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A SIMPLIFICATION OF ROSAY'S THEOREM ON GLOBAL SOLVABILITY OF TANGENTIAL CAUCHY-RIEMANN EQUATIONS

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In a recent paper by Rosay [6], the global solvability of the tangential Cauchy-Riemann complex $\bar{\partial}_b$ on the boundaries of weakly pseudo-convex domains is studied. He proved the following theorem:

THEOREM 1. *Let Ω be a weakly pseudo-convex domain in C^n with smooth boundary $b\Omega$. Assuming $n \geqq 2$ and $p \leqq n$, the equations*

$$(1) \qquad \bar{\partial}_b u = \alpha, \quad \text{where } \alpha \text{ is a smooth } (p, n-1) \text{ form on } b\Omega,$$

has a smooth solution u if and only if α satisfies

$$(2) \qquad \int_{b\Omega} \alpha \wedge \Phi = 0 \quad \text{for every } \bar{\partial}\text{-closed } (n-p, 0) \text{ form } \phi.$$

The same result for *strictly* pseudo-convex domains has been proved by Henkin in [2] using the integral representation for the $\bar{\partial}$ operator.

In his paper, Rosay also noted parenthetically that, following the work of Kohn and Rossi [5] and Kohn [3], the necessary and sufficient conditions for the solvability of the equations

$$(3) \quad \bar{\partial}_b u = \alpha, \quad \text{where } \alpha \text{ is a smooth } (p, q) \text{ form on } b\Omega \text{ and } q < n-1,$$

are

$$(4) \qquad \bar{\partial}_b \alpha = 0$$

Rosay's method for proving Theorem 1 is to use the solution of the $\bar{\partial}$-Neumann problem in an ingenious way. However, it is not the most direct one. In this note we shall show that, with a simple argument of integration by parts, Kohn and Rossi's method can be directly extended to $(p, n-1)$ forms, thus providing a unified approach to the solvability of equations (1) and (3).

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Let us first review how one can apply Kohn and Rossi's result to solve (3) under condition (4). Condition (4) allows one to extend α to a $\bar{\partial}$-closed form $\tilde{\alpha}$. On a weakly pseudo-convex domain, there exists a smooth $\tilde{u}$ (see [3] and [4]) such that $\bar{\partial}\tilde{u} = \tilde{\alpha}$ and the restriction of $\tilde{u}$ to the boundary gives a solution of (3). More specifically, assume r is the defining function of Ω, i.e., $\Omega = \{x | r(x) < 0\}$ and $r = 0$ on the boundary.

(i) Extend α to a smooth form α_1 on Ω such that $\alpha_1 \wedge \bar{\partial} r = \alpha \wedge \bar{\partial} r$ on $b\Omega$.
(ii) Set

$$\alpha_2 = - * \partial \overline{N * \bar{\partial}\alpha_1} \tag{5}$$

where n is the $\bar{\partial}$-Neumann operator on $(n-p, n-q-1)$ forms and $*$ is the Hodge star operator. Then by the result of Kohn and Rossi [5], for $q < n-1$, one has

$$\bar{\partial}\alpha_2 = \bar{\partial}\alpha_1 \text{ on } \Omega, \qquad \alpha_2 \wedge \bar{\partial} r = 0 \text{ on } b\Omega. \tag{6}$$

(iii) Set $\tilde{\alpha} = \alpha_1 - \alpha_2$. Then

$$\bar{\partial}\tilde{\alpha} = 0 \text{ on } \Omega, \qquad \tilde{\alpha} \wedge \bar{\partial} r = \alpha \wedge \bar{\partial} r \text{ on } b\Omega.$$

Then by the result of Kohn, one can find a smooth $\tilde{u}$ such that $\bar{\partial}\tilde{u} = \tilde{\alpha}$ and the restriction of $\tilde{u}$ to $b\Omega$ gives us a smooth solution of (3).

We note that the above argument, as in Rosay's paper, assumes the unproven existence of the $\bar{\partial}$-Neumann operator N on $(n-p, n-q-1)$ forms on a weakly pseudo-convex domain in general. Kohn [3] has proven that the weighted $\bar{\partial}$-Neumann operator exists and the proof should be limited to the weighted $\bar{\partial}$-Neumann operator and weighted L^2 space. However, since the arguments are similar with or without weight, we follow Rosay in assuming the existence of N on $(n-p, n-q-1)$ forms when $q < n-1$ for the sake of simplicity in presentation.

The three steps (i), (ii) and (iii) can also be extended to the case when α is a $(p, n-1)$ form and satisfies (2). The only part that needs justification is step (ii). The $\bar{\partial}$-Neumann operator for the $(n-p, 0)$ forms must exist and the α_2 defined by (5) must satisfy (6). A crucial observation is that if the $\bar{\partial}$-Neumann operator on $(n-p, 1)$ forms (denoted by N_1) exists, then the $\bar{\partial}$-Neumann operator on $(n-p, 0)$ forms (denoted by N_0) can also be defined (see Folland and Kohn [1] Theorem 3.1.19 and the remark after that). In fact, N_0 is defined by

$$N_0 = \vartheta N_1^2 \bar{\partial} \quad \text{for smooth } (n-p, 0) \text{ forms} \tag{7}$$

and

$$\vartheta \bar{\partial} N_0 = I - H \tag{8}$$

where ϑ is the adjoint operator of $\bar{\partial}$ and H is the Bergman Projection operator from square-integrable $(n-p, 0)$ forms into square-integrable $\bar{\partial}$-closed $(n-p, 0)$ forms.

We prove that (5) satisfies (6) in the following way. Using the relations $\vartheta = -*\partial*$ and $** = I$, we have

$$\begin{aligned}
\bar\partial\alpha_2 &= -\bar\partial(*\,\overline{\partial N_0 \overline{*\bar\partial\alpha_1}}) \\
&= \overline{*\vartheta\bar\partial N_0 \overline{*\bar\partial\alpha_1}} \\
&= *(*\bar\partial\alpha_1 - \overline{H(\overline{*\bar\partial\alpha_1})} \quad \text{by (8)} \\
&= \bar\partial\alpha_1 - \overline{*H(\overline{*\bar\partial\alpha_1})}\,.
\end{aligned}$$

We claim that under condition (2), $H(\overline{*\bar\partial\alpha_1}) = 0$. For every $\bar\partial$-closed $(n-p, 0)$ form ϕ, one has

$$\begin{aligned}
\langle \overline{*\bar\partial\alpha_1}, \phi\rangle &= \langle \vartheta * \bar\alpha_1, \phi\rangle \\
&= \langle *\bar\alpha_1, \bar\partial\phi\rangle - \int_{b\Omega}\overline{\alpha_1\wedge\phi} \quad \text{by Stoke's Theorem} \\
&= -\int_{b\Omega}\overline{\alpha\wedge\phi} \\
&= 0 \quad \text{by (2)}
\end{aligned}$$

which proves $\bar\partial\alpha_2 = \bar\partial\alpha_1$. It is easy to check that $\alpha_2 \wedge \bar\partial r = 0$ on $b\Omega$ by observing that $\bar\partial N \overline{*\bar\partial\alpha_1}$ belongs to the domain of the L^2-adjoint operator $\bar\partial^*$ of $\bar\partial$, thus its "normal part" vanishes on the boundary. This justifies step (ii) which allows one to have a $\bar\partial$-closed extension of α and hence completes our alternative proof to Theorem 1.

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