ILLINOIS JOURNAL OF MATHEMATICS
Volume 42, Number 2, Summer 1998

ON A FAMILY OF ORTHOGONAL POLYNOMIALS RELATED TO ELLIPTIC FUNCTIONS

MOURAD E. H. ISMAIL AND GALLIANO VALENT

1. Introduction

Chen and Ismail [7] studied orthogonal polynomials $\mathcal{F}_n(x)$ which satisfy the recurrence relation

$$x\mathcal{F}_n(x) = \mathcal{F}_{n+1}(x) + 4n^2(4n^2 - 1)\mathcal{F}_{n-1}(x), \quad n > 0, \tag{1.1}$$

and the initial data

$$\mathcal{F}_0(x) = 1, \quad \mathcal{F}_1(x) = x. \tag{1.2}$$

The polynomials $\mathcal{F}_n(x)$ are symmetric in the sense that $\mathcal{F}_n(-x) = (-1)^n\mathcal{F}_n(x)$. Berg and Ismail [4] derived the generating function

$$\sum_{n=0}^{\infty} \frac{\mathcal{F}_n(x)}{(2n)!}\, t^n = (1+t^2)^{-1/2} \cosh(\sqrt{xi}\, \Theta(\sqrt{t/i})), \tag{1.3}$$

where

$$\Theta(w) = \int_0^w \frac{du}{\sqrt{1-u^4}} = w\, {}_2F_1(1/2, 1/4; 5/4; w^4). \tag{1.4}$$

In this work we provide a nonsymmetric extension of the $\mathcal{F}_n$'s. Our extension is the one parameter system of polynomials $\{G_n(x;a)\}$ generated by

$$\begin{aligned} -xG_n(x;a) &= 2(n+1)(2n+1)G_{n+1}(x;a) + 2n(2n+1)G_{n-1}(x;a) \\ &\quad -2a(2n+1)^2G_n(x;a), \quad n > 0, \end{aligned} \tag{1.5}$$

with

$$G_0(x;a) := 1, \quad G_1(x;a) = a - x/2. \tag{1.6}$$

Received April 25, 1997.
1991 Mathematics Subject Classification. Primary 30E05; Secondary 33E05.
Research partly supported by an EPSRC grant, a NSF grant and a visiting fellowship from the Leverhulme Foundation.

In §2 we establish the generating function

$$\sum_{n=0}^{\infty} G_n(x;a)\, t^n = (1-2at+t^2)^{-1/2}\cos(\sqrt{x}\, g(t)), \tag{1.7}$$

where

$$g(t) = \frac{1}{2}\int_0^t u^{-1/2}(1-2au+u^2)^{-1/2}\, du, \tag{1.8}$$

as well as a generating function for the numerator polynomials $G_n^*(x;a)$. We then use (1.7) to develop the asymptotics of $\{G_n(x;a)\}$ for large n and fixed x in the complex plane.

It is easy to see from (1.5) and (1.6) that

$$G_n(x;a) = (-1)^n G_n(-x;-a); \tag{1.9}$$

hence there is no loss of generality in treating only the case

$$0 \le a < \infty. \tag{1.10}$$

The monic polynomials Q_n are given by

$$Q_n(x) = (-1)^n (2n)!\, G_n(x;a), \tag{1.11}$$

and satisfy

$$xQ_n(x) = Q_{n+1}(x) + 2a(2n+1)^2\, Q_n(x) + 4n^2(4n^2-1)Q_{n-1}(x), \quad n > 0. \tag{1.12}$$

When $a = 1$ and x is replaced by $x-4$ the G_n's are special continuous dual Hahn polynomials, [2], which come from a birth and death process [11] with

$$\lambda_n = 2(n+1)(2n+1), \quad \mu_n = 2n(2n+1). \tag{1.13}$$

In fact when $a > 1$ the G_n's come from a birth and death process with killing, or absorption, where the birth and death rates are as in (1.13) and the absorption rate is

$$\gamma_n = 2(a-1)(2n+1)^2. \tag{1.14}$$

General birth and death processes with killing were introduced by Karlin and Tavaré in [14] and [15]; see [10] for a survey of the connection with orthogonal polynomials.

From (1.5), (1.6) and the general theory of orthogonal polynomials it follows that there is a probability measure μ such that

$$\int_{-\infty}^{\infty} G_m(x;a)G_n(x;a)\, d\mu(x) = (2n+1)\,\delta_{m,n}. \tag{1.15}$$

In §3 we shall prove that the Hamburger moment problem associated with the G_n's is determinate; i.e., μ is unique for $a \in (1, \infty)$ (recall that $a \geq 0$ is assumed here). Using the generating functions given in §2 we work out the asymptotics of the polynomials $G_n(x; a)$ and $G_n^*(x; a)$. This gives the continued fraction and the (discrete) orthogonality measure μ.

In §4 we study the indeterminate case $a \in [0, 1)$. We find two of the four entire functions of the corresponding Nevanlinna matrix. This allows us to get the equations giving the spectrum. In two cases the spectrum and the explicit orthogonality measures are derived.

We also show that

$$W(x, a) := \frac{1/2}{\cos(\sqrt{x}K/2) + \cosh(\sqrt{x}K'/2)}, \tag{1.16}$$

with

$$\begin{aligned} k &= K\left(\frac{1+a}{2}\right), \\ K' &= K\left(\frac{1-a}{2}\right), \\ K(k^2) &= \frac{\pi}{2} F(1/2, 1/2; 1; k^2), \end{aligned} \tag{1.17}$$

is a weight function normalized by $\int w(x)dx = 1$. This weight function is only one of an infinite family of weight functions derived in §4. Observe that when $a = 0$, (1.16) reduces to the weight function derived in [7]:

$$w(x) = \frac{1/2}{\cos(K_0\sqrt{x}) + \cosh(K_0\sqrt{x})}, \qquad K_0 := K(1/2) = \frac{\Gamma^2(1/4)}{4\sqrt{\pi}}. \tag{1.18}$$

One reason for our interest in the weight function (1.16) is that for large $|x|$ it behaves as a Freud weight $\exp(-|x|^\alpha)$ for $\alpha = 1/2$. One can find the moments of the Freud weights but there are no explicit formulas known for the recursion coefficients or for generating functions of the polynomials, except in the case of Hermite polynomials when $\alpha = 2$. For a survey of the literature on the Freud weights, see [16], [17]. Our explicit results can then be viewed as results on modified Freud polynomials for $\alpha = 1/2$.

The generating functions (1.7) and (1.18), when combined with the orthogonality relation (1.15), lead to the evaluation of certain definite integrals. Indeed if we multiply (1.15) by $t^m s^n$ and add for all $m \geq 0$, $n \geq 0$ and then insert (1.7) in the resulting integral we get

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\cos(\sqrt{x}g(s))\cos(\sqrt{x}g(t))\,dx}{\cos(\sqrt{x}g(e^{-i\phi}))\cos(\sqrt{x}g(e^{i\phi}))} \\ &\qquad = (1 - 2as + s^2)^{-1/2}(1 - 2at + t^2)^{-1/2}\,\frac{1 + st}{(1 - st)^2}, \end{aligned} \tag{1.19}$$

for $|s| < 1$ and $|t| < 1$.

From (1.5) and (1.6) it is clear that for every fixed x, $G_n(x; a)$ is also a polynomial in a of degree n. In fact as functions of a, $\{G_n(x; a)\}$ is a system of orthogonal polynomials which contains the Legendre polynomials as the special case $x = 0$. These polynomials are orthogonal with respect to a measure whose absolutely continuous component is supported on $[-1, 1]$. Depending on the value of x, the measure may have a discrete part.

In §5 we study the spectral properties of $G_n(x; a)$ as polynomials in a and for fixed x. We evaluate the corresponding J-fraction and find their measure of orthogonality.

The continued J-fraction associated with the three term recurrence relation (1.12) for $a > 1$ was studied by L. J. Rogers as was noted by H. S. Wall [22]. Rogers noted the connection with elliptic functions. He proved

$$\int_0^\infty \frac{\operatorname{sn}(u, k)\operatorname{cn}(u, k)}{\operatorname{dn}(u, k)} e^{-zu}\, du \tag{1.20}$$

$$= \frac{1}{z^2 + b_1 -} \; \frac{a_1}{z^2 + b_2 -} \cdots \frac{a_n}{z^2 + b_{n+1} -} \cdots$$

where

$$a_n = (2n-1)(2n)^2(2n+1), \quad b_n = 2(2n-1)^2(2-k^2). \tag{1.21}$$

In §3 we shall provide an alternate representation of the continued fraction in (1.20), see Theorem 3.3. Our results on the case $-1 < a < 1$ correspond to having the modulus k of the elliptic function on the unit circle, $k \neq 1$. To the best of our knowledge this is the first analysis of continued fractions associated with Jacobian elliptic functions with modulus on the unit circle. Recently Steve Milne [18] has continued Rogers's analysis and applied it to sums of squares.

Since the recurrence relation for the G_n's is given explicitly one can use the Hellmann-Feynman theorem to study the monotonicity of their zeros as functions of a or use chain sequences to obtain inequalities satisfied by their extreme zeros, [12]. One can also use Markov's theorem because we know a weight function explicitly.

2. Generating functions

We first briefly indicate why (1.5), (1.6) are equivalent to the generating function (1.7). Let us observe that the right-hand side of (1.7) is analytic in t for $|t| < 1$ if $a \in [0, 1)$ and $|t| < a - \sqrt{a^2 - 1}$ if $a \geq 1$. Hence we can expand $G(x, t)$ in a power series of t. Let

$$G(x; t) = (1 - 2at + t^2)^{-1/2} \cos(\sqrt{x} g(t)) = \sum_{n=0}^{\infty} a_n(x) t^n. \tag{2.1}$$

Multiply both sides of (2.1) by $(1 - 2at + t^2)^{-1/2}$, differentiate with respect to t then multiply by $\sqrt{t(1 - 2at + t^2)}$ and differentiate once more. This produces the differential equation

$$4\sqrt{t}\frac{\partial}{\partial t}\left(\sqrt{t(1-2at+t^2)}\frac{\partial}{\partial t}\left(\sqrt{(1-2at+t^2)}G(x,t)\right)\right) = -xG(x,t),$$

or in expanded form

$$\begin{aligned} 4t(1-2at+t^2)\frac{\partial^2 G(x,t)}{\partial t^2} &+ 2(1-8at+7t^2)\frac{\partial G(x,t)}{\partial t} \\ &+ (x-2a+6t)G(x,t) = 0. \end{aligned} \tag{2.2}$$

Upon equating coefficients of different powers of t we see that $a_n(x)$ satisfies the three term recurrence relation (1.5). From (2.1) it is easy to verify that $a_0(x) = G_0(x; a)$ and $a_1(x) = G_1(x; a)$. Thus $G_n(x; a) = a_n(x)$ for all n and we have established that (1.7) implies (1.5) and (1.6). To show the converse one can simply reverse the steps in this argument.

Observe that another generating function for the G_n's follows by multiplying (2.1) by $t^{-1/2}$ and then integrating over t. The result is

$$\sum_{n=0}^{\infty} G_n(x; a)\frac{t^n}{2n+1} = \frac{\sin(\sqrt{x}g(t))}{\sqrt{xt}}. \tag{2.3}$$

Our next task is to find a generating function for the numerators $\{G_n^*(x; a)\}$. To do so we multiply (1.5) by t^n and then add for $n > 0$. After using the initial conditions

$$G_0^*(x; a) = 0, \quad G_1^*(x; a) = -1/2, \tag{2.4}$$

we see that the generating function

$$G^*(x,t) := \sum_{n=0}^{\infty} G_n^*(x; a)t^n \tag{2.5}$$

satisfies the differential equation

$$\begin{aligned} 4t(1-2at+t^2)\frac{\partial^2 G^*(x,t)}{\partial t^2} &+ 2(1-8at+7t^2)\frac{\partial G^*(x,t)}{\partial t} \\ &+ (x-2a+6t)G^*(x,t) = -1. \end{aligned}$$

Since

$$G(x,t) := (1-2at+t^2)^{-1/2}\cos(\sqrt{x}g(t)) \tag{2.6}$$

solves the corresponding homogeneous equation we set

$$G^*(x,t) = G(x,t)H(x,t). \tag{2.7}$$

Now H satisfies

$$\frac{\partial^2 H(x,t)}{\partial t^2} + \frac{\partial H(x,t)}{\partial t}\left[\frac{2}{G(x,t)}\,\frac{\partial G(x,t)}{\partial t} + \frac{1}{2t} + \frac{3(t-a)}{1-2at+t^2}\right]$$
$$= -\frac{\sec(\sqrt{x}\,g(t))}{4t\sqrt{1-2at+t^2}}.$$

Multiply the above equation by the integrating factor $G^2(x,t)\,t^{1/2}\,(1-2at+t^2)^{3/2}$ and then integrate over t to get

$$t^{1/2}\,(1-2at+t^2)^{3/2}\,G^2(x,t)\,\frac{\partial H(x,t)}{\partial t} = -\int_0^t \frac{\cos(\sqrt{x}\,g(u))}{4\sqrt{u}}\,du.$$

Then

$$H(x,t) = -\int_0^t \frac{\sec^2(\sqrt{x}\,g(v))}{4\sqrt{v(1-2av+v^2)}}\left(\int_0^v \frac{\cos(\sqrt{x}\,g(u))}{4\sqrt{u}}\,du\right)dv, \tag{2.8}$$

since $H(x,t)/t \to -1/2$ as $t \to 0$. In view of (1.2) we can integrate by parts in the v integral and obtain

$$H(x,t) = -\frac{\tan(\sqrt{x}\,g(t))}{2\sqrt{x}}\int_0^t \frac{\cos(\sqrt{x}\,g(u))}{\sqrt{u}}\,du + \int_0^t \frac{\sin(\sqrt{x}\,g(u))}{2\sqrt{xu}}\,du.$$

This and (2.8) establish the desired generating function

$$G^*(x,t) = -(1-2at+t^2)^{-1/2}\int_0^t \frac{\sin(\sqrt{x}(g(t)-g(u)))}{2\sqrt{xu}}\,du. \tag{2.9}$$

For $x = 0$ the polynomials $G_n(0,a)$ and $G_n^*(0,a)$ are multiples of the Legendre polynomials $P_n(a)$ and their numerators $P_n^{(1)}(a)$, respectively. Indeed

$$G_n(0,a) = P_n(a), \qquad G_n^*(0,a) = -\frac{1}{2}P_{n-1}^{(1)}(a). \tag{2.10}$$

These relations can be checked at the level of the generating functions since from (2.6) we have

$$G(0,t) = -(1-2at+t^2)^{-1/2}$$

and from (2.9) it follows that

$$G^*(0,t) = -(1-2at+t^2)^{-1/2}\int_0^t (g(t)-g(u))\frac{du}{2\sqrt{u}}.$$

Using the definition in (1.8) we get a double integral representation for $G_n^*(0,a)$, which after we reverse the order of the integration simplifies to

$$G^*(0,t) = -(1-2at+t^2)^{-1/2}\int_0^t \frac{du}{2\sqrt{1-2au+u^2}}, \tag{2.11}$$

in agreement with the result of Barrucand and Dickinson quoted on page 202 in [8].

3. The determinate case

Since the cases $a = \pm 1$ are essentially the continuous dual Hahn polynomials, their asymptotics follow from [13] or from the more general results in [11]. It is known that the moment problem is determinate in this case [11].

We therefore consider $a \in (1, \infty)$, which, in view of (1.10), also covers the remaining case $a \in (-\infty, -1)$. We shall use the notation

$$a = \cosh \phi, \qquad \phi > 0. \tag{3.1}$$

Let us begin with:

THEOREM 3.1. *If $a \in (1, \infty)$ then*

$$\begin{aligned} G_n(x; a) &= [2\pi n \sinh \phi]^{-1/2} e^{(n+1/2)\phi} \cos[\sqrt{x}\, g(e^{-\phi})]\, [1 + o(1)], \\ a &= \cosh \phi, \end{aligned} \tag{3.2}$$

as $n \to \infty$ for all x, real or complex.

Proof. Since $a > 1$, the generating function (1.7) has only one singularity, $t = e^{-\phi}$ with smallest absolute value. Hence Darboux's method [20] gives

$$G_n(x; a) = (1 - e^{-2\phi})^{-1/2} \frac{(1/2)_n}{n!} e^{n\phi} \cos(\sqrt{x} g(e^{-\phi})),$$

which simplifies to (3.2) due to the relation

$$\frac{\Gamma(a+n)}{\Gamma(b+n)} = n^{a-b}[1 + o(1)], \quad n \to \infty. \tag{3.3}$$

Using the first complete elliptic integral K one has

$$g(e^{-\phi}) = \frac{\pi}{2} e^{-\phi/2} F(1/2, 1/2; 1; e^{-2\phi}) = e^{-\phi/2} K(e^{-2\phi}). \tag{3.4}$$

THEOREM 3.2. *The Hamburger moment problem associated with the G_n's is determinate for $a \in (1, \infty)$ or $a \in (-\infty, -1)$.*

Proof. In view of (1.16) the polynomials $\{\hat{G}_n(x; a)\}$, defined by

$$\hat{G}_n(x; a) := G_n(x; a)/\sqrt{2n+1}, \tag{3.5}$$

are orthonormal. Theorem 2.9 in [19] asserts that the divergence of $\sum_{n=0}^{\infty} |\hat{G}_n(x; a)|^2$ for one complex x is sufficient for the determinacy of the Hamburger moment problem. If $a > 1$ then from (3.4) it is clear that $g(e^{-\phi})$ is positive when $\phi > 0$; hence there is a complex x for which $\cos(\sqrt{x}\, g(e^{-\phi})) \neq 0$ and the series $\sum_{n=0}^{\infty} |\hat{G}_n(x; a)|^2$ will indeed diverge for this complex x. □

THEOREM 3.3. *For $a > 1$ the continued J-fraction associated with the G_n's converges to $J_1(x; a)$, where*

$$J_1(x; a) := -\int_0^{e^{-\phi/2}} \frac{\sin(\sqrt{x}(g(e^{-\phi}) - g(u^2)))}{\sqrt{x}\ \cos(\sqrt{x}g(e^{-\phi}))}\, du, \qquad a = \cosh\phi. \tag{3.6}$$

Proof. The t-singularity of $G^*(x, t)$ of smallest absolute value is $t = e^{-\phi}$. Thus (2.9) and Darboux's method give

$$\lim_{n\to\infty} \frac{G_n^*(x; a)}{G_n(x; a)} = -\int_0^{e^{-\phi}} \frac{\sin(\sqrt{x}(g(e^{-\phi}) - g(u)))}{2\sqrt{xu}\cos(\sqrt{x}(g(e^{-\phi})))}\, du. \tag{3.7}$$

The continued J-fraction converges to the left-hand side of (3.7), hence (3.6) follows and the proof is complete. □

It is important to note that a formula equivalent to (3.6), due to Rogers, is stated as (94.21) in Wall [22].

Let μ be the measure with respect to which the G_n's are orthogonal. Then the Stieltjes transform of μ is $J_1(x; a)$ [19]. Thus

$$\int_{-\infty}^{\infty} \frac{d\mu(u)}{x - u} = -\int_0^{e^{-\phi/2}} \frac{\sin(\sqrt{x}(g(e^{-\phi}) - g(u^2)))}{\sqrt{x}\cos(\sqrt{x}(g(e^{-\phi})))}\, du, \quad x \notin [0, \infty). \tag{3.8}$$

The right-hand side of (3.6) is clearly a meromorphic function and its only singularities are isolated simple poles. Thus μ is purely discrete and has point masses at the poles of the right-hand side of (3.6), which are given by

$$x_n := \frac{(n + 1/2)^2\pi^2}{g^2(e^{-\phi})}, \quad n = 0, 1, \ldots. \tag{3.9}$$

Furthermore the mass at x_n is the residue of the right-hand side of (3.7) at $x = x_n$. Thus

$$\mu(x_n) = \frac{2}{g(e^{-\phi})} \int_0^{e^{-\phi/2}} \cos((n + 1/2)\pi g(u^2)/g(e^{-\phi}))\, du. \tag{3.10}$$

To evaluate $\mu(x_n)$ we need the theory of elliptic integrals. Set

$$g(e^{-\phi}) := \sqrt{k}\, K(k^2), \quad k := a - \sqrt{a^2 - 1} = e^{-\phi} < 1. \tag{3.11}$$

In this form the mass points in (3.9) are

$$x_n = \frac{(n + 1/2)^2\, \pi^2}{kK^2}, \quad n = 0, 1, \ldots. \tag{3.12}$$

Furthermore (3.10) becomes

$$\mu(x_n) = \frac{2}{\sqrt{k}\,K}\int_0^{\sqrt{k}} \cos\left[\left(n+\frac{1}{2}\right)\frac{\pi}{\sqrt{k}\,K}g(u^2)\right] du. \tag{3.13}$$

We now introduce the new variable

$$v := \frac{g(u^2)}{\sqrt{k}} = \frac{1}{\sqrt{k}}\int_0^{u^2}\frac{dt}{2\sqrt{t(1-2at+t^2)}} = \frac{1}{\sqrt{k}}\int_0^{u}\frac{d\tau}{\sqrt{1-2a\tau^2+\tau^4}}. \tag{3.14}$$

The Jacobi inversion theorem [23] gives

$$u/\sqrt{k} = \operatorname{sn}(v, k^2), \qquad du = \sqrt{k}\,\operatorname{cn} v\ \operatorname{dn} v\ dv.$$

Therefore the masses

$$\mu(x_n) = \frac{2}{K}\int_0^{K}\cos\left[\left(n+\frac{1}{2}\right)\frac{\pi v}{K}\right]\operatorname{cn} v\ \operatorname{dn} v\ dv \tag{3.15}$$

are the Fourier coefficients of $\operatorname{cn} v\ \operatorname{dn} v$. Starting from [23, §22.6],

$$\operatorname{sn} v = \frac{2\pi}{kK}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1-q^{2n+1}}\sin\left((2n+1)\frac{\pi v}{2K}\right); \tag{3.16}$$

upon differentiation we have the Fourier series

$$\operatorname{cn} v\ \operatorname{dn} v = \frac{\pi^2}{kK^2}\sum_{n=0}^{\infty}\frac{(2n+1)\,q^{n+1/2}}{1-q^{2n+1}}\cos\left((n+1/2)\frac{\pi v}{K}\right). \tag{3.17}$$

Thus (3.14) and (3.16) imply

$$\mu(x_n) = \frac{\pi^2}{kK^2}\,\frac{(2n+1)q^{n+1/2}}{1-q^{2n+1}}, \qquad n = 0, 1, \ldots. \tag{3.18}$$

Dividing (3.16) by $v \neq 0$ and then letting $v \to 0$ show that the total mass is indeed equal to 1. Thus we have established the following theorem.

THEOREM 3.4. *The orthogonality relation of the G_n's is*

$$\frac{\pi^2}{kK^2}\sum_{j=0}^{\infty}\frac{(2j+1)q^{j+1/2}}{1-q^{2j+1}}G_m(x_j;a)\,G_n(x_j;a) = (2n+1)\,\delta_{m,n}, \tag{3.19}$$

with x_j given by (3.9) *or* (3.12).

It is worth noting that the orthogonality relation (3.19) is equivalent to

$$4\sum_{j=0}^{\infty}\frac{q^{j+1/2}}{(1-q^{2j+1})(2j+1)}\sin(\sqrt{x_j}g(t))\sin(\sqrt{x_j}g(s)) \tag{3.20}$$

$$= \ln(1+\sqrt{st}) - st\ln(1-st),$$

with the x_j's as in (3.12).

4. The indeterminate case

From now on we shall use the notation

$$a = \cos\phi \in [0, 1), \quad \phi \in (0, \pi/2], \tag{4.1}$$

so that

$$e^{\pm i\phi} = a \pm \sqrt{a^2 - 1}, \tag{4.2}$$

and the branch of the square root in (4.2) is taken so that $\sqrt{x^2 - 1}/x \to 1$ as $x \to \infty$. This is equivalent to requiring

$$|e^{-i\phi}| \le |e^{i\phi}|. \tag{4.3}$$

Equality in (4.3) holds if and only if $a \in [-1, 1]$.

Our objective now is to determine the large n behavior of $G_n(x; a)$.

THEOREM 4.1. *Let $a \in [0, 1)$. Then for all x and large n we have*

$$G_n(x; a) = \sqrt{\frac{1}{2\pi\, n \sin\phi}}\, \left[e^{-i(n+1/2)\phi + i\pi/4} \cos(\sqrt{x} g(e^{i\phi})) \right. \tag{4.4}$$
$$\left. + e^{i(n+1/2)\phi - i\pi/4} \cos(\sqrt{x} g(e^{-i\phi}))\right] [1 + o(1)].$$

Proof. The t singularities of the generating function (1.7) are $t = e^{\pm i\phi}$ and $\cos(\sqrt{x}\, g(e^{i\phi}))$ is continuous as $t \to e^{\pm i\phi}$ from within the open unit disc of the t complex plane. Thus Darboux's method [20], gives

$$G_n(x; a) = (1 - e^{2i\phi})^{-1/2} \frac{(1/2)_n}{n!} e^{-in\phi} \cos(\sqrt{x} g(e^{i\phi}))\, [1 + o(1)] \tag{4.5}$$
$$+ (1 - e^{-2i\phi})^{-1/2} \frac{(1/2)_n}{n!} e^{in\phi} \cos(\sqrt{x} g(e^{-i\phi}))\, [1 + o(1)],$$

which simplifies to (4.4). □

It is useful to observe:

THEOREM 4.2. *For $a \in [0, 1)$ or $\phi \in (0, \pi/2]$ one has the relation*

$$g(e^{\pm i\phi}) = 1/2 \left(K(\cos^2\phi/2) \pm i K(\sin^2\phi/2)\right)$$
$$= 1/2 \left(K\left(\frac{1+a}{2}\right) \pm i K\left(\frac{1-a}{2}\right)\right),$$

where K is the first complete elliptic integral.

Proof. Let us first note that

$$\begin{aligned} g(e^{i\phi}) &= \frac{1}{2}\int_0^{e^{i\phi}} u^{-1/2}\,(1-ue^{i\phi})^{-1/2}\,(1-ue^{-i\phi})^{-1/2}\,du \\ &= \frac{e^{i\phi/2}}{2}\int_0^1 u^{-1/2}\,(1-ue^{2i\phi})^{-1/2}\,(1-u)^{-1/2}\,du. \end{aligned}$$

Thus

$$g(e^{\pm i\phi}) = \frac{\pi}{2}\,e^{\pm i\phi/2}\,{}_2F_1(1/2,\,1/2;\,1;\,e^{\pm 2i\phi}), \tag{4.6}$$

using Euler's integral representation [9, (2.1.3) p. 59]. □

In terms of the complete elliptic integral we have

$$g(e^{i\phi}) = e^{i\phi/2}K(e^{2i\phi}).$$

It is now possible to use the transformation theory of the elliptic integrals summarized in [9, p. 319]. We use successively the transformations

$$k \to \hat{k} = \frac{2\sqrt{k}}{1+k}, \qquad \text{and} \qquad \hat{k} \to \frac{1}{\hat{k}},$$

to first get

$$K\left(\frac{1}{\cos^2(\phi/2)}\right) = 2\cos(\phi/2)e^{i\phi/2}K(e^{2i\phi})$$

and then

$$K\left(\frac{1}{\cos^2(\phi/2)}\right) = \cos(\phi/2)\left(K(\cos^2\phi/2) + iK(\sin^2\phi/2)\right).$$

Combining these relations, and taking into account the fact that $g(e^{\pm i\phi})$ are complex conjugate, we have the theorem.

In the sequel we will use the simplified notations

$$\begin{aligned} K &= K(\cos^2\phi/2) = K\left(\frac{1+a}{2}\right), \\ K' &= K(\sin^2\phi/2) = K\left(\frac{1-a}{2}\right), \qquad a = \cos\phi. \end{aligned}$$

Let us now prove:

THEOREM 4.3. *The Hamburger moment problem associated with the $G_n(x;a)$ is indeterminate for $a \in (-1, 0]$ or $a \in [0, 1)$.*

Proof. From relation (1.9) we need consider only $a \in [0, 1)$. Theorem 2.9 in [19] asserts that the Hamburger moment problem is indeterminate if and only if $\sum_{n=0}^{\infty} |\hat{G}_n(x; a)|^2$ converges for all complex x. Now (4.4) shows that if $0 \leq a < 1$ then

$$|\hat{G}_n(x; a)|^2 = \frac{|G_n(x; a)|^2}{2n+1} = O(n^{-3/2});$$

hence $\sum_{n=0}^{\infty} |\hat{G}_n(x; a)|^2$ converges for every complex x and the indeterminacy follows. □

As a by-product, for $x = 0$, relation (4.4) simplifies to

$$G_n(0, a) = \sqrt{\frac{2}{\pi n \sin\phi}} \cos((n + 1/2)\phi - \pi/4)[1 + o(1)], \tag{4.7}$$

in agreement with Theorem 8.21.2 in [20].

We shall mostly follow the notation and terminology in Shohat and Tamarkin [19]. Let $\{Q_n(z)\}$ and $\{Q_n^*(z)\}$ be the solutions of the second order difference equation

$$\omega_{n+1}(z) = (z - \alpha_n)\omega_n(z) - \beta_n\omega_{n-1}(z), \quad n > 0, \tag{4.8}$$

which satisfy the initial data

$$Q_0(z) := 1, \quad Q_1(z) = z - \alpha_0, \quad Q_0^*(z) := 0, \quad Q_1^*(z) = 1. \tag{4.9}$$

Shohat and Tamarkin [19] use P_n instead of Q_n^*. We take $\beta_0 = 1$ in [19], so that all measures are normalized to have total mass equal to unity. The Q_n's are orthogonal with respect to a positive measure whose moments of all orders exist. If the moment problem associated with (3.1) and (3.2) is indeterminate then the polynomials $A_n(z)$, $B_n(z)$, $C_n(z)$ and $D_n(z)$, given by

$$A_{n+1}(z) := [Q_{n+1}^*(z)Q_n^*(0) - Q_{n+1}^*(0)Q_n^*(z)](\beta_1\beta_2\cdots\beta_n)^{-1}, \tag{4.10}$$

$$B_{n+1}(z) := [Q_{n+1}(z)Q_n^*(0) - Q_{n+1}^*(0)Q_n(z)](\beta_1\beta_2\cdots\beta_n)^{-1}, \tag{4.11}$$

$$C_{n+1}(z) := [Q_{n+1}^*(z)Q_n(0) - Q_{n+1}(0)Q_n^*(z)](\beta_1\beta_2\cdots\beta_n)^{-1}, \tag{4.12}$$

$$D_{n+1}(z) := [Q_{n+1}(z)Q_n(0) - Q_{n+1}(0)Q_n(z)](\beta_1\beta_2\cdots\beta_n)^{-1}, \tag{4.13}$$

converge uniformly on compact subsets of the complex plane to entire functions $A(z)$, $B(z)$, $C(z)$, $D(z)$, [19]. Furthermore the probability measures with respect to which the Q_n's are orthogonal are parameterized by functions $\sigma(z)$ which are analytic in the open upper and lower half planes, satisfy $\overline{\sigma(z)} = \sigma(\overline{z})$, and map the open upper (lower) half plane into $\Im z \le 0$ ($\Im z \ge 0$), respectively. The orthogonality measures $\psi(., \sigma)$ are related to A, B, C, D and σ through

$$\int_{-\infty}^{\infty} \frac{d\psi(t,\sigma)}{z-t} = \frac{A(z) - \sigma(z)C(z)}{B(z) - \sigma(z)D(z)}, \quad \Im z \neq 0. \tag{4.14}$$

The zeros of $A(z)$, $B(z)$, $C(z)$ and $D(z)$ are real and simple and the zeros of $B(z)$ and $D(z)$ interlace [19]. The orthogonality relation of the Q_n's is

$$\int_{-\infty}^{\infty} Q_m(x)\, Q_n(x) d\psi(x,\sigma) = \left[\prod_{k=1}^{n} \beta_k\right] \delta_{m,n}. \tag{4.15}$$

We now evaluate $B(x)$ and $D(x)$.

THEOREM 4.4. *When $a \in [0, 1)$, the functions $B(x)$ and $D(x)$ for the G_n's Hamburger moment problem are given by*

$$D(x) = -\frac{4}{\pi} \sin(\sqrt{x}K/2)\sinh(\sqrt{x}K'/2) \tag{4.16}$$

$$B(x) = \frac{2}{\pi} \log(\cot(\phi/2))\sin(\sqrt{x}K/2)\sinh(\sqrt{x}K'/2) + \cos(\sqrt{x}K/2)\cosh(\sqrt{x}K'/2). \tag{4.17}$$

Proof. From (1.12) it follows that $\beta_n = 4n^2(4n^2 - 1)$; hence

$$\prod_{k=1}^{n} \beta_k = (2n)!\,(2n+1)!.$$

Let us define the angle ψ by

$$\cos(\sqrt{x}\, g(e^{-i\phi})) = |\cos(\sqrt{x}\, g(e^{-i\phi}))| e^{i\psi}.$$

We then apply (4.4), (4.8) and (4.14) to see that for $x \ge 0$ we have

$$\begin{aligned} D(x) &= -\lim_{n\to\infty} \frac{4}{\pi \sin\phi} |\cos(\sqrt{x}\, g(e^{-i\phi}))| \\ &\quad \times [\cos((n+3/2)\phi - \psi - \pi/4)\cos((n+1/2)\phi - \pi/4) \\ &\quad - \cos((n+3/2)\phi - \pi/4)\cos((n+1/2)\phi - \psi - \pi/4)] \\ &= -\frac{4}{\pi} |\cos(\sqrt{x}\, g(e^{-i\phi}))| \sin\psi \\ &= -\frac{4}{\pi} \Im \cos(\sqrt{x}\, g(e^{-i\phi})), \end{aligned}$$

where we used (2.6). The result now simplifies to the right-hand side of (4.18) using Theorem 4.2. This establishes (4.18) for $x \geq 0$. Since both sides of (4.18) are entire functions, they must be equal for all x, by the identity theorem for analytic functions. We now come to (4.19). Relation (2.11) gives a generating function for the associated polynomials $G_n^*(0, a)$. Darboux's technique implies the asymptotics

$$G_n^*(0, a) = -2\,\Re\left[\frac{\exp(i(n-1/2)\phi - i\pi/4)}{\sqrt{2\pi n \sin\phi}}\; {}_2F_1(1/2, 1; 3/2; e^{-2i\phi})[1+o(1)]\right].$$

But

$$\begin{aligned} z\; {}_2F_1(1/2, 1; 3/2; z^2) &= \frac{1}{2}\left[\log(1+z) - \log(1-z)\right] \\ &= \frac{1}{2} e^{i\phi}\, [\log(\cot(\phi/2)) + i\pi/2], \end{aligned}$$

which is relation (16) in [9].

Thus we have established

$$\begin{aligned} G_n^*(0, a) &= -(2\pi n \sin\phi)^{-1/2}\,[1+o(1)] \\ &\quad\times\left[\log(\cot(\phi/2))\cos((n+1/2)\phi - \pi/4) - \frac{\pi}{2}\sin((n+1/2)\phi - \pi/4)\right]. \end{aligned} \tag{4.18}$$

For $x > 0$ the relationships (4.4), (4.11) and (4.18) lead to

$$\begin{aligned} B(x) &= \frac{2}{\pi}\left[\log(\cot(\phi/2))\sin\psi + \frac{\pi}{2}\cos\psi\right]\,|\cos(\sqrt{x}g(e^{i\phi}))|, \qquad (4.19) \\ &= \Re\cos(\sqrt{x}\,g(e^{-i\phi})) + 2/\pi\,\Im\cos(\sqrt{x}\,g(e^{-i\phi})) \end{aligned}$$

which proves (4.17) for $x > 0$ upon use of Theorem 4.2. Finally we invoke the identity theorem and extend (4.17) to the whole complex plane. Now the proof of Theorem 4.4 is complete. □

We now examine the so-called Nevanlinna extremal measures, for which

$$\sigma(z) = \sigma, \tag{4.20}$$

where σ is some real number, possibly ∞. The Stieltjes transform of the corresponding measures Ψ_σ, which are all discrete, is then given by (4.14) and it is known from [1], [19] that the polynomials $G_n(x; a)$ are dense in $L^2(d\Psi_\sigma)$.

The masses are located at x_n, determined by

$$B(x_n) - \sigma D(x_n) = 0, \tag{4.21}$$

while the masses are $\rho(x_n)$. Relation (2.24) in [6] gives the ρ function

$$\frac{1}{\rho(x)} = \sum_{n=0}^{\infty} \omega_n^2(x) = B'(x)D(x) - B(x)D'(x), \tag{4.22}$$

for real x. The next theorem records the simplest Nevanlinna extremal measures.

THEOREM 4.5. *When $a \in [0, 1)$ one has the orthogonality measures*

$$\Psi_\infty = \frac{\pi}{KK'}\delta_{(x_0)} + \frac{2\pi}{KK'}\sum_{n=1}^{\infty}\left\{\frac{2n\pi K'/K}{\sinh(2n\pi K'/K)}\delta_{(x_n)} + \frac{2n\pi K/K'}{\sinh(2n\pi K/K')}\delta_{(x'_n)},\right\} \tag{4.23}$$

with

$$x_0 = 0, \qquad x_n = 4\pi^2 n^2/K^2, \qquad x'_n = -4\pi^2 n^2/K'^2, \quad n = 1, 2, \ldots, \tag{4.24}$$

and

$$\psi_0 = \frac{2\pi}{KK'}\sum_{n=0}^{\infty}\left\{\frac{(2n+1)\pi K'/K}{\sinh((2n+1)\pi K'/K)}\delta_{(y_n)} + \frac{(2n+1)\pi K/K'}{\sinh((2n+1)\pi K/K')}\delta_{(y'_n)},\right\} \tag{4.25}$$

with

$$y_n = (2n+1)^2\pi^2/K^2, \qquad y'_n = -(2n+1)^2\pi^2/K'^2, \quad n = 0, 1, \ldots \tag{4.26}$$

Proof. For $\sigma = \infty$ the Stieltjes transform of Ψ_∞ is given by

$$\int_{-\infty}^{\infty}\frac{d\Psi_\infty}{z-u} = \frac{C(z)}{D(z)}.$$

It follows that the spectrum is given by (4.24). □

Relation (4.22) and Theorem 4.4 give for real x

$$\frac{1}{\rho(x)} = \frac{KK'}{2\pi}\left(\frac{\sin(\sqrt{x}K)}{\sqrt{x}K} + \frac{\sinh(\sqrt{x}K')}{\sqrt{x}K'}\right), \tag{4.27}$$

from which it follows that the measure ψ_0 is given by (4.25). Taking $\sigma = -1/2 \ln\cot(\phi/2)$ we get the measure Ψ_0 as given by (4.25) and (4.26).

We now mention some weight functions for the G_n's. It has been shown in [3] that the choice

$$\frac{1}{\sigma(z)} = \begin{cases} t + i\gamma, & \Im z > 0, \\ t - i\gamma, & \Im z < 0, \end{cases} \tag{4.28}$$

for real t and $\gamma > 0$ gives the weight function

$$w(x; t, \gamma) = \frac{\gamma/\pi}{(D(x) - tB(x))^2 + \gamma^2 B^2(x)}. \tag{4.29}$$

The function $w(x; t, \gamma)$ is a weight function for the Q_n's and is normalized to have unit total mass.

With the choice

$$\frac{t}{t^2+\gamma^2} = \frac{1}{2}\ln\cot(\phi/2),$$

the denominator in (4.29) simplifies and we obtain the weight function

$$w(x;\eta) = \frac{\zeta/4}{|\cos(\sqrt{x}K/2)\cosh(\sqrt{x}K'/2) + i\zeta\ \sin(\sqrt{x}K/2)\sinh(\sqrt{x}K'/2)|^2}, \tag{4.30}$$

with

$$\zeta = \frac{4\gamma}{\pi(t^2+\gamma^2)} > 0. \tag{4.31}$$

When $\zeta = 1$, (4.30) simplifies considerably and we obtain the weight function stated in (1.18).

5. The polynomials in a

Now, let us consider $G_n(x; a)$ as polynomials in the variable a, while x becomes a parameter.

The corresponding monic polynomials are

$$Q_n(x;a) = \frac{n!}{2^n(1/2)_n} G_n(x:a), \tag{5.1}$$

with the recurrence relation

$$Q_{n+1}(x;a) = \left(a - \frac{x/2}{(2n+1)^2}\right) Q_n(x;a) - \frac{n^2}{4n^2-1} Q_{n-1}(x;a), \quad n \geq 1, \tag{5.2}$$

and the initial conditions

$$Q_0(x;a) = 1, \qquad Q_1(x;a) = -\frac{x}{2} + a. \tag{5.3}$$

From (5.2), (5.3) and the general theory of orthogonal polynomials it follows that there is a probability measure ψ such that their orthogonality relation will be

$$\int_{-\infty}^{\infty} G_m(x;a)G_n(x;a)\, d\psi(a) = \frac{\delta_{m,n}}{2n+1}. \tag{5.4}$$

From the recurrence relation (5.2) it is clear that the orthogonality is now on a bounded subset of $(-\infty, \infty)$ and Theorems 3.1 and 4.1 indicate that the polynomials are oscillatory only if $a \in [-1, 1]$; hence the absolutely continuous component of the measure is supported on $[-1, 1]$.

A theorem of Maté, Nevai and Totik (see the survey [21, Theorem 6]) asserts that if

$$\sum_{n=0}^{\infty} \left(|1 - 4a_{n+1}^2| + 2|b_n| \right) < \infty,$$

then the polynomials $\{Q_n(x)\}$ generated by (5.2) and (5.3) are orthogonal with respect to a measure ψ whose absolutely continuous component ψ' is supported on $[-1, 1]$ and

$$Q_n(\cos\phi) = \sqrt{\frac{2}{\pi} \prod_{j=1}^{n} \beta_j} \, \frac{\cos(v(\phi, n))}{\sqrt{\sin(\phi)\ \psi'(\cos\phi)}} [1 + o(1)], \tag{5.5}$$

and $v(\phi, n)/n \to 0$ as $n \to \infty$. Here the previous hypotheses hold since we have

$$|b_n| = \frac{|x|/2}{(2n+1)^2}, \qquad |1 - 4a_n^2| = \frac{1}{4n^2 - 1}.$$

From (4.4) and (5.5) one obtains

$$\begin{aligned} \psi'(a) &= \frac{1}{\cos(\sqrt{x} g(e^{i\phi}))\ \cos(\sqrt{x} g(e^{-i\phi}))} \\ &= \frac{2}{\cos(\sqrt{x} K) + \cosh(\sqrt{x} K')}, \qquad a = \cos\phi \in (-1, 1), \end{aligned} \tag{5.6}$$

with

$$K = K\left(\frac{1+a}{2}\right), \qquad K' = K\left(\frac{1-a}{2}\right). \tag{5.7}$$

We have all the information needed to determine the Stieltjes transform of ψ. As polynomials in a, we replace (4.1) by

$$G_0^*(x; a) := 0, \quad G_1^*(x; a) = 1. \tag{5.8}$$

Hence the corresponding continued J-fraction, say $J_2(x; a)$ is

$$\begin{aligned} J_2(x; a) &= \int \frac{d\mu(u)}{a - u} \\ &= 2 \int_0^{e^{-i\phi/2}} \frac{\sin(\sqrt{x}(g(e^{-i\phi}) - g(u^2)))}{\sqrt{x}\ \cos(\sqrt{x} g(e^{-i\phi}))} \, du, \quad a \notin [-1, 1]. \end{aligned} \tag{5.9}$$

For $a > 1$, and $x > 0$ the masses are given by

$$g(e^{-\phi}) = \frac{(n + 1/2)\pi}{\sqrt{x}}, \tag{5.10}$$

where $g(e^{-\phi})$, given by (3.11), is monotonously decreasing from ∞ to 0. It follows that (5.10) has a unique solution for every n, which we denote by a_n. This agrees with the general theory [21]. Now the residue of $J_2(x, a)$ at $a = a_n$ is $\psi(a_n)$, where

$$\psi(a_n) = -\frac{2}{x\, g'(e^{-\phi})} \int_0^{e^{-\phi/2}} \cos((n + 1/2)\pi g(u^2)/g(e^{-\phi}))\, du,$$

where $g'(e^{-\phi})$ is the derivative with respect to the variable a. Comparing with (3.10) we get

$$\psi(a_n) = -\frac{g(e^{-\phi})}{x g'(e^{-\phi})}\mu(x_n),$$

and

$$\frac{g'(e^{-\phi})}{g(e^{-\phi})} = \frac{1}{2\sqrt{a^2-1}}\left(1 - \frac{2E(k^2)}{k'^2 K(k^2)}\right), \qquad k = a - \sqrt{a^2-1}.$$

For $x \in (0, +\infty]$, it follows from (1.9) that we get the spectrum $-a_n$. For large n, relation (5.10) shows that the $\pm a_n$ accumulate to ± 1.

Acknowledgments. This work was done while the authors were visiting Imperial College. We thank our friend Steve Milne for giving us a copy of his interesting work [18] which is related to the topic of this paper. M.I. gratefully acknowledges the hospitality and support from both Imperial College and the University of South Florida.

REFERENCES

1. N. I. Akhiezer, *The classical moment problem and some related questions in analysis*, English translation, Oliver and Boyd, Edinburgh, 1965.
2. G. E. Andrews and R. A. Askey, "Classical orthogonal polynomials" in *Polynomes Orthogonaux et Applications*, eds. C. Breziniski et al., Lecture Notes in Mathematics, vol. 1171, Springer-Verlag, Berlin, 1984, pp. 36–63.
3. C. Berg and J. P. R. Christensen, *Density questions in the classical theory of moments*, Ann. Inst. Fourier **31** (1981), 99–114.
4. C. Berg and M. E.H Ismail, in preparation.
5. C. Berg and H. L. Pedersen, *On the order and type of the entire functions associated with an indeterminate Hamburger moment problem*, Ark. Math. **32** (1994), 1–11.
6. C. Berg and G. Valent, *The Nevanlinna parameterization for some indeterminate Stieltjes moment problems associated with birth and death processes*, Methods and Applications of Analysis **1** (1994), 169–209.
7. Y. Chen and M. E. H. Ismail, *Some indeterminate moment problems and Freud-like weights*, Constructive Approximation, to appear.
8. T. S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach, New York, 1978.
9. A. Erdélyi, W. Magnus, F. Oberhettinger and G. F. Tricomi, *Higher transcendental functions*, volume 1, McGraw-Hill, New York, 1953.
10. M. E. H. Ismail, J. Letessier, D. Masson, and G. Valent, "Birth and death processes and orthogonal polynomials" in *Orthogonal polynomials: Theory and practice*, edited by P. Nevai, Proc. NATO ASI

on Orthogonal Polynomials and Their Applications, Kluwer Academic Publishers, Dordrecht, 1990, pp. 225–229.
11. M. E. H. Ismail, J. Letessier and G. Valent, *Quadratic birth and death processes and associated continuous dual Hahn polynomials*, SIAM J. Math. Anal. **20** (1989), 727–737.
12. M. E. H. Ismail and X. Li, *Bounds on the extreme zeros of orthogonal polynomials*, Proc. Amer. Math. Soc. **115** (1992), 131–140.
13. M. E. H. Ismail and J. Wilson, *Asymptotic and generating relations for the q-Jacobi and the* $_4\phi_3$ *polynomials*, J. Approximation Theory **36** (1982), 43–54.
14. S. Karlin and S. Tavaré, *Linear birth and death processes with killing*, J. Applied Prob. **19** (1982), 477–487.
15. ———, *A diffusion process with killing: the time to formation of recurrent deleterious mutant genes*, Stochastic Processes and their Applications **13** (1982), 249–261.
16. D. S. Lubinsky, *A survey of general orthogonal polynomials for weights on finite and infinite intervals*, Acta Applicandae Mathematicae **10** (1987), 237–296,
17. D. S. Lubinsky, *An update on orthogonal polynomials and weighted approximation on the real line*, Acta Applicandae Mathematicae **33** (1993), 121–164.
18. S. Milne, *Sums of squares, Jacobi elliptic functions, and Schur functions*, in preparation.
19. J. Shohat and J. D. Tamarkin, *The problem of moments*, revised edition, American Mathematical Society, Providence, 1950.
20. G. Szegő, *Orthogonal polynomials*, fourth edition, American Mathematical Society, Providence, 1975.
21. W. Van Assche, "Asymptotics for orthogonal polynomials and three-term recurrence relations" in *Orthogonal polynomials: Theory and practice*, P. Nevai, ed., NATO-ASI series **C 294**, Kluwer Academic Publishers, Dordrecht, 1990, pp. 435–462.
22. H. S. Wall, *Analytic theory of continued fractions*, Van Nostrand, Princeton, 1948.
23. E. T. Whittaker and G. N. Watson, *A course of modern analysis*, second edition, Cambridge University Press, Cambridge, 1927.

Mourad E. H. Ismail, Department of Mathematics, University of South Florida, Tampa, Florida 33620-5700
ismail@hahn.math.usf.edu

Galliano Valent, Laboratoire de Physique Theorique et des Hautes Energies, Université Paris 7, 2 Place Jussieu, 75251 Paris Cedex 05, France
valent@pthe.jussieu.fr