopt the following notation:

$R \; = \; \mathbb{Z}[\bar{B}]$

R^* the group of multiplicative units of R

x^{φ}	the image of the element x under the homomorphism φ
x^R	the R-submodule generated by x
$h_p^G(x)$	the p-height of x in G, i.e., the largest integer k such that p^k divides $x \in G$ in the torsion free abelian group G

LEMMA 3.2. *Suppose B is free metabelian with rank at least three and $\varphi \in \mathrm{Aut}(B')$. If $e^{\varphi} \in e^R$ for every e in every basis of B', then $\varphi \in R$.*

Proof. If B has rank at least three, B' has at least two R-independent elements, say e and f. Suppose $e^{\varphi} = e^r$ and $f^{\varphi} = f^s$, where $r, s \in R$. Then $(e \cdot f)^{\varphi} = (e \cdot f)^t$ for some $t \in R$, since $e \cdot f$ is in some basis of B'. Because $\varphi \in \mathrm{Aut}(B')$, $(e \cdot f)^t = e^r \cdot f^s$. It follows from the R-independence of e and f and Corollary 2.2 that $t = r = s$. Now consider an arbitrary element x of some basis of B'. Then either $\{x, e\}$ or $\{x, f\}$ is R-independent. By the foregoing argument, it is clear that $x^{\varphi} = x^r$. Moreover $\varphi \in R^*$. $\square$

LEMMA 3.3. *If $e \in B'$ and $\varphi \in \mathrm{Aut}(B')$ such that $e^{\varphi} \notin e^R$ and $h_p^{B'}(e) \neq 0$ for some prime p, then there exists $e_0 \in B'$ such that $e_0^{\varphi} \notin e_0^R$ and $h_p^{B'}(e_0) = 0$ for all primes p. Moreover e_0^R is pure in B'; i.e., if $x \in B'$ and $0 \neq n \in \mathbb{Z}$ such that $x^n \in e_0^R$ then $x \in e_0^R$.*

Proof. Write $e = f^n$ $(f \in B',\ 0 \neq n \in \mathbb{Z})$ such that $h_p^{B'}(f) = 0$ for all primes p. Take $e_0 = f$. $\square$

If A is a subgroup of an abelian, torsion free group G, we define the pure subgroup generated by A to be

$$\langle A \rangle_* = \{x \in G : x^n \in A \text{ for some } 0 \neq n \in \mathbb{Z}\}.$$

We use J_p to denote the group (or ring) of p-adic integers.

Lemma 3.4 is an adaptation of Lemma 3 in [5], which is a result in a strictly abelian setting, and will be used often in the constructions of the next propostion and section.

LEMMA 3.4. *Suppose B is free metabelian of rank λ with $3 \leq \lambda < 2^{\aleph_0}$. Let H be a pure, B-invariant subgroup of $\widehat{B'}$ with cardinality less than $2^{\aleph_0}$ and $B' \leq H < \widehat{B'}$. If Γ is a subset of $\widehat{B'}$ with cardinality less than $2^{\aleph_0}$ and $\varphi : B' \to \widehat{B'}$ is a monomorphism such that $H \cap \Gamma = \emptyset$ and $\varphi|_{B'} \notin R^*$, then there exists $g \in \widehat{B'}$ such that $g^{\varphi} \notin \langle H, g^R \rangle_*$ and $\Gamma \cap \langle H, g^R \rangle_* = \emptyset$.*

Proof. Since φ is defined on B', φ extends to $\widehat{B'}$ by continuity and restricts to H. If $H^{\varphi} \not\subseteq H$, then there exists $g \in H$ such that $g^{\varphi} \notin \langle H, g^R \rangle_* = H$ and

$\langle H, g^R\rangle_* \cap \Gamma = \emptyset$. Suppose $H^\varphi \subseteq H$. By Lemma 3.2, $\varphi|_{B'} \notin R^*$ implies there exists a basis element e of B' such that $e^\varphi \notin e^R$. Using Lemma 3.3, we can assume that $h_p^{B'}(e) = 0$ for every prime p and, hence, e^R is pure in B'. Since $|H| < 2^{\aleph_0}$, there exists $\xi \in J_p$ such that $H^\xi \cap H = 1$ (see [7]). Let $g = e^\xi$. If $g^\varphi \in \langle H, g^R\rangle_*$, then there exists a non-zero $n \in \mathbb{N}$ such that $g^{n\varphi} = (g^\varphi)^n = h \cdot g^r$ for some $h \in H$, $r \in R$. So $(e^{n\varphi - r})^\xi \in H^\xi \cap H = 1$ and $e^{n\varphi} = e^r$. Since e^R is pure, $e^\varphi \in e^R$, which is a contradiction. Therefore $g^\varphi \notin \langle H, g^R\rangle_*$.

There exist $2^{\aleph_0}$ algebraically independent ξ's in J_p such that $H^\xi \cap H = 1$. So there exist $2^{\aleph_0}$ g's ($g = e^\xi$) such that $g^\varphi \notin \langle H, g^R\rangle_* =: H_g$. Suppose each such H_g has $H_g \cap \Gamma \neq \emptyset$; i.e., there exists $x \in \Gamma$ such that $x^n = h \cdot g^r$ for some non-zero $n \in \mathbb{N}$, $h \in H$ and $r \in R \setminus \{0\}$. Since there are less than $2^{\aleph_0}$ choices for quadruples (n, x, h, r) and $2^{\aleph_0}$ choices for g, there must exist distinct ξ and μ with H_{e^ξ} and H_{e^μ} such that $x^n = h \cdot e^{\xi r}$ and $x^n = h \cdot e^{\mu r}$. The two equations yield $\xi = \mu$, since $r \neq 0$. This contradiction leads us to conclude that there exists $g \in \widehat{B'}$ such that $H_g \cap \Gamma = \emptyset$ and $g^\varphi \notin H_g$. □

Remark 3.5. The proof of Lemma 3.4 goes through if we restrict our choice of ξ's to any subset of E of J_p containing $2^{\aleph_0}$ algebraically independent elements.

PROPOSITION 3.6. *If B is free metabelian with rank at least three, λ a cardinal less than $2^{\aleph_0}$ and*

$$\{1 \neq \varphi_\alpha \in \mathrm{Out}(B) : \alpha < \lambda\},$$

then there exists a torsion free, metabelian group G into which B embeds such that each φ_α does not extend to an automorphism of G.

Proof. Suppose B is of rank less than $2^{\aleph_0}$. Apply Lemma 3.4 to $H_0 = B'$, $\Gamma_0 = \emptyset$ and $\varphi_0|_{B'}$ to obtain $g_0 \in \widehat{B'}$ such that $g_0^{\varphi_0} \notin \langle B', g_0^R\rangle_*$. Suppose $H_\alpha = \langle B', g_\beta^R : \beta < \alpha\rangle_* \subset \widehat{B'}$, $\Gamma_\alpha = \{g_\beta^{\varphi_\beta} : \beta < \alpha\}$ and $\varphi_\alpha|_{B'}$ such that $H_\alpha \cap \Gamma_\alpha = \emptyset$. By Lemma 3.4, there exists $g_\alpha \in \widehat{B'}$ such that $g_\alpha^{\varphi_\alpha} \notin \langle H_\alpha, g_\alpha^R\rangle_* =: H_{\alpha+1}$ and $H_{\alpha+1} \cap \Gamma_\alpha = \emptyset$. Let $H = \bigcup_{\alpha<\lambda} H_\alpha$ and $G = H \cdot B$. By Lemma 3.1, H is characteristic in G. Since H is torsion free, abelian and $G/H \cong B/B'$, G is torsion free metabelian. For each φ_α ($\alpha < \lambda$), there exists $g_\alpha \in H_{\alpha+1}$ such that $g_\alpha^{\varphi_\alpha} \notin H_{\alpha+1}$. Moreover, $\{g_\alpha^{\varphi_\alpha} : \alpha < \lambda\} \cap H = \emptyset$. Hence each φ_α ($\alpha < \lambda$) does not extend to G.

Assume B is of rank at least $2^{\aleph_0}$ with free generating set $\{x_\delta : \delta \in I\}$. For $x \in B$, let $x = x_{\alpha_1}^{e_1} \dots x_{\alpha_n}^{e_n}$ be the unique reduced word representing x, where $e_i \in \{\pm 1\}$ and x_{α_i}'s need not be distinct. Define for each $\delta \in I$,

$$\pi_\delta(x) = \begin{cases} x_\delta, & \text{if } \delta = \alpha_i \text{ for some } i \\ 1, & \text{otherwise.} \end{cases}$$

By hypothesis each $\varphi_\alpha \notin \mathrm{Inn}\, B$. For each α, choose $b_\alpha \in B'$ such that $b_\alpha^{\varphi_\alpha} \notin b_\alpha^R$. Each b_α is a product of commutators of a finite number of free generators x_δ. Call this finite set of generators S_α. Since each S_α is finite and $\lambda < 2^{\aleph_0}$, $|\bigcup_{\alpha<\lambda} S_\alpha| < 2^{\aleph_0}$. Let $A_0 = \langle S_\alpha : \alpha < \lambda\rangle$ and define inductively

$$A_n = \langle A_{n-1}, \pi_\delta(y^{\varphi_\alpha}) : \delta \in I, y \in A_{n-1}, \alpha < \lambda\rangle \text{ for } n \geq 1.$$

Set $A = \bigcup_{i\geq 0} A_i$ and $C = \langle x_\alpha : \alpha \in I,\ x_\alpha \notin A\rangle$. Then B is the free metabelian product of the free metabelian groups A and C, which we denote by $B = A * C$. Observe that A and C are free metabelian groups with rank less than $2^{\aleph_0}$ and at least $2^{\aleph_0}$, respectively. If $a \in A$, then $a \in A_i$ and $a^{\varphi_\alpha} \in A_{i+1}$ for each $\alpha < \lambda$, for some $i \geq 0$. Hence $\varphi_\alpha|_A \in \operatorname{Aut} A$ for $\alpha < \lambda$. Moreover $\varphi_\alpha|_A \notin \operatorname{Inn} A$ since each $b_\alpha \in A$ and $b_\alpha^{\varphi_\alpha} \notin b_\alpha^R$. The above argument for the case when the rank is less than $2^{\aleph_0}$ can now be applied to A to obtain a torsion free, abelian subgroup

$$H = \langle A', g_\alpha^a : \alpha < \lambda,\ a \in A\rangle_* \subset \widehat{A'}$$

such that $H \cap \{g_\alpha^{\varphi_\alpha} : \alpha < \lambda\} = \emptyset$. Hence each φ_α does not extend to $G = (H \cdot A) * C$. □

4. Complete, torsion free, metabelian groups

In the previous section we showed that for a given $1 \neq \varphi \in \operatorname{Out} B$, there exists a metabelian extension G of B to which φ does not extend. Here we show that for free metabelian B of rank less than $2^{\aleph_0}$, there exists a torsion free, metabelian extension G of B whose automorphisms are precisely its inner automorphisms. Recall the containment

$$B' < \oplus_{i\in I} I(\mathbb{Z}, \bar{B})t_i\,.$$

PROPOSITION 4.1. *Let B be a free metabelian group of rank at least two. Suppose H is a B-invariant subgroup of $\widehat{B'}$ and*

$$B' \leq H < \oplus_{i\in I} I(J_p, \bar{B})t_i\,.$$

If $\varphi \in \operatorname{Aut}(H\cdot B)$ such that $\varphi|_H = \operatorname{id}_H$, then $\varphi \in \operatorname{Inn}(H\cdot B)$.

Proof. Suppose $\varphi \in \operatorname{Aut}(H\cdot B)$ such that $\varphi|_H = id_H$. Let $b \in B$ and $a \in H$. Then $b^{-1}ab \in H$ and $b^{-1}ab = (b^{-1}ab)^\varphi = (b^{-1})^\varphi ab^\varphi$. So $b^\varphi b^{-1}$ commutes with every $a \in H$, and hence $b^\varphi b^{-1} \in H$. Thus φ induces the identity on the quotient $(H\cdot B)/H$.

Suppose $b, c \in B \setminus B'$, $b^\varphi = h \cdot b$ and $c^\varphi = k \cdot c$, where $h, k \in H$. Since $[b, c] \in B' \leq H$,

$$[b, c] = [b, c]^\varphi = h^{\bar{b}(\bar{c}-1)}k^{\bar{c}(1-\bar{b})}[b, c] \text{ implies } h^{\bar{b}(\bar{c}-1)} = k^{\bar{c}(\bar{b}-1)}\,.$$

In particular, if $\bar{b} = \bar{c}$, then $h = k$. By hypothesis, we identify H with a subgroup of $\oplus_{i\in I} I(J_p, \bar{B})t_i$ and let $h = \sum n_i t_i$ and $k = \sum m_i t_i$, for some $n_i, m_i \in I(J_p, \bar{B})$. Now

$$\sum n_i \bar{b}(\bar{c} - 1)t_i = \left(\sum n_i t_i\right)^{\bar{b}(\bar{c}-1)} = \left(\sum m_i t_i\right)^{\bar{c}(\bar{b}-1)} = \sum m_i \bar{c}(\bar{b} - 1)t_i$$

implies $n_i \bar{b}(\bar{c} - 1) = m_i \bar{c}(\bar{b} - 1)$ for each i. If n_i, m_i are nonzero, then $n_i = n_i'(\bar{b} - 1)$ and $m_i = m_i'(\bar{c} - 1)$, for some $n_i', m_i' \in J_p[\bar{B}]$, since $J_p[\bar{B}]$ is a unique factorization domain (see [12]) and elements of $I(J_p, \bar{B})$ are non-units. Hence, $m_i' = n_i'\bar{b}\bar{c}^{-1}$. Since $J_p[\bar{B}]$ has no zero divisors, $n_i = 0$ if and only if $m_i = 0$. Let $x = \sum n_i' t_i \in \oplus_{i\in I} I(J_p, \bar{B})t_i$. Then $h = \sum n_i'(\bar{b} - 1)t_i = (x)^{(\bar{b}-1)}$ and $k = \sum n_i'\bar{b}\bar{c}^{-1}(\bar{c} - 1)t_i = (x)^{\bar{b}\bar{c}^{-1}(\bar{c}-1)}$. So $b^\varphi = x^{(\bar{b}-1)}b = (b)^{x^{-b}}$ and $c^\varphi = x^{\bar{b}\bar{c}^{-1}(\bar{c}-1)}c = (c)^{x^{-b}}$. The elements $x^{(\bar{b}-1)}$ and $x^{(\bar{c}-1)}$ are in H, since H is B-invariant. If $\bar{b} \neq \bar{c}$, then $\bar{b} - 1$ and $\bar{c} - 1$ have 1 as greatest common divisor; i.e., there exist $r, s \in J_p[\bar{B}]$ such that $(\bar{b} - 1)r + (\bar{c} - 1)s = 1$. Hence $x = x^{(\bar{b}-1)r} \cdot x^{(\bar{c}-1)s} \in H$. Therefore φ is conjugation by x^{-b} for some $x \in H$ and $b \in B$. $\square$

COROLLARY 4.2. *Assume B and H have the same properties as in Proposition 4.1. If $\alpha, \beta \in \mathrm{Aut}(H \cdot B)$ such that $\alpha|_H = \beta|_H$ then $\alpha \in \beta \cdot \mathrm{Inn}(H \cdot B)$.*

Proof. Apply the proposition to $\beta^{-1}\alpha$. $\square$

LEMMA 4.3. *Suppose $G = H \cdot B$, where B is a free metabelian group of rank at least two, $B' \le H < \widehat{B}'$ and H is B-invariant. Then no automorphism of G induces inversion on H, i.e., if $\varphi \in \mathrm{Aut}(G)$ then $\varphi|_H \neq -1 \cdot \mathrm{id}_H$.*

Proof. Suppose $\varphi \in \mathrm{Aut}(G)$ and $\varphi|_H = -1 \cdot \mathrm{id}_H$. Let $a \in H$ and $b \in B$. Then $b^{-1}a^{-1}b = (b^{-1}ab)^\varphi = (b^{-1})^\varphi a^{-1} b^\varphi$ implies that $b^\varphi b^{-1}$ commutes with every element of H. Hence $b^\varphi b^{-1} \in H$. Since b is an arbitrary element of B, this means that φ induces the identity on G/H.

Let $b, c \in B$ and suppose $b^\varphi = h \cdot b$, $c^\varphi = k \cdot c$, for some $h, k \in H$. Then $b^{\varphi^2} = h^\varphi \cdot b^\varphi = h^{-1}h \cdot b = b$. Hence $\varphi^2 = \mathrm{id}_G$. Now $[b, c]^\varphi = [h \cdot b, k \cdot c] = [h, c]^b[b, c][b, k]^c$. Since $[h, c], [b, k] \in [G, H]$, it follows that $[b, c]^\varphi = [b, c] \bmod [G, H]$. Hence $[b, c]^2 \in [G, H]$. By commutator calculus, $[b, c]^2 = [b, c^2] \bmod [G, G']$. Since $G' \le H$, $[b, c]^2 = [b, c^2] \bmod [G, H]$. Thus $[b, c^2] \in [G, H]$. This yields a contradiction when b and c are chosen to be free generators of B. $\square$

We now have the necessary tools to prove our first main result.

THEOREM 4.4. *Let B be a free metabelian group of rank λ, where $3 \le \lambda < 2^{\aleph_0}$. Then there exists a torsion free, complete, metabelian group G embedding B, with G containing an abelian and characteristic subgroup A of cardinality $2^{\aleph_0}$ such that $G/A \cong B/B'$.*

Proof. Since $\aleph_0 \leq |B'| < 2^{\aleph_0}$ and $|\widehat{B'}| = 2^{\aleph_0}$, there are at most $2^{\aleph_0}$ monomorphisms from B' to $\widehat{B'}$. Let $\mathcal{I} = \{\varphi_\alpha \in \mathrm{Mon}(B', \widehat{B'}) : \varphi_\alpha \notin R,\ \alpha < 2^{\aleph_0}\}$, where $\mathrm{Mon}(B', \widehat{B'})$ denotes the monomorphisms $B' \to \widehat{B'}$. Note that $\mathcal{I}$ contains the automorphisms of B' induced by $\mathrm{Out}\, B$.

Assume inductively that for some $\alpha < 2^{\aleph_0}$ we have a continuous ascending chain of pure R-submodules H_β $(\beta < \alpha)$ of $\widehat{B'}$ and elements $g_\gamma \in \widehat{B'}$ $(\gamma + 1 < \alpha)$ with the property that for each $\beta < \alpha$,

$$H_\beta = \langle B', g_\gamma^R : \gamma < \beta \rangle_* \qquad I(\beta)$$

$$H_\beta \cap \{g_\gamma^{\varphi_\gamma} : \gamma < \beta\} = \emptyset \qquad II(\beta)$$

The assumption is vacuously true for $\alpha = 0$. If α is a limit ordinal, define $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$. So $I(\alpha)$ and $II(\alpha)$ clearly hold. If α is a successor ordinal, say $\alpha = \beta + 1$, apply Lemma 3.4 to $H = H_\beta$, $\varphi = \varphi_\beta$ and $\Gamma = \{g_\gamma^{\varphi_\gamma} : \gamma < \beta\}$. If we take g_β to be the element g provided by the lemma, then $I(\alpha)$ and $II(\alpha)$ are satisfied. Define

$$H_{2^{\aleph_0}} = \langle B', g_\alpha^R : \alpha < 2^{\aleph_0} \rangle_* \text{ and } G = H_{2^{\aleph_0}}.B\,.$$

G is clearly torsion free, metabelian with trivial center. By Lemma 3.1, $H_{2^{\aleph_0}}$ is characteristic. Moreover, it is abelian and $G/H_{2^{\aleph_0}} \cong B/B'$. It suffices to show $\mathrm{Aut}\, G = \mathrm{Inn}\, G$. Let $\varphi \in \mathrm{Aut}\, G$. Then $(g_\alpha)^\varphi \in H_{2^{\aleph_0}}$ for all $\alpha < 2^{\aleph_0}$. By the choice of the g_α's, $\varphi \notin \mathcal{I}$ and $\varphi|_{B'}$ is an element of R. By continuity, $\varphi|_{H_{2^{\aleph_0}}}$ is an element of R. Since the units of R consist of the trivial ones, i.e., $R^* = \{\pm 1\} \times \bar{B}$, then, by Lemma 4.3, $\varphi|_H$ is an element of $\bar{B}$. Recall from Lemma 3.4 that each element $g_\alpha = e^{\xi_\alpha}$ for some $e \in B'$ and $\xi_\alpha \in J_p$. Since B' is a subgroup of $\oplus_{i\in I} I(\mathbb{Z}, \bar{B})t_i$, each $g_\alpha \in \oplus_{i\in I} I(J_p, \bar{B})t_i$ and $H_{2^{\aleph_0}}$ is a subgroup of $\oplus_{i\in I} I(J_p, \bar{B})t_i$. Therefore, by Corollary 4.2, $\varphi \in \mathrm{Inn}(G)$. □

Remark 4.5. Let E be a set of algebraically independent elements of J_p with cardinality $2^{\aleph_0}$. We can find a family $\{E_\alpha \subset E : \alpha < 2^{2^{\aleph_0}}\}$ of almost disjoint sets E_α with $|E_\alpha| = 2^{\aleph_0}$ and $|E_\alpha \cap E_\beta| < 2^{\aleph_0}$ for all $\alpha, \neq \beta < 2^{2^{\aleph_0}}$. By Lemma 3.1 and Remark 3.5, we can construct a rigid system of $2^{2^{\aleph_0}}$ groups G_α satisfying Theorem 4.4; i.e., $\mathrm{Hom}(G_\alpha, G_\beta) = 0$ for $\alpha \neq \beta$. In particular, these groups are pairwise nonisomorphic.

5. Realizing abelian and UP groups as Out(G)

The results in this section are motivated by a naturally occurring and simple construction, namely the quotient of the automorphism group of a direct sum of two abelian groups modulo its center, which we illustrate below. This translates the problem to a question in ring and module theory. We generalize this example to obtain a realization theorem using a result (see [4] and [6]) concerning the endomorphism rings of *cotorsion free abelian groups*, i.e., a group containing no copies of $\mathbb{Q}$, $\mathbb{Z}/p\mathbb{Z}$ or J_p for each prime p.

Example 5.1. Let $S = \{\frac{m}{2^n} : m, n \in \mathbb{Z}\}$ be a subring of $\mathbb{Q}$ and let M be an abelian group such that $M \not\cong S$ and $\operatorname{End} M \cong S$. Define $N = S \oplus M$. Then $G = \operatorname{Aut} N/Z(\operatorname{Aut} N)$ is metabelian and complete.

Proof. Let M be an abelian group such that $M \not\cong S$ and $\operatorname{End} M \cong S$. Then M is an S-module and $\operatorname{Hom}(M, S) = \operatorname{Hom}_S(M, S) = 0$. Since $N = S \oplus M$,

$$\operatorname{End} N \cong \begin{pmatrix} S & 0 \\ M & S \end{pmatrix} \text{ and } \operatorname{Aut} N \cong \begin{pmatrix} S^* & 0 \\ M & S^* \end{pmatrix}$$

where S^* is the group of multiplicative units of S. Observe that $G = (\operatorname{Aut} N)/Z(\operatorname{Aut} N)$ is isomorphic to the semi-direct product $M \rtimes S^*$. Identifying G with $M \rtimes S^*$, we see that $G' = M$. Clearly G has trivial center. It suffices to show that $\operatorname{Aut} G = \operatorname{Inn} G$.

If x, $n \in M$ and r, $s \in S^*$, the action of $xr \in G$ is as follows:

$$n^{xr} = n^r \text{ and } s^{xr} = (x^{-1}x^{s^{-1}})^r s\,.$$

Suppose $\varphi \in \operatorname{Aut} G$. Since M is characteristic in G and $\operatorname{End} M = S$, $\varphi|_M = t \in S^*$. Let $s \in S^*$ and $s^\varphi = mr$ for some $m \in M$, $r \in S^*$. If $n \in M$, then

$$mn^{tr^{-1}}r = mr \cdot n^t = (sn)^\varphi = (n^{s^{-1}}s)^\varphi = n^{s^{-1}t}mr = mn^{s^{-1}t}r$$

implies $n^{tr^{-1}} = n^{s^{-1}t}$. Hence $r = s$ and $s^\varphi = ms$. Note that $S^* = \{\pm 2^k : k \in \mathbb{Z}\}$. In particular, $2^\varphi = m \cdot 2$ and $(-1)^\varphi = x \cdot (-1)$ for some m, $x \in M$. Since $(2 \cdot (-1))^\varphi = ((-1) \cdot 2)^\varphi$, $m \cdot x^{2^{-1}} = xm^{-1}$ and so $x = m^4$. We now have

$$(-1)^\varphi = m^4.(-1)\ ,\ n^\varphi = n^t\ (n \in M)\ ,\ 2^\varphi = m \cdot 2\,.$$

An easy calculation verifies that φ is a conjugation by the element $m^{-2t^{-1}} \cdot t \in M \rtimes S^*$. □

We remark that there exists a proper class of non-isomorphic, torsion free, abelian groups M such that $\operatorname{End} M \cong S$ (see [4]). An abelian group A is said to be *p-divisible* (*p-torsion free*) if $A^p = A$ ($a^p = 1$ iff $a = 1$). If H is a subgroup of an arbitrary group G, we denote the *normalizer of H in G* by $N_G(H)$.

PROPOSITION 5.2. *Let M be a 2-divisible, 2-torsion free, abelian group and P be an abelian subgroup of* $\operatorname{Aut} M$ *with* $-1 \cdot \mathrm{id}_M$ *or* $2 \cdot \mathrm{id}_M$ *in P. Define $G = M \rtimes P$. Then the center of G is trivial and* $\operatorname{Out} G \cong N_{\operatorname{Aut} M}(P)/P$.

Proof. Since P is abelian, $G' \subseteq M$. Now

$$\langle a^y a^{-1} = [y, a^{-1}] : a \in M,\ y \in P\rangle \subseteq G'\,.$$

If either $-1 \cdot \mathrm{id}_M$ or $2 \cdot \mathrm{id}_M$ is in P, then $G' = M$ since M is 2-divisible and 2-torsion free. Hence M is characteristic in G. Since the action of P on M is faithful, the center of G is trivial.

Observe that if $\varphi \in \mathrm{Inn}\, G$, then $\varphi|_M \in P$. Moreover, if $\varphi \in \mathrm{Aut}\, G$ then $\varphi|_M \in N_{\mathrm{Aut}\, M}(P)$. If $m \in M$, $p \in P$ and $\varphi|_M = f \in \mathrm{Aut}\, M$,

$$m^f p^\varphi = (mp)^\varphi = (pm^p)^\varphi = p^\varphi m^{pf}.$$

This implies $m^f = m^{pf(p^{-1})^\varphi}$ for all $m \in M$. If $(p^{-1})^\varphi = m_0 p_0$ for some $m_0 \in M$, $p_0 \in P$, then $f = pfp_0$. Hence $f^{-1}pf = p_0^{-1}$ and $f \in N_{\mathrm{Aut}\, M}(P)$. Conversely, if $f \in N_{\mathrm{Aut}\, M}(P)$, then there exists $\varphi \in \mathrm{Aut}\, G$ such that $\varphi|_M = f$ and $p^\varphi = p^f$ for all $p \in P$.

We now have the following homomorphism:

$$\begin{array}{ccccc} \mathrm{Out}\, G & \to & N_{\mathrm{Aut}\, M}(P)/P & \to & 1 \\ \varphi & \mapsto & \varphi|_M & & . \end{array}$$

It suffices to show that the kernel is trivial. Suppose $\varphi \in \mathrm{Out}\, G$ such that $\varphi|_M = \mathrm{id}_M$. Substituting $f = \mathrm{id}_M$ in the preceding calculation, we get $m = m^{p(p^{-1})^\varphi}$ for all $m \in M$ and $p \in P$. This means that $p^\varphi \in p.M$; i.e., φ induces the identity on G/M. Let $p, q \in P$ with $p^\varphi = pm$ and $q^\varphi = qn$ for some $m, n \in M$. Since $(pq)^\varphi = (qp)^\varphi$, $m^q n = n^p m$. If $p = -1 \cdot \mathrm{id}_M$, then $n^2 = mm^{-q}$ and $q^\varphi = q(mm^{-q})^{2^{-1}} = q^{m^{2^{-1}}}$ for all $q \in P$. If $p = 2 \cdot \mathrm{id}_M$, then $n = m^q m^{-1}$ and $q^\varphi = q^{m^{-1}}$ for all $q \in P$. In either case, we get $\varphi \in \mathrm{Inn}\, G$. Hence the above map is an isomorphism. $\square$

At this point it is clear that the problem reduces to determining the units in End M. A ring R is said to be a *cotorsion free ring* if $(R, +)$ is a cotorsion free group. We recall the following result on endomorphism rings.

THEOREM 5.3 ([4], [6]). *Suppose R is a cotorsion free ring, $\mu^{\aleph_0} = \mu > |R|$. Then there exists a cotorsion free abelian group A of cardinality μ such that* End $A \cong R$.

In particular, if F is any group, there exists a cotorsion free, abelian group M such that End $M \cong S[F]$, where $S = \{\frac{m}{2^n} : m, n \in \mathbb{Z}\}$. The group M is necessarily 2-divisible and 2-torsion free. The following theorems now follow from Proposition 5.2 and Theorem 5.3.

THEOREM 5.4. *Every abelian group can be realized as the outer automorphism group of a metabelian group with trivial center.*

Proof. Let A be any abelian group. Then there exists a free abelian group F with $F/U = A$ for some $U \le F$. By Theorem 5.3, there exists a cotorsion free, abelian group M such that End $M \cong S[F]$. Now Aut $M = (S^* \times F)$. Define

$G = M \rtimes (S^* \times U)$. Clearly G satisfies the hypotheses of Proposition 5.2. So it follows that

$$\operatorname{Out} G \cong N_{\operatorname{Aut} M}(S^* \times U)/(S^* \times U) = (S^* \times F)/(S^* \times U) \cong F/U = A. \quad \square$$

UP groups (see [13]) are necessarily torsion free. If K is a UP group, then the units of $S[K]$ are the trivial ones, that is $S^* \times K$.

THEOREM 5.5. *Every UP group is the outer automorphism group of a metabelian group with trivial center.*

Proof. Let K be a UP group. By Theorem 5.3, there exists a cotorsion free, abelian group M such that $\operatorname{End} M = S[K]$. Define $G = M \rtimes S^*$. By Proposition 5.2,

$$\operatorname{Out} G \cong (S^* \times K)/S^* \cong K. \quad \square$$

Since $S^* \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$, the metabelian groups G in Theorems 5.4 and 5.5 realizing a given abelian or UP group have torsion part $t(G) \cong \mathbb{Z}/2\mathbb{Z}$. If we were to insist that G be torsion free, then the resulting outer automorphism group $\operatorname{Out} G$ is isomorphic to the direct sum of $\mathbb{Z}/2\mathbb{Z}$ and the prescribed abelian or UP group.

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