ILLINOIS JOURNAL OF MATHEMATICS
Volume 31, Number 1, Spring 1987

TWO SPACE SCATTERING AND PROPAGATIVE SYSTEMS

BY

MARTIN SCHECHTER

1. Introduction

Wilcox [27] showed that many wave propagation phenomena of classical physics are governed by systems of partial differential equations of the form

$$E(x)\frac{\partial u}{\partial t} = \sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j} \equiv -iAu \tag{1.1}$$

where $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $u(x, t)$ is a column vector of length m describing the state of the medium at position x and time t, and $E(x)$ and the A_j are $m \times m$ matrices with the following properties:

(a) $E(x)$ is real, symmetric and uniformly positive definite.

(b) The A_j are real, symmetric and constant.

From the point of view of spectral and scattering theory it is desirable that the solution of (1.1) be of the form

$$u = e^{-itH}u_0, \quad u_0(x) = u(x,0) \tag{1.2}$$

where H is a self adjoint operator. This would require that H be an extension of $E^{-1}A$. When $E = 1$, one can easily obtain a self adjoint realization H_0 of A in $\mathscr{H} = (L^2)^m$ using Fourier transforms. On the other hand, if $E \neq 1$, the operator $E^{-1}A$ need not be Hermitian on $\mathscr{H}$. However, it is Hermitian on the Hilbert space $\mathscr{H}_1$ with scalar product

$$(u, v)_1 = \int v(x)^* E(x) u(x)\, dx. \tag{1.3}$$

If $E(x)$ is uniformly bounded, it can be shown that $E^{-1}H_0$ is self adjoint on $\mathscr{H}_1$ (cf. [27]). However, when $E(x)$ is unbounded, it need not be self adjoint.

In the present paper we give sufficient conditions on the matrix $E(x)$ for the operator $E^{-1}A$ to have a self adjoint extension H on $\mathscr{H}_1$. We then study the

Received January 28, 1985.

spectrum of H and develop a scattering theory for it. In particular, we give sufficient conditions for the wave operators to exist and be complete.

Let $e_{ij}(x)$ denote the elements of the matrix $E(x)$, and let

$$\rho(x) = (1 + |x|^2)^{1/2}.$$

Our first assumption is:

(I) There are constants a, C such that

$$\sum_{i,j} \int_{|x-y|<1} |e_{ij}(y)|\, dy \leq C\rho(x)^{2a}. \tag{1.4}$$

We have:

THEOREM 1.1. *Under hypothesis* (I), *the operator $E^{-1}A$ has a self adjoint extension H on $\mathscr{H}_1$.*

Next we study the spectrum of H. It is easily checked that

$$\sigma(H_0) = (-\infty, \infty) = \mathbf{R}. \tag{1.5}$$

We give sufficient conditions that the same hold for H. We let P_0 be the projection onto $N(H_0)^{\perp}$ and put $F(x) = E(x)^{1/2}$. We have:

THEOREM 1.2. *If $D(H_0^k) \subset \mathscr{H}_1$ and $(F-1)(H_0 - i)^{-k}P_0$ is a compact operator for some k, then*

$$\sigma(H) = \mathbf{R}. \tag{1.6}$$

Another way of stating the hypothesis of Theorem 1.2 is to say that $(F-1)P_0$ is H_0^k-compact. Sufficient conditions for this to be true can be found in [16].

Next we turn to scattering theory. Let J be a bounded linear map from $\mathscr{H}$ to $\mathscr{H}_1$. We shall say that $u \in D(W_{\pm}(H, H_0, J))$ if

$$e^{itH}Je^{-itH_0}u \to f \text{ in } \mathscr{H}_1 \quad \text{as } t \to \pm\infty \tag{1.7}$$

and we set $W_{\pm}(H, H_0, J)u = f$.

First we have:

THEOREM 1.3. *Assume*

$$(Ju)^{\wedge} = J(p)\hat{u}(p) \tag{1.8}$$

where $J(p)$ is a matrix function of p which commutes with each A_j. Assume also *that there are constants* $\alpha > 1$ *and* q, $2 \le q \le \infty$, *such that*

$$(1.9) \qquad (1 + |x|)^{\alpha}\left(\int_{|x-y|<1} |F(y) - 1|^2\, dy\right)^{1/2} \in L^q.$$

Then

$$(1.10) \qquad D(W_{\pm}(H, H_0, J)) = \mathscr{H}.$$

THEOREM 1.4. *Let* $J = J(x)$ *be a matrix function of* x *and suppose there is a constant matrix* J_0 *such that* $J_0 H_0 \subset H_0 J_0$ *and*

$$(J - J_0)(i - H_0)^{-k} P_0$$

is compact for some $k \ge 0$. *If* $D(H_0^k) \subset \mathscr{H}_1$ *and* (1.9) *holds, then*

$$(1.11) \qquad N(H_0)^{\perp} \subset D(W_{\pm}(H, H_0, J)).$$

Now we strengthen our hypotheses on A. This will allow us to weaken our hypotheses on E. We shall say that the system (1.1) has *constant deficit* if the matrix

$$(1.12) \qquad A(p) = -\sum_{j=1}^{n} A_j p_j$$

has constant rank for $0 \ne p = (p_1, \ldots, p_n) \in \mathbf{R}^n$. Let S be the self adjoint realization of $(1 - \Delta)^{1/2}$ in L^2, where Δ is the Laplacian in $\mathbf{R}^n$. We shall prove the following stronger version of Theorem 1.2.

THEOREM 1.5. *If* $D(S^k) \subset \mathscr{H}_1$ *and* $(F - 1)S^{-k}P_0$ *is a compact operator for some* k, *then* (1.6) *holds.*

COROLLARY 1.6. *If*

$$\int_{|x-y|<1} |F(y) - 1|^2\, dy$$

is bounded and tends to 0 *as* $|x| \to \infty$, *then* (1.6) *holds.*

We also have the following generalization of Theorem 1.4.

THEOREM 1.7. *Assume that* (1.1) *has constant deficit and that*

$$(1.13) \qquad \int_{|x-y|<1} |F(y)|^2\, dy \in L^{\infty}.$$

Let $J = J(x)$ be a matrix function of x and suppose there is a constant matrix J_0 such that $J_0 H_0 \subset H_0 J_0$ and

$$\int_{|x-y|<1} |F(y)(J(y) - J_0)|^2 \, dy \tag{1.14}$$

is bounded and tends to 0 *as* $|x| \to \infty$. *Then* (1.9) *implies* (1.11).

Following Wilcox we shall call the system (1.1) *uniformly progagative* if the roots of

$$\det(\lambda I - A(p)) = 0 \tag{1.15}$$

have constant multiplicities and constant algebraic signs for real vectors $p \neq 0$. (A uniformly propagative system has constant deficit.) For such systems we have:

THEOREM 1.8. *Let* (1.1) *be a uniformly propagative system and let J satisfy* (1.8). *If there are constants q, α satisfying*

$$2 \leq q \leq \infty, \quad \alpha > 1 - \frac{n-1}{q} \tag{1.16}$$

such that (1.9) *holds, then* (1.10) *is true.*

THEOREM 1.9. *If* (1.1) *is uniformly propagative, then the hypotheses of Theorem* 1.7 *can be weakened to allow* (1.9) *to hold for some q, α satisfying* (1.16).

Now we turn to the question of completeness. We shall say that the wave operators $W_\pm(H, H_0, J)$ are *complete* if their domains contain $\mathscr{H}_{ac}(H_0)$ and their ranges are dense in $\mathscr{H}_{ac}(H)$. Here $\mathscr{H}_{ac}(H)$ denotes the subspace of absolute continuity of H (cf. [12]).

THEOREM 1.10. *Assume that* (1.1) *is uniformly propagative and that* (1.8) *holds. If* $D(|H_0|^{1/2}) \subset \mathscr{H}_1, C_0^\infty \subset R(J), |E(x) - 1|^{1/2}$ *is* $S^{1/2}$*-compact and, for some* $\alpha > 1$,

$$(1 + |x|)^\alpha \int_{|x-y|<1} |E(y) - 1| \, dy \in L^\infty \tag{1.17}$$

then the wave operators are complete.

THEOREM 1.11. *Let* (1.1) *be uniformly propagative, and let the hypotheses of Theorem* 1.7 *hold. Assume in addition that* $D(|H_0|^{1/2}) \subset \mathscr{H}_1$, J_0 *is invertible,*

$|E(x) - 1|^{1/2}$ *is* $S^{1/2}$ *compact and that* (1.17) *holds for some* $\alpha > 1$. *Then the wave operators are complete.*

Theorem 1.1 will be proved in the next section while Theorems 1.2 and 1.5 are proved in Section 3. The remaining proofs are given in Section 5. In Section 4 we present a new criterion for the existence of wave operators. It generalizes the work of the author in [18], [19], [21], [22]. Subsequent to our work in [18], [19], several authors generalized or simplified our results. These include Simon [25], Kato [13], Davies [16], Enss [10], Ginibre [11] and Combes-Weder [4]. Our present theorem (Theorem 4.1) generalizes all of their results.

Spectral and scattering theory for uniformly propagative systems were studied by Wilcox [27], [28], [29]. He proved the existence of the wave operators under the assumption

$$E(x) - 1 = 0(|x|^{-\alpha}) \quad \text{as } |x| \to \infty \tag{1.18}$$

for some $\alpha > 1$. Completeness was proved by Mochizuki [14], Birman [2], Deic [7], Suzuki [26], Yajima [30] under this assumption and various others. It was proved by Schulenberger-Wilcox [23], Birman [3], Deic [8] and Schulenberger [24] under the assumption

$$\int (1 + |x|)^{\beta} |E(x) - 1|^2 \, dx < \infty \tag{1.19}$$

for some $\beta > n$ together with various other stipulations. Deift [9] was able to remove the other assumptions. Schechter [17] proved completeness under assumption (1.9) with $q = \infty$ and $\alpha > 1$. This includes the other results. In all of these results it is assumed that $E(x)$ is bounded and J is the identity operator. The author's paper [21] was the first to allow $E(x)$ to be unbounded. For systems not uniformly propagative very little work has hitherto been done. Avila [1] proved the existence of the wave operators under condition (1.19) with $\beta = 4$ in addition to (a), (b) and the boundedness of $E(x)$. Nenciu [15] has considered eigenfunction expansions under the conditions that system (1.1) has constant deficit.

2. The self adjoint extension

In this section we shall construct the self adjoint extension H of $E^{-1}A$. If $A(p)$ is the matrix given by (1.12) and A is the operator on the righthand side of (1.1), then $(Au)\hat{} = A(p)\hat{u}(p)$ for any test function $u \in C_0^\infty$, where $\hat{u}$ denotes the Fourier transform of u. We shall say that $u \in D(H)$ and $Hu = f \in \mathscr{H}_1$ if $u \in \mathscr{H}_1$ and

$$\int \hat{v}(p)^* A(p) \hat{u}(p) \, dp = (f, v)_1, \quad v \in \mathscr{H}_1. \tag{2.1}$$

Clearly f is unique. Since the bilinear form on the left of (2.1) is Hermitian, the same is true of H. It is clear that H is an extension of the operator $E^{-1}A$.

To prove that H is self adjoint on $\mathscr{H}_1$, consider the norm

$$\|u\|_Y = \sup_{v \in \mathscr{H}_1} \frac{|(u, v)|}{\|v\|_1} \tag{2.2}$$

and let Y be the completion of $\mathscr{H} = [L^2]^m$ with respect to this norm. We shall show that Y consists of tempered distributions. Assume this for the moment, and note that $\mathscr{H}_1 \subset \mathscr{H} \subset Y$ with continuous inclusions. We can define the scalar product (u, v) for $u \in Y$ and $v \in \mathscr{H}_1$ with the inequality

$$|(u, v)| \leq \|u\|_Y \|v\|_1, \quad u \in Y, v \in \mathscr{H}_1. \tag{2.3}$$

If $u \in Y$, we can extend $\hat{u}$ to apply to the Fourier transforms of functions in $\mathscr{H}_1$ with

$$(\hat{u}, \hat{v}) = (u, v), \quad u \in Y, v \in \mathscr{H}_1. \tag{2.4}$$

Let $F = E^{1/2}$. Then F is a bijective map of $\mathscr{H}_1$ onto $\mathscr{H}$. Moreover, (2.2) implies

$$\|Fu\|_Y = \|u\| \tag{2.5}$$

for $u \in \mathscr{H}$. Thus we can extend F to a bijective map of $\mathscr{H}$ onto Y satisfying (2.5).

Now we turn to the proof of Theorem 1.1. Suppose $u, f \in \mathscr{H}_1$ and

$$(u, Hv)_1 = (f, v)_1, \quad v \in D(H). \tag{2.6}$$

By (2.1) this is the same as

$$\int [A(p)\hat{v}(p)]^* \hat{u}(p)\,dp = (f, v)_1, \quad v \in D(H)$$

This gives

$$\int \hat{v}(p)^* A(p)\hat{u}(p)\,dp = (g, v), \quad v \in D(H) \tag{2.7}$$

where $g = F(Ff) \in Y$. Let $\mathscr{S}$ denote the space of rapidly decreasing functions. We shall show that $\mathscr{S} \subset D(H)$. By (2.7) this implies that

$$\int \hat{v}(p)^* [A(p)\hat{u}(p) - \hat{g}(p)]\,dp = 0 \tag{2.8}$$

holds for $v \in \mathcal{S}$. Consequently $A(p)\hat{u}(p) = \hat{g}(p)$ is the Fourier transform of an element in Y. Thus (2.8) holds for all $v \in \mathcal{S}_1$. Hence

$$\int \hat{v}(p)^* A(p)\hat{u}(p)\, dp = (g, v) = (f, v)_1, \quad v \in \mathcal{H}_1 \tag{2.9}$$

This shows that $u \in D(H)$ and $Hu = f$.

It remains to show that Y consists of tempered distributions and $\mathcal{S} \subset D(H)$. We do this by proving that there are constants C, k such that

$$\|F\phi\| \leq C \sum_{|\mu| \leq k} \sup_{\mathbf{R}^n} \left|\rho(x)^k D^\mu \phi(x)\right| \tag{2.10}$$

holds for all $\phi \in \mathcal{S}$. Once we have (2.10), we see that $\mathcal{S} \subset \mathcal{H}_1$ and consequently Y consists of tempered distributions. Also, if $u \in \mathcal{S}$, then $A(p)\hat{u}(p)$ is in $\mathcal{H}$. Hence the lefthand side of (2.1) is a bounded linear functional on $\mathcal{H}_1$. Thus there is an $f \in \mathcal{H}_1$ such that (2.1) holds. Consequently $u \in D(H)$.

To prove (2.10) we let $S = (1 - \Delta)^{1/2}$, where Δ is the self adjoint realization of the Laplacian in L^2. We note that (1.4) implies that there are constants N, C_1 such that

$$\|F\rho^{-a}\phi\| \leq C_1\|S^N\phi\|, \quad \phi \in \mathcal{S}$$

(cf. [16, p. 105]). Thus

$$\|F\rho^{-a}\phi\| \leq C' \sum_{|\mu| \leq 2N} \|D^\mu\phi\| \leq C'' \sum_{|\mu| \leq 2N} \sup \rho(x)^{n/2+1}|D^\mu\phi(x)|.$$

This implies (2.10), and the proof of Theorem 1.1 is complete. □

3. Spectral theory

In proving Theorems 1.2 and 1.5 we shall use a few lemmas. We denote the essential spectrum of an operator T by $\sigma_e(T)$ (cf. [16]). Let $\hat{H}_0$ be the restriction of H_0 to $R(P_0) = N(H_0)^\perp$. We have:

LEMMA 3.1. $\sigma_e(H_0) \subset \sigma_e(\hat{H}_0) \cup \{0\}$.

Proof. If $0 \neq \lambda \in \sigma_e(H_0)$, then there is a sequence $\{u_k\} \subset D(H_0)$ such that

$$\|u_k\| = 1, \quad u_k \rightharpoonup 0, \quad (\lambda - H_0)u_k \to 0 \tag{3.1}$$

(cf. [16]). Thus

$$\lambda(1 - P_0)u_k = (\lambda - H_0)(1 - P_0)u_k = (1 - P_0)(\lambda - H_0)u_k \to 0$$

and consequently,

$$\|P_0 u_k\| \to 1, \quad P_0 u_k \rightharpoonup 0, \quad (\lambda - H_0) P_0 u_k = P_0(\lambda - H_0) u_k \to 0.$$

Thus $\lambda \in \sigma_e(\hat{H}_0)$ (cf. [16]). □

LEMMA 3.2. *If r is an even integer and λ is real, then*

$$M_r = \sum_{j=0}^{r} \lambda^j H_0^{r-j} \geq \frac{1}{2}(\lambda^r + H_0^r). \tag{3.2}$$

Thus M_r has a bounded inverse of $\lambda \neq 0$.

Proof. We use induction on r. The lemma is true if $r = 0$. Assume it is true for r. Then

$$\begin{aligned} M_{r+2} &= \lambda^{r+2} + \lambda^{r+1} H_0 + H_0^2 M_r \\ &\geq \lambda^{r+2} + \lambda^{r+1} H_0 + \frac{1}{2}\lambda^r H_0^2 + \frac{1}{2} H_0^{r+2} \\ &= \lambda^r\left(\lambda^2 + \lambda H_0 + \frac{1}{2} H_0^2\right) + \frac{1}{2} H_0^{r+2} \\ &\geq \frac{1}{2}\left(\lambda^{r+2} + H_0^{r+2}\right) \end{aligned}$$

since

$$\lambda^2 + \lambda H_0 + \frac{1}{2} H_0^2 \geq \frac{1}{2}\lambda^2. \tag{3.3}$$ □

LEMMA 3.3. *For $j < r$ we have*

$$\|H_0^j u\|^2 \leq C\left(\|H_0^r u\|^2 + \|u\|^2\right), \quad u \in D(H_0^r) \tag{3.4}$$

Proof. For any $a > 0$ we have

$$2\|H_0 u\|^2 = 2\left(H_0^2 u, u\right) \leq a\|H_0^2 u\|^2 + a^{-1}\|u\|^2.$$

An induction down gives

$$\|H_0^j u\|^2 \leq C\left(\|H_0^r u\|^2 + \|H_0^{j-1} u\|^2\right).$$

An induction up now gives (3.4). □

LEMMA 3.4. *If the hypotheses of Theorem 1.2 hold, then*

$$\sigma_e(H_0) \subset \sigma_e(H) \cup \{0\} \tag{3.5}$$

Proof. Let $\lambda \neq 0$ be real. Then $\lambda \in \sigma(H_0)$. We may assume that k is a positive odd integer. By the spectral mapping theorem $\lambda^k \in \sigma(H_0^k) = \sigma_e(H_0^k)$. By Lemma 3.1 there is a sequence $\{v_j\} \subset D(H_0^k) \cap N(H_0)^\perp$ such that

$$\|v_j\| = 1, \quad v_j \rightharpoonup 0, \quad (\lambda^k - H_0^k)v_j \to 0. \tag{3.6}$$

Now $(\lambda^k - H_0^k) = M_{k-1}(\lambda - H_0)$, where M_{k-1} is given by (3.2). By Lemma 3.2, M_{k-1} has a bounded inverse. Thus

$$(\lambda - H_0)v_j \to 0 \tag{3.7}$$

Note that $v_j \in \mathscr{H}_1$ and consequently it is in $D(H)$. Moreover

$$\begin{aligned} F(\lambda - H)v_j &= F(\lambda - E^{-1}H_0)v_j = F^{-1}(\lambda E - H_0)v_j \\ &= \lambda(1 + F^{-1})(F - 1)(H_0 - i)^{-k} \\ &\quad \times P_0(H_0 - i)^k v_j + F^{-1}(\lambda - H_0)v_j. \end{aligned} \tag{3.8}$$

Since

$$(H_0 - i)^k + \sum_{j=0}^{k} \binom{k}{j} (-i)^{k-j} H_0^j$$

we see from Lemma 3.3 and (3.6) that $\|(H_0 - i)^k v_j\| \leq C$ for some constant C. Thus there is a subsequence (also denoted by $\{v_j\}$) such that $(H_0 - i)^k v_j$ converges weakly. Since $v_k \rightharpoonup 0$, we must have

$$(H_0 - i)^k v_j \rightharpoonup 0 \tag{3.9}$$

This together with (3.7), (3.8) and the hypothesis implies

$$F(\lambda - H)v_j \to 0 \tag{3.10}$$

Moreover since

$$(F - 1)v_j = (F - 1)(H_0 - i)^{-k} P_0 (H_0 - i)^k v_j \to 0$$

we see that $Fv_j \rightharpoonup 0$. Since

$$1 = \|v_j\| = \|F^{-1}Fv_j\| \leq \|F^{-1}\| \, \|v_j\|,$$

we see that $\lambda \in \sigma_e(H)$. □

It is now a simple matter to give the following proof.

Proof of Theorem 1.2. By (1.5) we know that $\sigma_e(H_0) = \mathbf{R}$ (cf. [16]). Lemma 3.4 tells us that $\mathbf{R} - \{0\} \subset \sigma_e(H)$. Since $\sigma_e(H)$ is a closed set, we must have $\sigma_e(H) = \mathbf{R}$. □

In proving Theorem 1.5 we shall use:

LEMMA 3.5. *If the system* (1.1) *has constant deficit, then there is a self adjoint operator* H_1 *such that* $D(H_1) = D(S)$, $P_0 H_1 \subset H_1 P_0 = H_0$, *and* $H_1 S = SH_1$.

Proof. For $p \subset \mathbf{R}^n - \{0\}$, let $\hat{P}_0(p)$ be the orthogonal projection of $\mathbf{C}^m$ onto $N(A(p))^{\perp}$. Since the system (1.1) has constant deficit, the dimension of $R(\hat{P}_0(p))$ is constant. It is easily checked that $\hat{P}_0(p)$ is analytic and homogeneous of degree 0. Put

$$B(p) = A(p) + |p|(1 - \hat{P}_0(p)). \tag{3.10}$$

Note that $B(p)$ is homogeneous of degree 1 and det $B(p) \neq 0$. Thus there are constants $C > c_0 > 0$ such that

$$c_0|p|\,|u| \leq |B(p)u| \leq C|p|\,|u|. \tag{3.11}$$

Hence $B(p)\hat{u} \in \mathscr{H}$ iff $|p|\hat{u} \in \mathscr{H}$. Define the operator H_1 by

$$(H_1 u)\hat{} = B(p)\hat{u}$$

Thus $u \in \mathscr{H}$ is in $D(H_1)$iff $B(p)\hat{u} \in \mathscr{H}$. By (3.9),

$$\|Su\| \leq C(\|H_1 u\| + \|u\|) \leq C'\|Su\|. \tag{3.12}$$

Thus $D(S) = D(H_1)$. Itis easily checked that $(P_0 u)\hat{} = \hat{P}_0(p)\hat{u}$. Since

$$\hat{P}_0(p)B(p) \subset B(p)\hat{P}_0(p) = A(p)$$

we see that $P_0 H_1 \subset H_1 P_0 = H_0$. □

Now it is a simple matter to give the next proof.

Proof of Theorem 1.5. By Lemma 3.5 we have

$$(H_0 - i)^{-k} P_0 = (H_1 - i)^{-k} P_0 = S^{-k} P_0 \left[S(H_1 - i)^{-1}\right]^k.$$

Since $D(S) = D(H_1)$, and they are closed operators, it follows that $S(H_1 - i)^{-1}$ is bounded. Hence the hypotheses of Theorem 1.2 are satisfied.

Corollary 1.6 follows from Theorem 1.5 if we note that $D(S^k) = H^{k,2}$. We take k large and apply the results of [16]. □

4. Existence of wave operators

In proving Theorem 1.3 we shall use an abstract theorem which we state and prove below. A weaker form was proved by Schechter [19]. Several generalizations of this have appeared. They are all special cases of our Theorem 4.1. They are given as corollaries.

Let H_0, H be self adjoint operators on Hilbert spaces $\mathscr{H}_0$, $\mathscr{H}$, respectively. Let J be a bounded operator from $\mathscr{H}_0$ to $\mathscr{H}$. Set

$$u_{0t} = e^{-itH_0}u, \quad v_t = e^{-itH}v, \quad W(t)u = e^{itH}Ju_{0t} \tag{4.1}$$

For a particular element $u \in \mathscr{H}_0$ we assume that there are complex valued functions $f(\lambda)$, $g(\lambda)$, a real number a and a function $\phi(t)$ from $[a, \infty)$ to $\mathscr{H}$ such that:

(1) $D(g(H)^*)$ is dense in $\mathscr{H}$.
(2) $u \in D(H_0) \cap D[f(H_0)]$.
(3) $W(a)u \in D(g(H))$.
(4) For $t \geq a$,

$$(Ju_{0t}, Hg(H)^*v) - (JH_0u_{0t}, g(H)^*v) = (\phi(t), v), \qquad v \in D(Hg(H)^*). \tag{4.2}$$

(5) The function $\phi(t)$ satisfies $\int_a^\infty \|\phi(t)\|\, dt < \infty$.

THEOREM 4.1. *Under the above hypotheses the following conclusions hold*:
(a) *$g(H)W(t)u$ converges to some element h in $\mathscr{H}$*;
(b)

$$\limsup_{t\to\infty} \| W(t)f(H_0)u - h\| \leq \limsup_{t\to\infty} \| [g(H)J - Jf(H_0)]\, u_{0t}\|;$$

(c)

$$\limsup_{s,t\to\infty} \| [W(s) - W(t)]\, f(H_0)u\| \leq 2 \limsup_{t\to\infty} \| [g(H)J - Jf(H_0)]\, u_{0t}\|.$$

Proof. We observe that for $a \leq s \leq t$,

$$\begin{aligned}(Ju_{0t}, g(H)^*v_t) &- (Ju_{0s}, g(H)^*v_s)\\ &= i\int_s^t \{(Ju_{0\sigma}, Hg(H)^*v_\sigma) - (JH_0u_{0\sigma}, g(H)^*v_\sigma)\}\, d\sigma\\ &= i\int_s^t (\phi(\sigma), v_\sigma)\, d\sigma, \quad v \in D(Hg(H)^*)\end{aligned} \tag{4.3}$$

by hypothesis (4). In particular, we have

$$(W(t)u, g(H)^*v) = (W(a)u, g(H)^*v) + i\int_a^t (\phi(\sigma), v_\sigma)\, d\sigma.$$

By hypotheses (3) and (5) we see that there is a constant C such that

$$|(W(t)u, g(H)^*v)| \le C\|v\|, \quad v \in D(Hg(H)^*) \tag{4.4}$$

A density argument now shows that (4.4) holds for all $v \in D(g(H)^*)$. Thus it follows that $W(t)u \in D(g(H))$ for $t \ge a$. Moreover (4.3) implies

$$|(g(H)[W(t) - W(s)]u, v)| \le \|v\| \int_s^t \|\phi(\sigma)\|\, d\sigma$$

for $v \in D(Hg(H)^*)$. Hypothesis (1) now gives

$$\|g(H)[W(t) - W(s)]u\| \le \int_s^t \|\phi(\sigma)\|\, d\sigma \to 0 \tag{4.5}$$

as $s, t \to \infty$. This proves conclusion (a). To prove (b) and (c) note that

$$W(t)f(H_0)u = e^{itH}(Jf(H_0) - g(H)J)u_{0t} + g(H)W(t)u \qquad \square$$

Let M be the set of those $u \in \mathscr{H}_0$ such that $W(t)u$ converges in $\mathscr{H}$. We have:

COROLLARY 4.2. *If in addition to the hypotheses of Theorem* 4.1 *we assume*

$$[g(H)J - Jf(H_0)]u_{0t} \to 0 \quad \text{as } t \to \infty, \tag{6}$$

then $f(H_0)u \in M$.

COROLLARY 4.3 (Combes-Weder [4]). *Assume that there are a bounded complex valued function* $f(\lambda)$, *an operator* A *from* $D(H_0)$ *to a Banach space* $\mathscr{K}$, *a bounded operator* B *from* $\mathscr{H}$ *to* $\mathscr{K}'$ *and a set* $D \subset D(H_0 f(H_0))$ *such that*:

(i) $f(H_0)$ *on* D *has a range dense in* $\overline{D}$;

(ii) *for* $u \in D$, $v \in D(H)$ *we have*

$$(f(H)Jf(H_0)u, Hv) - (f(H)JH_0f(H_0)u, v) = \langle Au, Bv\rangle; \tag{4.6}$$

(iii) *for* $u \in D$,

$$[f(H)J - Jf(H_0)]u_{0t} \to 0; \tag{4.7}$$

(iv) *for each* $u \in D$ *there is an* $a > 0$ *such that*

$$\int_a^\infty \|Au_{0t}\|_{\mathscr{K}}\, dt < \infty. \tag{4.8}$$

Then $\overline{D} \subset M$.

Proof. By hypothesis (i), we can replace u by $f(H_0)u$ in (4.7). Thus $f(H_0)^2 u \in M$ for each $u \in D$ by Corollary 4.2. The result now follows from hypothesis (i). □

COROLLARY 4.4 (Ginibre [11]). *Assume that*

$$\|E(CI)JE(I_0)\| \to 0 \quad \text{as } |I| \to \infty \tag{4.9}$$

for each bounded interval I_0 *and that there is a dense subset* $D \subset \mathscr{H}_{ac}(H_0)$ *such that*

$$\int_1^\infty \|E(I)[HJ - JH_0]E_0(I_0)u_{0t}\|\, dt < \infty \tag{4.10}$$

for each $u \in D$ *and bounded intervals* I, I_0. *Then* $\mathscr{H}_{ac}(H_0) \subset M$.

Proof. Let I_0 be any bounded interval, and let $\varepsilon > 0$ be given. Let I be a bounded interval such that $\|E(CI)JE_0(I_0)\| < \varepsilon$. Let $f(\lambda)$, $g(\lambda)$ be the characteristic functions of the intervals I_0, I, respectively. For $u \in \mathscr{H}_{ac}(H_0)$, $v \in \mathscr{H}$ we have

$$\begin{aligned}
&(JE_0(I_0)u_{0t}, HE(I)v) - (JH_0E_0(I_0)u_{0t}, E(I)v)\\
&\quad = (E(I)[HJ - JH_0]E_0(I_0)u_{0t}, v)\\
&\quad = (\phi(t), v)
\end{aligned}$$

For $u \in \mathscr{H}_{ac}(H_0)$ we apply Theorem 4.1 to $E_0(I_0)u$. Hypotheses (1)–(5) are easily verified. From conclusion (c) we have

$$\begin{aligned}
&\limsup_{s,t\to\infty} \|[W(s) - W(t)]E_0(I_0)u\|\\
&\quad \le 2\|[E(I)J - JE_0(I_0)]E_0(I_0)\|\,\|u\|\\
&\quad = 2\|E(CI)JE_0(I_0)\|\,\|u\|\\
&\quad < 2\varepsilon\|u\|
\end{aligned}$$

Since ε was arbitrary, we conclude that $E_0(I_0)u \in M$. Since I_0 was arbitrary we conclude that $u \in M$. □

COROLLARY 4.5 (Kato [13]). *For* Im $z \neq 0$ *let*

$$C(z) = R(z)J - JR_0(z).$$

If

$$\int_1^\infty \|C(z)u_{0t}\|\,dt < \infty, \tag{4.11}$$

then $u \in M$.

Proof. Put $f(\lambda) = (z - \lambda)^{-1}$ in Corollary 4.3. Note that

$$\begin{aligned}(R(z)JR_0(z)w, Hv) &- (R(z)JR_0(z)H_0w, v)\\ &= -(JR_0(z)w, v) + (R(z)Jw, v)\\ &= (C(z)w, v)\end{aligned}$$

for $w \in \mathscr{H}_0$, $v \in D(H)$. Let D be the set of those $u \in \mathscr{H}_0$ satisfying (4.11). We note that

$$C(z)u_{0t} \to 0 \text{ as } t \to \infty, \quad u \in D \tag{4.12}$$

To see this, let $u \in D$, and let

$$v_a = \frac{1}{a}\int_0^a u_{0t}dt \to u \quad \text{as } a \to 0.$$

Then

$$\begin{aligned}\|C(z)v_{a,0\sigma}\| &= \frac{1}{a}\left\|\int_0^a C(z)u_{o,\,t+\sigma}dt\right\|\\ &\leq \frac{1}{a}\int_\sigma^\infty \|C(z)u_{0\tau}\|d\tau \to 0 \quad \text{as } \sigma \to \infty\end{aligned}$$

Since $C(z)$ is bounded, we get (4.12). The result now follows from Corollary 4.3.

COROLLARY 4.6 (Davies [6]). *Assume* $u \in D(H_0)$ *and there are constants* $\alpha \geq 1, \beta \geq 1$ *such that* $R(z)^\alpha[HJ - JH_0]R_0(z)^\beta u_{0t} \in L^1$. *Assume also that there is a function* $F(\lambda)$ *such that* $|F(\lambda)| \geq 1$ *and* $|F(\lambda)| \to \infty$ *as* $|\lambda| \to \infty$ *and*

$$\limsup_{t\to\infty} \|F(H)JR_0(z)^\beta u_{0t}\| < \infty$$

Then $u \in M$.

Proof. Setting $f(z) = (z - \lambda)^{-\beta}$, $g(z) = (z - \lambda)^{-\alpha}$ in Theorem 4.1 we see that $R(z)^{\alpha}W(t)R_0(z)^{\beta}u$ converges. For each bounded I, $(z - H)^{\alpha}E(I)$ is bounded. Consequently $E(I)W(t)R_0(z)^{\alpha}u$ converges as well. On the other hand,

$$\|E(CI)W(t)R_0(z)^{\beta}u\| \leq \sup_{CI}|F(\lambda)|^{-1}\|F(H)JR_0(z)^{\beta}u_{0t}\|.$$

Thus $W(t)R_0(z)^{\beta}u$ converges. Hence $R_0(z)^{\beta}u \in M$. Therefore $u \in M$. □

COROLLARY 4.7 (Enss [10]). *Assume that* $g(H)[HJ - JH_0]g(H_0)u_{0t} \in L^1$ *for every* $g \in C_0^{\infty}$ *and every* u *such that* $u = g(H_0)u$. *If there is a* z *such that* $C(z)$ *is a compact operator, the* $\mathscr{H}_{ac}(H_0) \subset M$.

Proof. If $C(z)$ is compact, the same is true of $g(H)J - Jg(H_0)$. We apply Corollary 4.2 with $f(\lambda) = g(\lambda) \in C_0^{\infty}$, $u \in \mathscr{H}_{ac}(H_0)$ such that $u = g(H_0)u$. Note that hypothesis (6) holds. Hence M contains all such u. Since they are dense in $\mathscr{H}_{ac}(H_0)$, the result follows. □

COROLLARY 4.8 (Schechter [18], [19]). *Assume that there are a Banach space* $\mathscr{K}$ *and linear operators* A *from* $\mathscr{H}_0$ *to* $\mathscr{K}$ *and* B *from* $\mathscr{H}$ *to* $\mathscr{K}'$ *such that* $D(H_0) \subset D(A)$, B *is* H*-bounded and*

$$(Ju_{0t}, Hg) - (JH_0u_{0t}, g) = \langle Au_{0t}, Bg\rangle \tag{4.13}$$

holds for some $u \in D(H_0)$ *and all* $g \in D(H)$. *If* u *satisfies* (4.8), *then* $u \in M$.

Proof. Note that (4.13) implies

$$[BR(\bar{z})]^{*}Au_{0t} = C(z)(z - H_0)u_{0t}. \tag{4.14}$$

Thus (4.8) implies that $(z - H_0)u \in M$ by Corollary 4.5. Since M is a reducing subspace of H_0, we see that $u \in M$. □

5. Scattering

Now we prove Theorem 1.3 by applying Theorem 4.1. We note that

$$\begin{aligned}(Ju, Hv)_1 - (JH_0u, v)_1 &= (Ju, H_0v) - (FJH_0u, Fv)\\ &= ([F^{-1} - F]JH_0u, Fv)\\ &= ([1 - F]JH_0u, (1 + F)v), \quad u \in D(H_0), v \in D(H).\end{aligned} \tag{5.1}$$

Let $\lambda_1(p), \ldots, \lambda_r(p)$ denote the roots of

$$\det(\lambda I - A(p)) = 0 \tag{5.2}$$

which do not vanish identically. We shall need the following result.

LEMMA 5.1. *There exists a closed set N of measure 0 and measurable functions $v_1(p), \ldots, v_m(p)$ from $\mathbf{R}^n$ to $\mathbf{R}^m$ such that:*

(a) *The $\lambda_j(p)$ are analytic on $CN = \mathbf{R}^n - N$ with $\lambda_j(p) \neq 0, \nabla\lambda_j(p) \neq 0$ there.*

(b) *The $v_j(p)$ are analytic on CN and orthonormal in $\mathbf{R}^m$ for each $p \in CN$.*

(c) $[A(p) - \lambda_j(p)]v_j(p) = 0$, *and every function $u \in \mathscr{H}$ can be written in the form*

$$\hat{u}(p) = \sum_{j=1}^{m} w_j(p)v_j(p) \quad \textit{where } \{w_1(p), \ldots, w_m(p)\} \in \mathscr{H}. \tag{5.3}$$

Proof. This was essentiallly proved by Avila [1] and Wilcox [27], [28]. Avila showed that the set where the $\lambda_j(p)$ vanishes has measure 0.

Wilcox [27] showed that the $\lambda_j(p)$ are continuous and that their multiplicities are constant on the complement of a closed set of measure 0. He showed how to construct the $v_j(p)$ on this complement. It follows from his work [29] that the $\lambda_j(p)$ and $v_j(p)$ are analytic there as well. The set of points where $\nabla\lambda_j(p) = 0$ is also closed and of measure 0. □

Proof of Theorem 1.3. Let Q be the set of those $u \in \mathscr{H}$ which satisfy (5.3) with the $w_j(p) \in C_0^\infty(CN)$. This set is dense in $\mathscr{H}$. Assume for the moment that $J = J(p) \in C^\infty$. Then we have

$$\left[e^{itH_0}JH_0u\right]^\wedge = \sum_{j=1}^{m} e^{-it\lambda_j(p)}J(p)\lambda_j(p)w_j(p)v_j(p). \tag{5.4}$$

Note that all components of this vector are in $C_0^\infty(CN)$. By a lemma of Veselic and Weidmann [31] for each real s there is a constant C_s such that

$$|e^{itH_0}JH_0u| \le C_s|t|^{-s}\rho(x)^s, \quad |t| \ge 1 \tag{5.5}$$

where $\rho(x) = 1 + |x|$. Let $\phi(x)$ be a function in $\mathscr{S}$ such that $\hat{\phi}(p) \in C_0^\infty(CN)$ and $\hat{\phi}(p) = 1$ on the support of $\hat{u}(p)$. Let $V(y) = |F(y) - 1|^2$. We are going to show that (1.9) implies

$$\rho(x)^\alpha\left(\int V(y)|\phi(y-x)|\,dy\right)^{1/2} \in L^q. \tag{5.6}$$

Assume this for the moment. Put $h = JH_0u$. Then $\hat{h}_{0t} = \hat{\phi}\hat{h}_{0t}$ and consequently $h_{0t} = \phi * h_{0t}$. Hence

$$(5.7)\quad |([F-1]h_{0t}, v)| \le \iint V(y)^{1/2}|\phi(y-x)h_{0t}(x)v(y)|\,dx\,dy$$
$$\le \left(\iint V(y)|\phi(y-x)||h_{0t}(x)|^2\,dx\,dy\right)^{1/2} \|\phi\|_1\|v\|$$

Now assume that (5.6) holds for $\alpha > 1$, $q = 2$. Take $s = \alpha$ in (5.5). Then (5.7) gives

$$(5.8)\qquad \|[F-1]h_{0t}\| \le C\|\rho^\alpha V_\phi\|\,|t|^{-\alpha}, \quad |t| \ge 1$$

where

$$(5.9)\qquad V_\phi(x) = \left(\int V(y)|\phi(y-x)|\,dy\right)^{1/2}.$$

This shows that the lefthand side of (5.8) is in $L^1(|t| \ge 1)$. We can now apply Theorem 4.1 to obtain the desired conclusion.

Next suppose (5.6) holds for $\alpha > 1$, $q = \infty$. Take s in (5.5) so that

$$(5.10)\qquad s > \max\left[1, \frac{n}{2(\alpha-1)}\right]$$

and let ν satisfy

$$(5.11)\qquad \alpha^{-1} < \nu < \frac{2s-2}{2s-2\alpha+n}.$$

Then we have

$$(5.12)\qquad \int_{\rho<|t|^\nu} V_\phi(x)^2|h_{0t}(x)|^2\,dx$$
$$\le C|t|^{-2s}\int_{\rho<|t|^\nu} \rho(x)^{2(s-\alpha)}\,dx \le C|t|^{(2s-2\alpha+n)\nu-2s}$$

and

$$(5.13)\qquad \int_{\rho>|t|^\nu} V_\phi(x)^2|h_{0t}(x)|^2\,dx \le C|t|^{-2\alpha\nu}\|h\|.$$

In view of inequalities (5.10) and (5.11), we see from (5.7), (5.12) and (5.13) that the lefthand side of (5.8) is again in $L^1(|t| \ge 1)$. If (5.6) holds for

$\alpha > 1, 2 < q < \infty$, the same result follows from interpolation. Thus by Theorem 4.1 we see that (1.10) holds.

Now assume $J(p)$ arbitrary. Since it maps $\mathscr{H}$ into $\mathscr{H}_1$ it must be a bounded function of p. For any u such that $\hat{u} \in C_0^\infty$ and any $\varepsilon > 0$ we can find a matrix $J_1(p) \in C^\infty$ commuting with $A(p)$ such that

$$\int |J(p) - J_1(p)|^2 |\hat{u}(p)|^2 \, dp < \varepsilon^2.$$

If we let

$$W(J, t) = e^{itH} J e^{itH_0}, \tag{5.14}$$

we have

$$\begin{aligned} \| W(J, t)u - W(J, \tau)u\|_1 & \\ \le \| W(J - J_1, t)u\|_1 &+ \| W(J_1, t)u - W(J_1, \tau)u\|_1 \\ &+ \| W(J_1 - J, \tau)u\|_1 \end{aligned}$$

The first and third terms on the right are bounded by $2\|u\|$ while the middle term tends to 0 as $t, \tau \to \infty$ by what has just been proved. Since such u are dense in L^2, (1.10) holds for J.

It remains to show that (1.9) implies (5.6). This follows from the next result.

LEMMA 5.2. *Let $\phi(x)$ be a function satisfying $|\phi(x)| \le C\rho(x)^{-b}$ for some $b > \alpha + n$, where $\alpha \ge 0$. If*

$$\tilde{h}(x) = \int_{|x-y|<1} |h(y)| \, dy, \tag{5.15}$$

then for any $p, 1 \le p \le \infty$,

$$\rho^\alpha [h * \phi]\|_p \le C \|\rho^\alpha \tilde{h}\|_p \tag{5.16}$$

Proof. Let

$$A_k(x) = \int_{k<|x-y|<k+1} |h(y)\phi(x-y)| \, dy,$$

$$\tilde{h}_\alpha(x) = \int_{|x-y|<1} |h(y)| \rho(y)^\alpha \, dy$$

Since $\rho(y) \le \rho(x)\rho(x-y)$, we know that

$$\tilde{h}_\alpha(x) \le 2^\alpha \rho(x)^\alpha \tilde{h}(x). \tag{5.17}$$

Let $z_1^{(k)}, \ldots, z_{N(k)}^{(k)}$ be points in the set $k < |z| < k + 1$ such that this set is covered by the $N(k)$ balls of radius 1 centered at the $z_j^{(k)}$. We know that $N(k) \leq Ck^{n-1}$. Now

$$\rho(x)^{\alpha} A_k(x) \leq \int_{k<|x-y|<k+1} |h(y)\rho(y)^{\alpha}\rho(x-y)^{\alpha}\phi(x-y)|\,dy$$

$$\leq C\rho(k)^{\alpha-b} \sum_{j=1}^{N(k)} \tilde{h}_{\alpha}\left(x + z_j^{(k)}\right)$$

Thus

$$\|\rho^{\alpha}A_k\|_p \leq C'\rho(k)^{\alpha-b+n-1}\|\tilde{h}_{\alpha}\|_p$$

The lefthand side of (5.16) is bounded by

$$\sum_{k=1}^{\infty} \|\rho^{\alpha}A_k\|_p \leq C''\|\tilde{h}_{\alpha}\|_p \sum_{k=1}^{\infty} \rho(k)^{\alpha+n-b-1}.$$

The result now follows from (5.17). □

Proof of Theorem 1.4. The operator $J_0(i - H_0)^{-k}$ is a bounded operator from $\mathscr{H}$ to $\mathscr{H}_1$ and it satisfies (1.8). Thus

$$D\left(W_{\pm}\left(H, H_0, J_0(i - H_0)^{-k}\right)\right) = \mathscr{H}$$

by Theorem 1.3. On the other hand $R(P_0) = \mathscr{H}_{ac}(H_0)$, the subspace of absolute continuity of H_0 (cf. [12]). Thus

$$(5.18) \qquad e^{itH_0}P_0u \rightharpoonup 0 \text{ in } \mathscr{H}, \quad u \in \mathscr{H}.$$

By hypothesis, this implies

$$(J - J_0)(i - H_0)^{-k}P_0e^{itH_0}P_0u \to 0 \quad \text{in } \mathscr{H}$$

This shows that

$$D\left(W_{\pm}\left(H, H_0, J(i - H_0)^{-k}P_0\right)\right) = D\left(W_{\pm}\left(H, H_0, J_0(i - H_0)^{-k}P_0\right)\right) = \mathscr{H}$$

and

$$W_{\pm}\left(H, H_0, J(i - H_0)^{-k}P_0\right) = W_{\pm}\left(H, H_0, J_0(i - H_0)^{-k}P_0\right)$$

Thus $W(J, t)(i - H_0)^{-k}P_0u$ converges as $t \to \pm\infty$, where $W(J, t)$ is given by

(5.14). This means that $W(J, t)v$ converges for $v \in R(P_0) \cap D(H_0^k)$. Since this set is dense in $R(P_0)$ and $\| W(J, t)\| \leq \|J\|$, we see that (1.11) holds. □

To prove Theorem 1.7 we note that its hypotheses imply those of Theorem 1.4 in the case of a system with constant deficit (see the proof of Corollary 1.6 in Section 3).

Proof of Theorem 1.8. We follow the proof of Theorem 1.3. In this case we can replace (5.4) with

$$(5.19) \qquad \left[e^{-itH_0}JH_0u\right]^{\wedge} = \sum_{j=1}^{r} e^{-it\lambda_j(p)}J(p)\lambda_j(p)P_j(p)\hat{u}(p)$$

where r is the rank of the matrix (1.12) and the $P_j(p)$ are homogeneous matrix functions of degree 0 (cf. [27]). It was shown in [31] that there is a dense set Q in $\mathscr{H}$ such that each $u \in Q$ satisfies $\hat{u} \in C_0^\infty$ and for each u in Q and each real number s there is a constant C_s such that

$$(5.20) \qquad |e^{-itH_0}JH_0u| \leq C_s|t|^{-s-(n-1)/2}\rho(x)^s, \quad |t| \geq 1$$

We use this inequality in place of (5.5). If we use this in (5.7), we obtain

$$(5.21) \qquad \|[F-1]h_{0t}\| \leq C\|\rho^\alpha V_\phi\|\,|t|^{-\alpha-(n-1)/2}, \quad |t| \geq 1$$

in place of (5.8). Thus in this case all we need is $\alpha + 1/2(n-1) > 1$, which is the new inequality for α when $q = 2$. If we now proceed as in the proof of Theorem 1.3 we get the desired result via interpolation (cf. [17]). □

To prove Theorem 1.9, follow the proof of Theorem 1.4 replacing Theorem 1.3 with Theorem 1.8. Note also that the hypotheses of Theorem 1.7 imply those of Theorem 1.4 (see the proof of Corollary 1.6 in Section 3).

In proving Theorem 1.10 we shall make use of the following theorem proved in [2].

LEMMA 5.3. *Let $H_0(H)$ be a self adjoint operator on a Hilbert space $\mathscr{H}_0(\mathscr{H})$ and let J be a bounded linear operator from $\mathscr{H}_0$ to $\mathscr{H}$. Assume*:

(I) *There are a Hilbert space $\mathscr{K}$ and linear operators A from $\mathscr{H}_0$ to $\mathscr{K}$ and B from $\mathscr{H}$ to $\mathscr{K}$ such that $D(H_0) \subset D(A)$, $D(H) \subset D(B)$ and*

$$(5.22) \quad (Ju, Hv) - (JH_0u, v) = (Au, Bv)_{\mathscr{K}}, \quad u \in D(H_0), v \in D(H)$$

(II) *$AR_0(i)$ and $BR(i)$ are bounded operators.*

(III) *There is an open subset Q of* $\mathbf{R}$ *with complement of measure* 0 *such that*

$$(5.23) \quad \limsup_{a\to 0^+} \sup_{s\in I} \left\{a\|AR_0(s+ia)\|^2 + a\|BR(s+ia)\|^2\right\} < \infty$$

for each interval I with compact closure in Q.

Then $\mathscr{H}_{ac}(H) \subset D(W(H, H_0, J))$ *and* $\mathscr{H}_{ac}(H) \subset D(W(H_0, H, J^*))$.
We shall also use:

LEMMA 5.4. *There exist Hermitian matrices* $V(x), W(x)$ *such that*

$$|E(x) - 1|^{1/2} = |W(x)| \le V(x) \le F(x) + 1,$$

$V(x)$ *is invertible and* $S^{1/2}$*-compact,*

$$V(x)W(x) = W(x)V(x) = 1 - E(x) \tag{5.24}$$

and

$$(1 + |x|)^{\alpha} \int_{|x-y|<1} \left\{ |V(y)|^2 + |W(y)|^2 \right\} dy \in L^{\infty} \tag{5.25}$$

LEMMA 5.5. *If* V, W *satisfy* (5.24), *then*

$$R(z)E^{-1} - R_0(z) = zR(z)VWE^{-1}R_0(z) = zR_0(z)VWR(z)E^{-1}$$

Proof. By (2.1) we have

$$(H_0 u, v) = (u, Hv)_1, \quad u \in D(H_0), v \in D(H). \tag{5.26}$$

Thus

$$([H_0 - z]u, v) = (u, [H - \bar{z}]v)_1 + z([E - 1]u, v)$$

or

$$\left(E^{-1}f, R(\bar{z})g\right)_1 = \left(R_0(z)f, g\right)_1 + z\left(WF^{-1}R_0(z)f, VF^{-1}R(\bar{z})g\right)_1 \tag{5.27}$$

where $f = (z - H_0)u$ and $g = (\bar{z} - H)v$. This gives the first identity. To obtain the other we note that (5.27) is equivalent to

$$\left(f, R(\bar{z})g\right) = \left(R_0(z)f, Eg\right) + z\left(WR_0(z)f, VR(\bar{z})g\right)$$

or

$$\left(f, R(\bar{z})E^{-1}h\right) = \left(f, R_0(\bar{z})h\right) + \left(WR_0(z)f, VR(\bar{z})E^{-1}h\right)$$

where $h = Eg$. This gives the other identity. □

LEMMA 5.6. *Let*

$$G_0(z) = 1 - zVR_0(z)W, \quad G(z) = 1 + zVR(z)WE^{-1}.$$

Then

(5.28) $$G(z)G_0(z) = G_0(z)G(z) = 1, \quad \operatorname{Im} z \neq 0.$$

Proof. The first product equals

$$1 + zV\big(R(z)E^{-1} - R_0(z)\big)W - z^2VR(z)WVE^{-1}R_0(z)W = 1$$

by Lemma 5.5. The same result is obtained if we reverse the order of multiplication. □

LEMMA 5.7. *For* Im $z \neq 0$,

(5.29) $$G_0(z)VR(z)E^{-1} = VR_0(z)$$

and

(5.30) $$R(z)E^{-1}V(1 - zWR_0V) = R_0(z)V.$$

Proof. By one identity in Lemma 5.5,

$$(1 - zR_0(z)WV)R(z)E^{-1} = R_0(z)$$

Applying (V) to both sides, we obtain (5.29). By the other identity in Lemma 5.5,

$$R(z)E^{-1}(1 - zVWR_0(z)) = R_0(z)$$

Applying this to V, we obtain (5.30). □

LEMMA 5.8. *If $V(x)$ is $S^{1/2}$-bounded and satisfies* (5.25), *then for each bounded interval I bounded away from* 0 *there is a constant C_I such that*

(5.31) $$a\|S^{1/2}R_0(s + ia)Vu\|^2 \leq C_I\|u\|^2, \quad s \in I, 0 < a < 1.$$

Proof. Note that

$$a\|R_0(s + ia)Vu\|^2 = \int |s + ia - A(p)|^{-2}|\hat{V}u(p)|^2\, dp.$$

As $a \to 0$, this converges to $\int_{S_s}|\hat{V}u|^2 dS$ where $S_s = \{p \in \mathbf{R}|A(p) = s\}$. If $s \neq 0$, S_s consists of smooth bounded sheets [27]. On the other hand, the limit of the lefthand side of (5.31) is a.e. equal to

$$\pi\frac{d}{ds}(E_0(s)Vu, Vu)$$

where $E_0(s)$ is the spectral projection of H_0. We can now apply Lemma 3.7 of [17] to conclude that

$$\frac{d}{ds}(E_0(s)Vu, Vu) = (T(s)u, u) \quad \text{a.e.} \tag{5.32}$$

where $T(s)$ is a locally Holder continuous map from $\mathbf{R} - \{0\}$ to $B(L^2)$ (the bounded operators on L^2). Thus we have

$$\begin{aligned} a\|S^{1/2}(R_0(s+ia) - R_0(i))Vu\|^2 &= a|s+ia-i|^2\big(\|S^{1/2}R_0(i)R_0(s+ia)P_0Vu\|^2 \\ &\qquad + |s+ia|^{-2}\|(1-P_0)Vu\|^2\big) \\ &\leq a|s+ia-i|^2\Big(c\int |s+ia-t|^{-2}(T(t)u,u)\,dt \\ &\qquad + |s+ia|^{-2}\|(1-P_0)Vu\|^2\Big) \end{aligned}$$

where we used the fact that $R(P_0) = \mathscr{H}_{ac}(H_0)$. Since $(T(t)u, u)$ is locally Holder continuous, we see that (5.31) holds. □

LEMMA 5.9. *If V, W are $S^{1/2}$-bounded and satisfy* (5.25), *then*

$$[VR_0(s+ia)W]$$

is uniformly continuous in any rectangle $s \in I, 0 < a < 1$ when the interval I is bounded and bounded away from 0.

Proof. By (5.32) we have

$$\begin{aligned} (V[R_0(z_1) - R_0(z)]Wu, v) &= (z-z_1)\int (z-t)^{-1}(z_1-t)^{-1}(T(t)u,v)\,dt \\ &\quad + (z-z_1)z^{-1}z_1^{-1}(V(1-P_0)(1-P_0)Wu, v). \end{aligned}$$

Since $T(t)$ is locally Holder continuous, there are positive constants α, C such that

$$\|V[R_0(z_1) - R_0(z)]Wu\| \leq C|z_1 - z|^\alpha \|u\|$$

for z, z_1 in the rectangle. □

LEMMA 5.10. *If $V(x)$ is $S^{1/2}$-compact, then $G_0(z_1) - G_0(z)$ is a compact operator for non-real z, z_1.*

Proof. We have

$$\begin{aligned} G_0(z_1) - G_0(z) &= V[zR_0(z) - z_1R_0(z_1)]W \\ &= (z_1 - z)VH_0R_0(z)R_0(z_1)W \\ &= (z_1 - z)VS^{-1/2}H_1S(z - H_1)^{-1}(z_1 - H_1)^{-1}P_0S^{-1/2}W \end{aligned}$$

where H_1 is the operator constructed in Lemma 3.5. Since $VS^{-1/2}$ is compact and the remaining combination of operators is bounded, the result follows. □

Now we can give the next proof.

Proof of Theorem 1.10. We apply Lemma 5.3. By (5.24) and (5.26) we have

$$(Ju, Hv)_1 - (JH_0u, v)_1 = (VJH_0u, Wv) = (Au, Bv)$$

Now, $AR_0(i) = VJH_0R_0(i)$ is bounded on L^2 since $\|VJ\| \le \|(1 + F)J\|$ and J is a bounded operator from L^2 to $\mathscr{H}_1$. Moreover, by (5.28) and (5.29),

$$\begin{aligned} BR(i)F^{-1} &= WV^{-1}G(i)VR_0(i)F \\ &= WV^{-1}G(i)V|R_0(i)|^{1/2}|i - H_0|R_0(i)\left[|R_0(i)|^{1/2}F\right]. \end{aligned}$$

This is bounded since $|W| \le V$ and V is $|H_0|^{1/2}$-bounded. Thus hypotheses (I) and (II) are satisfied. To verify (III), note that

$$\begin{aligned} &a\|[R_0(s + ia) - R_0(i)]A^*u\|^2 \\ &\quad = a|s + ia - i|^2\|H_0J'R_0(i)R_0(s + ia)Vu\|^2 \\ &\quad \le Ca\|R_0(s + ia)Vu\|^2 \\ &\quad \le C_I\|u\|^2 \end{aligned}$$

by Lemma 5.8. Thus the first part of (5.23) holds. To verify the second, note that

$$BR(z)F^{-1} = WV^{-1}G(z)V|i - H_0|^{1/2}R_0(z)\left[|R_0(i)|^{1/2}F\right]. \tag{5.33}$$

By Lemma 5.9, $G_0(s + ia)$ is uniformly continuous in every bounded rectangle $s \in I, 0 < a < 1$ as long as I is bounded away from 0. Thus it can be extended to be continuous on the boundary. Moreover, if z_1 is any fixed point in the rectangle,

$$G(z_1)G_0(z) - 1 = G(z_1)[G_0(z) - G_0(z_1)] \equiv K(z) \tag{5.34}$$

by Lemma 5.6. By Lemma 5.10, $K(z)$ is compact. Since $G_0(z)$ depends analytically on z, its limit as z approaches the real axis must have a bounded inverse on an open set with complement of measure 0 (cf. [32], [33]). Thus there is an open subset Q of $\mathbf{R}$ with complement of measure 0 such that $G_0(z)$ has a bounded inverse in any closed bounded rectangle with base in Q. Since $G(z)$ is the inverse of $G_0(z)$, we have

$$\|G(s+ia)\| \leq C_I, \quad s \in \bar{I} \subset Q, 0 \leq a \leq 1 \tag{5.35}$$

where C_I depends on I but not on s or a. Thus for such rectangles, by (5.33), we have

$$\|BR(z)F^{-1}\| \leq C\|V|i - H_0|^{1/2}R_0(z)\|. \tag{5.36}$$

In view of (5.31), this implies

$$a\|BR(s-ia)F^{-1}\|^2 \leq C_I, \quad s \in \bar{I} \subset Q, 0 \leq a \leq 1. \tag{5.37}$$

Thus (5.23) holds. Hence we may apply Lemma 5.3 to conclude that $\mathscr{H}_{ac}(H_0)$ is contained in the domains of $W_\pm(H, H_0, J)$. To show that their ranges are dense in $\mathscr{H}_{ac}(H)$, let g be any element in $\mathscr{H}_{ac}(H)$ which is orthogonal to the range of $W_+(H, H_0, J)$. Let u by any function such that Vu is in C_0^∞. Then there is a $w \in \mathscr{H}$ satisfying $Jw = Vu$. Hence

$$a\int_I (JR_0(z)w, R(z)g)_1\, ds \to \pi(W_+E_0(I)w, g)_1 = 0 \tag{5.38}$$

as $a \to 0$ for any interval I (cf. [20]). On the other hand, in view of Lemma 5.7 we have

$$\begin{aligned}(JR_0(z)w, R(z)g)_1 &= (u, VR_0(\bar{z})ER(z)g)\\ &= (u, G_0(\bar{z})VR(\bar{z})R(z)g)\\ &= (VG_0(\bar{z})^*u, R(\bar{z})R(z)g)\\ &= \left(R(z)E^{-1}VG_0(\bar{z})^*u, R(z)g\right)_1\end{aligned}$$

Let Q be the set described above and let $\bar{I} \subset Q$. Then the lefthand side of (5.38) converges to

$$\pi\int_I m\left(s, E^{-1}VG_0(s)^*u, g\right) ds \tag{5.39}$$

where $m(s, f, g)$ is a measurable function of s, defined everywhere, and equal to $d(E(s)f, g)/ds$ whenever the latters exists (cf. [20], [34]). Thus we can conclude that (5.39) vanishes for all u such that Vu is in C_0^∞. Such u are dense in $\mathscr{H}$. For if h is orthogonal to all such u, then $V^{-1}h = 0$ and consequently $h = 0$. By (5.31),

$$a\|R_0(z)Vu\|_1^2 \leq a\|FS^{-1/2}\|^2\|S^{1/2}R_0(z)Vu\|^2 \leq C_I\|u\|^2, \quad u \in I, 0 < a < 1$$

provided I is bounded and away from 0. Thus the expression (5.39) is bounded by $C_I\|u\|\,\|g\|_1$. Since $V^{-1}C_0^\infty$ is dense in $\mathscr{H}$, we see that (5.39) vanishes for all u in $\mathscr{H}$. This is true for all I such that $\bar{I} \subset Q$. Hence

$$(5.40)\qquad m\big(s, E^{-1}VG_0(s)^*u, g\big) = 0, \quad s \in Q, u \in \mathscr{H}.$$

Since $G_0(s)$ has a bounded inverse on $\mathscr{H}$ for each $s \in Q$, this implies

$$m\big(s, E^{-1}Vh, g\big) = 0, \quad s \in Q, h \in \mathscr{H}.$$

Consequently, $(E^{-1}Vh, g)_1 = 0$, $h \in \mathscr{H}$ or $(Vh, g) = 0$, $h \in \mathscr{H}$. If we take $h = Vg$, we have $Vg = 0$ and hence $g = 0$. This completes the proof. □

Proof of Theorem 1.11. For k sufficiently large,

$$(5.41)\qquad (J - J_0)S^{-k}$$

is a compact operator on $\mathscr{H}$. On the other hand J_0S^{-k} satisfies the hypotheses of Theorem 1.10. Hence the wave operators $W_\pm(H, H_0, J_0S^{-k})$ are complete. Since (5.41) is compact, we see that

$$W_\pm\big(H, H_0, JS^{-k}\big)P_0 = W_\pm\big(H, H_0, J_0S^{-k}\big)P_0$$

in view of (5.18). Since the domain of S^k is dense in $\mathscr{H}$, we have

$$W_\pm(H, H_0, J)P_0 = W_\pm\big(H, H_0, JS^{-k}\big)P_0.$$

Thus the wave operators $W_\pm(H, H_0, J)$ are complete. □

References

1. G.S.S. Avila, *Spectral resolution of differential operators associated with symmetric hyperbolic systems*, Applicable Anal., vol. 1 (1972), pp. 283–299.
2. M.S. Birman, *Some applications of a local criterion for the existence of wave operators*, Dokl. Akad. Nauk. SSSR, vol. 185 (1969), pp. 735–738 (Russian).
3. ______, *Scattering problems for differential operators with perturbation of the space*, Izv. Akad. Nauk. SSSR, vol. 35 (1971), pp. 440–455 (Russian).
4. J.M. Combes and R.A. Weder, *New criterion for existence and completeness of wave operators and applications to scattering by unbounded obstacles*, Comm. Partial Differential Equations, vol. 6 (1981), pp. 1179–1223.
5. J.M. Cook, *Convergence of the Moller wave matrix*, J. Mathematical Phys., vol. 36 (1957), pp. 82–87.
6. E.B. Davies, *On Enss' approach to scattering theory*, Duke Math. J., vol. 47 (1980), pp. 171–185.
7. V.G. Deic, *The local stationary method in the theory of scattering with two spaces*, Dokl. Akad. Nauk. SSSR, vol. 197 (1971), pp. 1247–1250 (Russian).
8. ______, *Application of the method of nuclear perturbations in two space scattering theory*, Izv. Vyss. Ucebn. Zaved Mathematika (1971), pp. 33–42.

9. P. DEIFT, *Classical scattering theory with a trace condition*, Thesis, Princeton University, 1976.
10. V. ENSS, "Scattering theory of Schrodinger operators" in *Rigorous Atomic and Molecular Physics*, G. Velo and A.S. Wightman (editors), Plenum, N.Y., 1980/81.
11. J. GINIBRE, *La method 'dependent du temps' dans le problem de la completude asymptotique,* to appear.
12. T. KATO, *Perturbation theory for linear operators*, Springer-Verlag, N.Y., 1966.
13. ______, *On the Cook-Kuroda criterion in scattering theory*, Comm. Math. Phys., vol. 67 (1979), pp. 85–90.
14. K. MOCHIZUKI, *Spectral and scattering theory for symmetric hyperbolic systems in an exterior domains*, Publ. Research Institute of Mathematical Sciences, Kyoto Univ., vol. 5 (1969), pp. 219–258.
15. G. NENCIU, *Eigenfunction expansions for wave propagation problems in classical physics*, Comm. Stat. Pen. En. Nuc., Inst. Fiz. Atom., Bucharest, FF-113, 1975.
16. M. SCHECHTER, *Spectra of Partial Differential Operators*, North-Holland, Amsterdam, 1986.
17. ______, *A unified approach to scattering*, J. Math. Pures. Appl., vol. 53 (1974), pp. 373–396.
18. ______, *The existence of wave operators in scattering theory*, Bull. Amer. Math. Soc., vol. 83 (1977), pp. 381–383.
19. ______, *A new criterion for scattering theory*, Duke Math. J., vol. 44 (1977), pp. 863–882.
20. ______, *Completeness of wave operators in two Hilbert space*, Ann. Inst. H. Poincare, vol. 30 (1979), pp. 109–127.
21. ______, *Scattering in two Hilbert spaces*, J. London Math. Soc., vol. 19 (1979), pp. 175–186.
22. ______, *Wave operators for pairs of spaces and the Klein-Gordan equations*, Aequationes Math., vol. 20 (1980), pp. 38–50.
23. J.R. SCHULENBERGER AND C.H. WILCOX, *Completeness of the wave operators for perturbations of uniformly propagative systems*, J. Functional Anal., vol. 7 (1971), pp. 447–474.
24. J.R. SCHULENBERGER, *A local compactness theorem for wave propagation problems of classical physics*, Indiana Univ. Math. J., vol. 22 (1972), pp. 429–432.
25. B. SIMON, *Scattering theory and quadratic forms: On a theorem of Schechter*, Comm. Math. Phys., vol. 53 (1977), pp. 151–153.
26. T. SUZUKI, *Scattering theory for a certain non-self adjoint operator*, Mem. Fac. Liberal Arts and Education, vol. 23 (1974), pp. 14–18.
27. C.H. WILCOX, *Wave operators and asymptotic solutions of wave propagation problems in classical physics*, Arch. Rational Mech. Anal., vol 22 (1966), pp. 37–78.
28. ______, *Steady-state wave propagation in homogeneous anisotropic media*, Arch. Rational Mech. Anal., vol. 25 (1967), pp. 201–242.
29. ______, *Transient wave propagation in homogeneous anisotropic media*, Arch. Rational Mech. Anal., vol. 37 (1970), pp. 323–343.
30. K. YAJIMA, *Eigenfunction expansions associated with uniformly propagative systems and their applications to scattering theory*, J. Fac. Sci. Univ. Tokyo, vol. 22 (1975), pp. 121–151.
31. K. VESELIC AND J. WEIDMANN, *Asymptotic estimates of wave functions and existence of wave operators*, J. Functional Anal., vol. 17 (1974), pp. 61–71.
32. T. KATO AND S.T. KURODA, *The abstract theory of scattering*, Rocky Mountain J. Math., vol. 1 (1971), pp. 127–171.
33. M. SCHECHTER, *Operator Methods in Quantum Mechanics*, North-Holland, Amsterdam, 1981.
34. ______, *Self adjoint realizations in another Hilbert space*, Amer. J. Math., vol. 106 (1984), pp. 43–65.

UNIVERSITY OF CALIFORNIA
IRVINE, CALIFORNIA