ILLINOIS JOURNAL OF MATHEMATICS
Volume 42, Number 3, Fall 1998

A PARTIAL CONNECTION ON COMPLEX FINSLER BUNDLES AND ITS APPLICATIONS

TADASHI AIKOU

Introduction

Let $\pi: E \to M$ be a holomorphic vector bundle over a complex manifold M, and $p: PE \to M$ the projective bundle of E. If a complex Finsler structure F is given on E, a natural Hermitian structure H on the pull-back $\tilde{E} = p^{-1}E$ is defined, and the differential geometry of $(\tilde{E}, H)$ has been studied (cf. Abate-Patrizio[1], Aikou [2], [3], [4], [5], Faran [7], Kobayashi [8], [10], Royden [11]). In particular, Kobayashi [8] gave a differential geometric characterization of negative holomorphic vector bundles, and proved a vanishing theorem for holomorphic sections of a complex Finsler bundle.

In this paper, we are also concerned with a holomorphic vector bundle with a convex Finsler structure. The main purpose of this paper is to prove vanishing theorems for holomorphic sections and cohomology groups, which are generalizations of the ones in Hermitian geometry (Theorem 3.1 and 3.2). For this purpose, we shall introduce a partial connection so that local calculations have an invariant meaning. This partial connection means a covariant derivation in transversal direction to the fibres of PE. By using this partial connection, for example, the curvature of a canonical tautological Hermitian line bundle is expressed in a simple form(Proposition 2.2).

First of all, we shall introduce some basic notations. Let M be a complex manifold of dimension n, and E a holomorphic vector bundle of rank r over M. Each fibre E_z is a complex vector space of complex dimension r. We denote by $p: PE \to M$ the projective bundle associated to E, and by $\tilde{E}$ the induced bundle $p^{-1}E$ over PE. The tautological line subbundle $\pi_L: LE \to PE$ of $\tilde{E}$ is defined by

$$LE := \{(v, V) \in E \times PE; v \in V\}.$$

We denote by $E^{\times}$ (resp. $LE^{\times}$) the open submanifold of E (resp. LE) consisting of the non-zero elements. The holomorphic map $\tau: E^{\times} \to PE \times E$ defined by $\tau(v) = ([v], v) \in LE^{\times}$ maps $E^{\times}$ biholomorphically to $LE^{\times}$.

Let $\{U, (z^{\alpha})\}$ be a complex coordinate system of M, and $\{\pi^{-1}(U), (z^{\alpha}, \xi^{i})\}$ the induced complex coordinate system on E with respect to a holomorphic frame field $\{s_1, \dots, s_r\}$ on U. We denote by $[\xi]$ the point of PE corresponding to $(z, \xi) \in E^{\times}$.

Received July 21, 1997.
1991 Mathematics Subject Classification. Primary 53C56, 53C60.

We define $t_a: V_a \to LE$ by

$$t_a([\xi]) := \left([\xi], \left(\frac{\xi^1}{\xi^a}, \dots, \frac{\xi^r}{\xi^a}\right)\right)$$

on $V_a := \{[\xi] \in PE; \xi^a \neq 0\}$. Then $\{V_a, t_a\}$ defines a local trivialization $\Phi_a: V_a \times \mathbb{C} \to \pi_L^{-1}(V_a)$ on LE by $\Phi_a([\xi], \lambda) = \lambda t_a([\xi])$. Hence the biholomorphism τ can be written as

$$\tau(z, \xi) = ([\xi], \xi) = \Phi_a([\xi], \xi^a) \cong ([\xi], \xi^a)$$

on V_a.

We shall use the following notation throughout this paper:

A^p (resp. $A^{p,q}$) is the space of p-forms (resp. (p, q)-forms) on PE;

$A^p(\tilde{E})$ (resp. $A^{p,q}(\tilde{E})$) is the space of $\tilde{E}$-valued p-forms (resp. (p, q)-forms) on PE.

1. Finsler structures and partial connections

1.1. *Partial connection.* First we shall make the following definition.

DEFINITION. A *complex Finsler structure* F on E is a real valued function satisfying the following conditions:

(1) F is C^∞-class on $E^\times$;
(2) $F(z, \xi) \geq 0$, and equals 0 if and only if $\xi = 0$;
(3) $F(z, \lambda\xi) = |\lambda|^2 F(z, \xi)$ for all $\lambda \in \mathbb{C}$.

Since $E^\times$ is biholomorphic to $LE^\times$, there exists a one-to-one correspondence between Hermitian structures on LE and Finsler structures on E via the holomorphic map τ (cf. [7]). A complex Finsler structure F is said to be *convex* if the Hermitian matrix $(F_{i\bar{j}})$ defined by

$$F_{i\bar{j}} := \frac{\partial^2 F}{\partial \xi^i \partial \bar{\xi}^j} \tag{1.1}$$

is positive-definite. In the following, we always assume the convexity of F, and call (E, F) a *convex Finsler vector bundle*. By the condition (3) in the definition, matrix components $F_{i\bar{j}}$ defined by (1.1) are functions on PE.

Putting $Z^i = \xi^i \circ p$, we take $([\xi], Z) = (z^1, \dots, z^n, \xi^1 : \cdots : \xi^r, Z^1, \dots, Z^r)$ as a local coordinate system for $\tilde{E}$. Here and in the following, $(\xi^1 : \cdots : \xi^r)$ is considered as a homogeneous coordinate system for fibres. For the convenience for local calculations, we use the homogeneous coordinate system $(\xi^1 : \cdots : \xi^r)$. Then the line bundle LE is characterized by $(\xi^1 : \cdots : \xi^r) = (Z^1 : \cdots : Z^r)$.

For all Z, $W \in A^0(\tilde{E})$, we shall define a Hermitian structure H on $\tilde{E}$ by

$$H(Z, W) := \sum_{i,j} F_{i\bar{j}} Z^i \bar{W}^j.$$

The Hermitian connection ∇: $A^0(\tilde{E}) \to A^1(\tilde{E})$ in $(\tilde{E}, H)$ is given by the form

$$\theta^i_j := \sum_l F^{\bar{l}i} \partial F_{j\bar{l}} = \sum_{l,\alpha} F^{\bar{l}i} \frac{\partial F_{j\bar{l}}}{\partial z^\alpha} dz^\alpha + \sum_{l,k} F^{\bar{l}i} \frac{\partial F_{j\bar{l}}}{\partial \xi^k} d\xi^k. \tag{1.2}$$

For the cotangent bundle T^*_{PE} of PE, we shall introduce a C^∞-splitting σ of the exact sequence

$$0 \longrightarrow \mathcal{H}^* \longrightarrow T^*_{PE} \underset{\sigma}{\overset{\longrightarrow}{\longleftarrow}} (\ker dp)^* \longrightarrow 0 \tag{1.3}$$

by

$$\sigma(d\xi^i) := d\xi^i + \sum_j \theta^i_j \xi^j. \tag{1.4}$$

The set of local 1-forms $\{dz^1, \dots, dz^n, \theta^1, \dots, \theta^r\}$, $\theta^i := \sigma(d\xi^i)$, is a local co-frame field for T^*_{PE}, and it defines a C^∞-splitting

$$T^*_{PE} = \mathcal{H}^* \oplus \mathcal{V}^*, \tag{1.5}$$

where $\mathcal{H}^*$ is locally spanned by $\{dz^1, \dots, dz^n\}$, and $\mathcal{V}^*$ by $\{\theta^1, \dots, \theta^r\}$. We denote by $p'_{\mathcal{H}}$: $A^0(T^*_{PE}) \to A^0(\mathcal{H}^*)$ and $p''_{\mathcal{H}}$: $A^0(T^*_{PE}) \to A^0(\bar{\mathcal{H}}^*)$ the natural projections respectively. Then we shall define a *partial connection* $D := D' + D''$: $A^0(\tilde{E}) \to A^0\left((\mathcal{H} \oplus \bar{\mathcal{H}})^* \otimes \tilde{E}\right)$ by

$$D := (p_{\mathcal{H}} \otimes 1) \circ \nabla.$$

The following diagram is commutative:

$$\begin{array}{ccc} A^0(\tilde{E}) & \xrightarrow{\nabla} & A^0\left((T_{PE} \oplus \bar{T}_{PE})^* \otimes \tilde{E}\right) \\ \| & & \downarrow{\scriptstyle p_{\mathcal{H}} \otimes 1} \\ A^0(\tilde{E}) & \xrightarrow[D]{} & A^0\left((\mathcal{H} \oplus \bar{\mathcal{H}})^* \otimes \tilde{E}\right) \end{array}$$

We put $d_{\mathcal{H}} := \partial_{\mathcal{H}} + \bar{\partial}_{\mathcal{H}}$, $\partial_{\mathcal{H}} = p'_{\mathcal{H}} \circ d$, and $\bar{\partial}_{\mathcal{H}} = p''_{\mathcal{H}} \circ d$. Using this notation, the connection form ω^i_j of D is given by

$$\omega^i_j = \sum_{m=1}^r F^{i\bar{m}} \partial_{\mathcal{H}} F_{j\bar{m}}.$$

We shall give a local expression for the operator $\partial_{\mathcal{H}}$. If we put

$$N_\alpha^i = \sum_{j,m} F^{i\bar{m}} \frac{\partial F_{j\bar{m}}}{\partial z^\alpha} \xi^j,$$

for any function f, $\partial_{\mathcal{H}} f$ is defined by

$$\partial_{\mathcal{H}} f = \sum_\alpha \frac{\delta f}{\delta z^\alpha} dz^\alpha := \sum_\alpha \left(\frac{\partial f}{\partial z^\alpha} - \sum_m N_\alpha^m \frac{\partial f}{\partial \xi^m} \right) dz^\alpha.$$

Since the Hermitian connection ∇ is compatible with the Hermitian structure H, we have:

PROPOSITION 1.1.

$$d_{\mathcal{H}} H(Z, W) = H(DZ, W) + H(Z, DW)$$

for all $Z, W \in A^0(\tilde{E})$.

The following are also important in our local calculations.

$$\partial_{\mathcal{H}} \omega + \omega \wedge \omega \equiv 0, \quad \partial_{\mathcal{H}} \circ \partial_{\mathcal{H}} \equiv 0. \tag{1.6}$$

These identities are proved by direct calculations (cf. Lemma 2.1 and 2.2 in [5]).

1.2. *Curvature of partial connections.* Extending the partial connection D to the space $A^p(\tilde{E})$ in the usual way, $D \circ D$ determines an $\mathrm{End}(\tilde{E})$-valued $(1,1)$-form R called the *curvature* of D. From (1.6), the partial connection $D = D' + D''$ satisfies $D' \circ D' = D'' \circ D'' \equiv 0$, and so

$$D \circ D = D' \circ D'' + D'' \circ D'. \tag{1.7}$$

For all $Z \in A^0(\tilde{E})$, if we put $D^2 Z = R(Z)$, the right-hand side is written as

$$R(Z) = \sum_{i,j} Z^j \Omega_j^i \otimes s_i,$$

where the curvature form Ω_j^i is given by

$$\Omega_j^i = \bar{\partial}_{\mathcal{H}} \omega_j^i =: \sum_{\alpha,\beta} R^i_{j\alpha\bar{\beta}}([\xi]) dz^\alpha \wedge d\bar{z}^\beta.$$

The curvature tensor $R_{i\bar{j}\alpha\bar{\beta}} := \sum_m F_{m\bar{j}} R^m_{i\alpha\bar{\beta}}$ is defined by

$$R_{i\bar{j}\alpha\bar{\beta}} = -\frac{\delta^2 F_{i\bar{j}}}{\delta z^\alpha \delta \bar{z}^\beta} + \sum_{k,l} F^{k\bar{l}} \frac{\delta F_{k\bar{j}}}{\delta \bar{z}^\beta} \frac{\delta F_{i\bar{l}}}{\delta z^\alpha}. \tag{1.8}$$

From Proposition 1.1, we have:

PROPOSITION 1.2. *The partial connection D satisfies*

$$\partial_{\mathcal{H}}\bar{\partial}_{\mathcal{H}} H(Z, Z) = H(DZ, DZ) - H(R(Z), Z) \tag{1.9}$$

for any holomorphic section Z of $\tilde{E}$.

2. Negative vector bundles and Kobayashi's theorem

In this section, we shall discuss negative holomorphic vector bundles, and recall the theorem due to Kobayashi [8] from our point of view. For a convex Finsler vector bundle (E, F), the curvature form of the corresponding Hermitian metric on LE is given by $\bar{\partial}\partial \log F$. For the natural projection $\rho\colon E^{\times} \to PE$, direct calculations implies that $\ker d\rho$ is locally spanned by $\epsilon := \sum \xi^j \frac{\partial}{\partial \xi^j}$, and so $\ker d\rho$ is a holomorphic line subbundle of $TE^{\times}$. If we put

$$ann.\bar{\partial}\partial \log F := \{X \in TE^{\times};\, i(X)\bar{\partial}\partial \log F = 0\},$$

we see easily that $ann.\bar{\partial}\partial \log F = \ker d\rho$. Moreover we have

$$\mathcal{L}_{\epsilon}\bar{\partial}\partial \log F = 0,$$

where $\mathcal{L}_{\epsilon}$ is the Lie derivative with respect to ϵ. This means that $\bar{\partial}\partial \log F$ is invariant under complex multiplication. Hence $\bar{\partial}\partial \log F$ is invariant along the fibres of $ann.\bar{\partial}\partial \log F$. This fact implies that $\bar{\partial}\partial \log F$ may be considered as a $(1, 1)$-form on PE, that is, there exists a $(1, 1)$-form Φ on PE such that

$$\rho^*\Phi = \bar{\partial}\partial \log F. \tag{2.1}$$

For the convenience of local calculations, however, we shall use the form $\bar{\partial}\partial \log F$ in stead of Φ. The real $(1, 1)$-form $\frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log F$ on PE represents the first Chern class $c_1(LE)$ of LE.

2.1. *Negative vector bundles.* The ampleness of a holomorphic vector bundle E is important in algebraic geometry, and it is well known that it is equivalent to weak positivity in the sense of Griffith. By definition, E is said to be *negative* if its dual E^* is *ample*, and it is equivalent to the negativity of its tautological line bundle LE. Hence LE admits a Hermitian structure F satisfying

$$\frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log F < 0. \tag{2.2}$$

Let E be a negative vector bundle over M. We shall construct a complex Finsler structure on E satisfying (2.2). By definition, the tautological line bundle LE is negative, and so its dual LE^* is ample. Hence there exists an $m \gg 0$ such that

$L := (LE^*)^m$ is very ample. Then, by definition, we can choose $f^0, \dots, f^N \in H^0(PE, L)$ so that

$$\varphi\colon PE \ni [\xi] \to (f^0([\xi]) : \cdots : f^N([\xi])) \in P^N(\mathbb{C})$$

is a holomorphic embedding. It is well known that $L \cong \varphi^* H$ for the hyperplane bundle H over $P^N(\mathbb{C})$. Since $P^N(\mathbb{C})$ admits the Fubini-Study metric, the first Chern form of H is given by

$$\frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log \left(\frac{\sum_{k=0}^N |T^k|^2}{|T^j|^2} \right)^{-1}$$

on $U_j = \{[T^1 : \cdots : T^r] \in P^N(\mathbb{C}); T^j \neq 0\}$. On $V_a \subset PE$, we put $f^k = \{f_a^k\}$, $(k = 0 \dots, N)$, where f_a^k are holomorphic functions on V_a. Then a canonical Hermitian metric $F_{\varphi^* H}$ of $\varphi^* H$ is defined by

$$F_{\varphi^* H, j}([\xi]) = \left(\frac{\sum_{k=0}^N |f_a^k([\xi])|^2}{|f_a^j([\xi])|^2} \right)^{-1}$$

on $\varphi^{-1}(U_j) \cap V_a$. Since H is ample and φ is holomorphic embedding, we have

$$\frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log F_{\varphi^* H, j}([\xi]) > 0. \tag{2.3}$$

The corresponding Hermitian metric F_L on L is given by the functions

$$F_{L,a}([\xi]) = \left(\sum_{k=0}^N |f_a^k([\xi])|^2 \right)^{-1}$$

on each V_a. Since $L = (LE^*)^m$, the corresponding Hermitian metric on LE is given by the functions

$$F_{LE,a}([\xi]) = \sqrt[m]{\sum_{k=0}^N |f_a^k[\xi]|^2}$$

on V_a. Then, since $\tau(z, \xi) = \xi^a t_a([\xi]) \cong ([\xi], \xi^a)$ on V_a, we shall define a complex Finsler structure F on E by

$$F(z, \xi) := F_{LE,a} |\xi^a|^2 = \sqrt[m]{\sum_{k=0}^N |f_a^k([\xi])(\xi^a)^m|^2}. \tag{2.4}$$

This definition is independent of the choice of the neighborhood V_a, since on $V_a \cap V_b$ we have $F_{LE,a} |\xi^a|^2 = F_{LE,b} |\xi^b|^2$. Because of (2.3), the Finsler structure F defined by (2.4) satisfies (2.2). Hence we have proved:

PROPOSITION 2.1. *Let E be a negative vector bundle over a compact complex manifold M. For the holomorphic embedding φ: $PE \ni [\xi] \to (f^0([\xi]) : \cdots : f^N([\xi])) \in P^N(\mathbb{C})$, the function F defined by* (2.4) *is a convex Finsler structure with negative curvature.*

2.2. *Kobayashi's theorem.* Let F be a convex Finsler structure on E. We shall express the curvature form $\bar{\partial}\partial \log F$ with respect to the basis $\{dz^\alpha, \theta^i\}$. The following proposition shows the advantage of taking the splitting (1.4).

PROPOSITION 2.2. *The curvature form $\bar{\partial}\partial \log F$ of (LE, F) is given by*

$$\bar{\partial}\partial \log F = \frac{1}{F} \sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\bar{\beta}} \xi^i \bar{\xi}^j dz^\alpha \wedge d\bar{z}^\beta - \sum_{i,j} \frac{\partial^2 \log F}{\partial \xi^i \partial \bar{\xi}^j} \theta^i \wedge \bar{\theta}^j, \tag{2.5}$$

where $R_{i\bar{j}\alpha\bar{\beta}}$ is the curvature tensor of the partial connection D of (E, F).

Proof. This identity is obtained by calculations by using the identity $DF = \partial_{\mathcal{H}} F \equiv 0$ and the homogeneity of F (See also [6], Proposition 2.3). □

If we put $(\log F)_{i\bar{j}} := \partial^2 \log F / \partial \xi^i \partial \bar{\xi}^j$ and $\Psi_{\alpha\bar{\beta}} := \sum_{i,j} R_{i\bar{j}\alpha\bar{\beta}}([\xi])\xi^i \bar{\xi}^j$, the matrix representation of (2.5) is given by

$$\bar{\partial}\partial \log F = \begin{pmatrix} \frac{1}{F}\Psi_{\alpha\bar{\beta}} & O \\ O & -(\log F)_{i\bar{j}} \end{pmatrix}.$$

In the following, we define the $(1, 1)$-form Ψ by

$$\Psi := \sum \Psi_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta.$$

Then we have:

THEOREM 2.1(Kobayashi [8]). *A holomorphic vector bundle E is negative if and only if it admits a convex Finsler structure F with negative Ψ.*

Proof. Because of

$$\frac{1}{F} F_{i\bar{j}} = (\log F)_{i\bar{j}} + \frac{1}{F^2} \sum_{k,l} F_{i\bar{l}} F_{k\bar{j}} \xi^k \bar{\xi}^l, \tag{2.6}$$

the convexity of F is equivalent to the fact that Hermitian matrix $(\log F)_{i\bar{j}}$ has $r - 1$ positive eigenvalues and one zero eigenvalue. Hence, if the bundle E admits a convex Finsler structure F with negative $\Psi := \frac{\sqrt{-1}}{2\pi} \sum \Psi_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$, then E is negative.

Conversely, if (2.2) is satisfied, then we first obtain the convexity of F from (2.6). Hence we get the expression (2.5), which shows the negativity of Ψ. □

Here we note that the curvature tensor $R_{i\bar{j}\alpha\bar{\beta}}$ is different from the one defined by (3.15) in Kobayashi [8]. Our curvature Ψ, however, coincides with Kobayashi's Ψ defined by (3.19) in [8].

EXAMPLE. A typical example of M with negative TM is a compact Kähler manifold with negative bi-sectional curvature. In fact, if the metric F is Hermitian, say $F(z,\xi) = \sum h_{i\bar{j}}(z)\xi^i\bar{\xi}^j$, the partial connection D coincides with the Hermitian connection of $h_{i\bar{j}}$. So we have $\Psi_{i\bar{j}} = \sum_{k,l} R_{i\bar{j}k\bar{l}}(z)\xi^k\bar{\xi}^l$. Hence, if M is a compact Kähler manifold with negative bi-sectional curvature, then its tangent bundle TM is negative.

Here we shall give another proof of Kobayashi's theorem (cf. [8], Corollary 6.3) as an application of Proposition 2.2 and Theorem 2.1.

We suppose that a complex Finsler structure F is defined on the holomorphic tangent bundle TM of a compact complex manifold M. Royden [11] defined the *holomorphic sectional curvature* $K_F(z,\xi)$ at $(z,\xi) \in TM$ as follows. Let Δ be the unit disk in $\mathbb{C}$. For all $(z,\xi) \in TM$, there exists a holomorphic map $\psi\colon \Delta \to M$ satisfying $\psi(0) = z$ and $\psi_*(0) = \xi$. Since ψ^*F is a Hermitian metric on Δ, we can calculate its Gaussian curvature K_{ψ_*F} at the origin. Then, $K_F(z,\xi)$ is defined by

$$K_F(z,\xi) := \sup_{\psi}\{K_{\psi^*F}\}.$$

Then (cf. [2]) we have:

LEMMA 2.1. *If F is a convex Finsler structure on TM, then its holomorphic sectional curvature $K_F(z,\xi)$ at $(z,\xi) \in TM$ is given by*

$$K_F(z,\xi) = \frac{2}{F^2}\sum_{i,j}\Psi_{i\bar{j}}\xi^i\bar{\xi}^j = \frac{2}{F^2}\sum_{i,j,k,l} R_{i\bar{j}k\bar{l}}\xi^i\bar{\xi}^j\xi^k\bar{\xi}^l$$

for the curvature $R_{i\bar{j}k\bar{l}}$ of D.

Hence the negativity of Ψ implies the negativity of K_F, and so, since PE is compact, K_F is bounded above by a negative constant. The generalized Shcwarz lemma implies $F_K \geq kF$ for a positive constant k (cf. [3], Theorem 5.1), where F_K is the *Kobayashi metric* on M. Hence we have proved

PROPOSITION 2.3 (Kobayashi [8]). *Let M be a compact complex manifold. If its holomorphic tangent bundle TM is negative, then M is Kobayashi hyperbolic.*

3. Applications-Vanishing theorems

3.1. *A vanishing theorem for holomorphic sections.* In this section, we shall state a Bochner-type vanishing theorem for holomorphic sections of a convex Finsler bundle (E, F) (cf. [5]).

Let $\zeta = \sum_i \zeta^i(z)s_i$ be a non-vanishing holomorphic section over an open set U. We denote by $PE_{\zeta(U)} \subset PE$ the image of $\zeta(U)$ by the natural projection $\rho\colon E^\times \to PE$; that is,

$$PE_{\zeta(U)} := \{[\zeta(z)] \in PE;\ z \in U\}.$$

We also denote by ζ_P the corresponding holomorphic section of LE over $PE_{\zeta(U)}$. For the holomorphic mapping $f_\zeta\colon z \in U \to [\zeta(z)] \in PE$, we get the following commutative diagram:

$$\begin{array}{ccc} LE^\times & \xleftarrow{\ \tau\ } & E^\times \\ {\scriptstyle \zeta_P}\uparrow & & \uparrow{\scriptstyle \zeta} \\ PE_{\zeta(U)} & \xleftarrow[f_\zeta]{} & U \end{array}$$

We say that a holomorphic section $\zeta = \sum_i \zeta^i(z)s_i$ is *parallel with respect to D* if it satisfies $D\zeta_P = 0$ on $PE_{\zeta(U)}$, that is, if

$$D_\alpha \zeta^i := \frac{\partial \zeta^i}{\partial z^\alpha} + \sum_{m=1}^{r} \zeta^m(z)\Gamma^i_{m\alpha}([\zeta(z)]) = 0,$$

where $\omega^i_j := \sum \Gamma^i_{j\alpha} dz^\alpha$.

For a holomorphic section ζ of E, we put $f(z) = F(z, \zeta(z)) = H(\zeta_P, \zeta_P)$. Applying (1.9) to the function $f(z)$, we can give the complex Hessian of $f(z)$ by

$$\partial\bar\partial f(z) = -H(R(\zeta_P), \zeta_P) + H(D'\zeta_P, D'\zeta_P),$$

or, in local coordinates,

$$\text{(3.1)}\quad \frac{\partial^2 f(z)}{\partial z^\alpha \partial \bar z^\beta} = -\sum_{i,j} R_{i\bar j\alpha\bar\beta}([\zeta(z)])\zeta^i(z)\overline{\zeta^j(z)} + \sum_{i,j} F_{i\bar j}([\zeta(z)])D_\alpha\zeta^i\overline{D_\beta\zeta^j}.$$

If $\Psi_{\alpha\bar\beta}$ has at least one negative eigenvalue at every point of PE, the complex Hessian $\partial\bar\partial f$ has a positive eigenvalue at every point of PE. Hence, by (3.1) we have Kobayashi's vanishing theorem as follows.

PROPOSITION 3.1(Kobayashi [8]). *Let (E, F) be a convex Finsler vector bundle over a compact complex manifold. If $\Psi_{\alpha\bar\beta}$ has at least one negative eigenvalue at every point of PE, then there exists no nonzero holomorphic sections*:

$$H^0(M, E) = 0.$$

Remark 3.1. This vanishing theorem is due to Kobayashi [8]. The curvature $\overset{K}{R}_{i\bar{j}\alpha\bar{\beta}}$ used in [8] is different from our $R_{i\bar{j}\alpha\bar{\beta}}$. In fact, our curvature $R_{i\bar{j}\alpha\bar{\beta}}$ is defined by (1.8), and his curvature $\overset{K}{R}_{i\bar{j}\alpha\bar{\beta}}$ is defined by

$$\overset{K}{R}_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial F_{i\bar{j}}}{\partial z^\alpha \partial \bar{z}^\beta} + \sum_{k,l,} F^{k\bar{l}} \frac{\partial F_{k\bar{j}}}{\partial \bar{z}^\beta} \frac{\partial F_{i\bar{l}}}{\partial z^\alpha}.$$

But the negativity or positivity of $\sum_{i,j} \overset{K}{R}_{i\bar{j}\alpha\bar{\beta}} \xi^i \bar{\xi}^j$ coincides with the one of $\Psi_{\alpha\bar{\beta}}$, since we can take a normal coordinate system at each point of PE.

Let (E, F) be a convex Finsler vector bundle over a compact Hermitian manifold (M, g), where $g = \sum g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta$. For the curvature tensor $R_{i\bar{j}\alpha\bar{\beta}}$ of D, we put $K_{i\bar{j}} := \sum g^{\alpha\bar{\beta}} R_{i\bar{j}\alpha\bar{\beta}}$ and call it the *mean curvature* of (E, F) (cf. [5]). Then we shall define a Hermitian form K by

$$K(Z, W) := \sum_{i,j} K_{i\bar{j}} Z^i \bar{W}^j$$

for all $Z, W \in A^0(\tilde{E})$. Then we get the following Bochner-type vanishing theorem for holomorphic sections.

THEOREM 3.1. *Let (E, F) be a convex Finsler vector bundle over a compact Hermitian manifold (M, g).*

(1) *If the mean curvature K is negative semi-definite on PE, then every holomorphic section ζ of E is parallel with respect to D; that is,*

$$D\zeta_P = 0,$$

and satisfies

$$K(\zeta_P, \zeta_P) = 0.$$

(2) *If K is negative definite on PE, then E admits no nonzero holomorphic sections: $H^0(M, E) = 0$.*

Proof. By taking the g-trace of (3.1), we have

$$\Box f(z) = \left\| D'\zeta_P \right\|^2 - K(\zeta_P, \zeta_P) \tag{3.2}$$

for any holomorphic section ζ of E, where

$$\left\| D'\zeta_P \right\|^2 := \sum_{\alpha,\beta,i,j} g^{\bar{\beta}\alpha}(z) F_{i\bar{j}}([\zeta(z)]) D_\alpha \zeta^i \overline{D_\beta \zeta^j},$$

and

$$\Box f(z) = -\sum_{\alpha,\beta} g^{\alpha\bar{\beta}} \frac{\partial^2 f(z)}{\partial z^\alpha \partial \bar{z}^\beta}.$$

From (3.2) and the maximum principle of E. Hopf (cf. [8], Theorem 1.10) imply our assertions. □

3.2. *A vanishing theorem for cohomology groups.* In this last sub-section, we shall show a vanishing theorem for cohomology groups as an application of Proposition 2.2.

Let (E, F) be a convex Finsler vector bundle over a compact Kähler manifold (M, ω), where ω is its Kähler form. We assume that Ψ is semi-negative with rank $\geq k$. Then the first Chern class $c_1(LE)$ is semi-negative with rank $\geq k + r - 1$. Hence the bundle E is *semi-negative of rank* $\geq k$ (cf. [9], p. 83).

Moreover, since M is compact Kähler, the projective bundle PE is also compact Kähler. In fact, since PE is compact, we can take a sufficiently positive ϵ such that

$$\omega_{PE} := \pi^*\omega - \epsilon\sqrt{-1}\Phi$$

defines a Kähler form on PE, where Φ is the $(1, 1)$-form defined by (2.1). Then, Theorem 6.17 of [9] may be generalized as follows:

THEOREM 3.2. *Let (E, F) be a convex Finsler bundle of rank r over a complex manifold M.*

(1) *The curvature Ψ of D is semi-negative of rank $\geq k$ if and only if the curvature $\bar{\partial}\partial \log F$ of the corresponding Hermitian structure in LE is semi-negative of rank $\geq k + r - 1$.*
(2) *If the curvature Ψ of D is semi-negative of rank $\geq k$, then E is semi-negative of rank $\geq k$.*
(3) *If the curvature Ψ of D is semi-negative of rank $\geq k$, then*

$$H^q(M, \Omega^p(E)) = 0$$

for $p + q \leq k - r$, provided that M is compact Kähler.

Proof. The first and second are trivial from the definition of semi-negativity. Hence we shall prove the third assertion. If we apply the Gigante's vanishing theorem (cf. [9], p. 69) to the Hermitian line bundle (LE, F), we get

$$H^Q(PE, \Omega^P(LE)) = 0$$

for $P + Q \leq k + r - 2$, where $\Omega^P(LE)$ denotes the sheaf of LE-valued holomorphic

P-forms. On the other hand, we have the following isomorphism (cf. [9], p. 84):

$$\begin{aligned} H^q(M, \Omega^p(E)) &\sim H^{n-q}(M, \Omega^{n-p}(E^*)) \\ &\approx H^{n-q}(PE, \Omega^{n-p}((LE)^*)) \\ &\sim H^{q+r-1}(PE, \Omega^{p+r-1}(LE)), \end{aligned}$$

where $\sim$ and $\approx$ denote the Serre duality and Le Potier isomorphism respectively. This implies the third assertion. □

As a special case, if Ψ is negative definite, Theorem 2.1 implies that E is negative. Then Corollary 5.10 in [9] can be written as follows:

COROLLARY 3.1. *Let E be a holomorphic vector bundle of rank r over a compact complex manifold M of dimension n. If E admits a convex Finsler structure F with negative Ψ, then*

$$H^q(M, \Omega^p(E)) = 0$$

for $p + q \leq n - r$.

In the case where E is a holomorphic line bundle, any Finsler structure on E is a Hermitian structure, so Corollary 3.1 (Corollary 5.10 in [9]) is a generalization of Nakano's vanishing theorem.

REFERENCES

[1] M. Abate and G. Patrizio, *Finsler metrics — A Global approach with applications to geometric function theory*, Lecture Notes 1591, Springer, 1994.

[2] T. Aikou, *On complex Finsler manifolds*, Rep. Fac. Sci. Kagoshima Univ. **35** (1991), 9–25.

[3] ———, *Complex manifolds modeled on a complex Minkowski space*, J. Math. Kyoto Univ. **24** (1995), 83–101.

[4] ———, *Some remarks on locally conformal complex Berwlad spaces*, Contemporary Math. **196** (1996), 109–120.

[5] ———, *Einstein-Finsler vector bundles*, to appear in Publ. Math. Debrecen (1997).

[6] J. Bland and M. Kalka, *Variations of holomorphic curvature for Kähler Finsler metrics*, Contemporary Math. **196** (1996), 121–132.

[7] J. J. Faran, *Hermitian Finsler metric and Kobayashi metric*, J. Differential Geometry **31** (1990), 601–625.

[8] S. Kobayashi, *Negative vector bundles and complex Finsler structures*, Nagoya Math. J. **57** (1975), 153–166.

[9] ———, *Differential geometry of complex vector bundles*, Iwanami-Princeton Univ. Press, 1987.

[10] ———*Complex Finsler vector bundles*, Contemporary Math. **196** (1996), 145–153.

[11] H. L. Royden, *Complex Finsler metrics*, Contemporary Math. **49** (1986), 261–271.

1-2-35 Korimoto Kagoshima, 890 Japan
aikou@sci.kagoshima-u.ac.jp