ILLINOIS JOURNAL OF MATHEMATICS
Volume 31, Number 1, Spring 1987

MODULES OF FINITE LENGTH AND CHOW GROUPS OF SURFACES WITH RATIONAL DOUBLE POINTS

BY

V. SRINIVAS

Let R be a local ring, and let $\mathscr{C}_R$ denote the category of R-modules of finite length and finite projective dimension. The Grothendieck group $K_0(\mathscr{C}_R)$ is defined, as usual, to be the quotient $\mathscr{F}/\mathscr{R}$ where $\mathscr{F}$ is the free abelian group on isomorphism classes of objects of $\mathscr{C}_R$, and $\mathscr{R}$ is the subgroup of $\mathscr{F}$ generated by elements $[M] - [M'] - [M'']$ corresponding to exact sequences

$$0 \to M' \to M \to M'' \to 0$$

in $\mathscr{C}_R$. Note that $\mathscr{C}_R$, and hence $K_0(\mathscr{C}_R)$, depends only on the analytic isomorphism class (i.e., the completion $\hat{R}$) of R, since modules of finite length are complete. Also, if R is regular, $K_0(\mathscr{C}_R)$ is just $\mathbf{Z}$, since in $K_0(\mathscr{C}_R)$ a module of length l is equivalent to l copies of the residue field.

If $\dim R = 2$, we say that R has a *rational double point* if the completion $\hat{R}$ is isomorphic to $\hat{\mathcal{O}}_{P,X}$ where $P \in X$ is the local ring of a rational double point P on a surface X over an algebraically closed field k. Thus if k has characteristic 0, $\hat{R}$ is isomorphic to $k[[x, y, z]]/(f(x, y, z))$ where f is one of the following:

$$\begin{array}{ll} z^{n+1} + xy & (A_n) \\ z^2 + xy^2 + x^{n+1} & (D_{n+2}),\ n \geq 2 \\ z^2 + y^3 + x^4 & (E_6) \\ z^2 + x^3y + y^3 & (E_7) \\ z^2 + x^3 + y^5 & (E_8). \end{array}$$

We can now state our main result.

THEOREM 1. *Let R be a 2-dimensional (noetherian) local ring with algebraically closed residue field k of characteristic* 0. *Suppose that R has a rational double point. Then $K_0(\mathscr{C}_R) = \mathbf{Z}$.*

We have the following geometric consequence. For a normal quasi-projective surface X, the Chow group of zero cycles $F_0K_0(X)$ is defined to be the

Received January 8, 1985.

subgroup of the Grothendieck group of vector bundles $K_0(X)$ generated by the classes of smooth points. From results of Collino [3], $F_0K_0(X)$ can also be described as the free abelian group on the smooth points of X modulo divisors of functions on curves contained in the smooth locus. A singular point P on a normal surface X is called a *quotient singularity* if the complete local ring $\hat{\mathcal{O}}_{P,X}$ is the ring of invariants of a finite group of k-linear automorphisms of $k[[x, y]]$, obtained from automorphisms of the vector space $k.x \oplus k.y$. We assume below that k is algebraically closed of characteristic 0.

THEOREM 2. *Let X/k be a normal quasi-projective surface with only quotient singularities, and let f: $Y \to X$ be a resolution of singularities. Then the natural map f^*: $K_0(X) \to K_0(Y)$ induces an isomorphism of Chow groups*

$$F_0K_0(X) \xrightarrow{\cong} F_0K_0(Y)$$

This generalizes a result of Levine [12], where the theorem is proved for cyclic quotient singularities. Indeed, parts of our arguments are inspired by his work. The proof of Theorem 2 is fairly easy, given Theorem 1. To prove Theorem 1, we first make an easy reduction to the case $k = \mathbf{C}$. The proof then falls naturally into 2 parts. The case of E_8 is dealt with separately (this case is already settled by Theorem 1 of [19, I]; for completeness we give a different proof here). For the other cases, we prove (while proving Lemma (1.2)) a sort of proper Artin approximation result for the ideal class group:

Let R be a semi-local ring, essentially of finite type over an algebraically closed field k of characteristic 0, with only rational double point singularities which are not of type E_8 (i.e., for $\mathfrak{m} \in \operatorname{Spec} R$, either $R_{\mathfrak{m}}$ is regular or has a rational double point). Then there exists a semi-local ring S, étale and finite over R, such that for any maximal ideal $\mathfrak{m}$ of S such that $S_{\mathfrak{m}}$ is not regular, the natural map of ideal class groups $\mathrm{Cl}(S) \to \mathrm{Cl}(S_{\mathfrak{m}})$ is non-zero.

This is based on Levine's ideas. Then using K-theory, we show:

PROPOSITION (1.4). *Let R be the local ring of a rational double point P on a surface $X/\mathbf{C}$. Suppose that P is not of type E_8, and R is a unique factorisation domain. Then $K_0(\mathscr{C}_R) = \mathbf{Z}$.*

This concludes the first part of the proof. Since $K_0(\mathscr{C}_R)$ depends only on the analytic isomorphism class of R it suffices to exhibit one algebraic local ring S in each analytic isomorphism class which is a unique factorisation domain. The second half of the proof is devoted to constructing such examples on elliptic surfaces.

I wish to thank Mangala Nori for suggesting that elliptic surfaces with maximal Picard number might provide the required examples, and for showing me the example with E_7 and E_8 singular fibers. The other examples were

constructed along lines suggested by Madhav Nori. I also wish to thank him for explaining Kodaira's construction of elliptic surfaces with prescribed singular fibers using monodromy. Finally, I thank Mohan Kumar for explaining his ideas on modules of finite length and finite projective dimension, and for stimulating discussions on this work.

The methods used to prove Theorem 1 are unsatisfactory for (at least) two reasons. Firstly, the particular K-theoretic argument only works for unique factorisation domains; presumably $K_0(\mathscr{C}_R) = \mathbf{Z}$ if R has any rational singularity. Further, many of the steps do not readily generalize to higher dimensions, even for Gorenstein quotient singularities. It seems reasonable in this direction to expect the following: let R be a complete equicharacteristic normal Cohen Macaulay local ring of dimension n with algebraically closed residue field k. Suppose there exists a power series ring $k[[X_1, \dots, X_n]]$ which is finite over R; then $K_0(\mathscr{C}_R) = \mathbf{Z}$.

Such a result would follow, at least up to torsion, if we had a transfer map f_*: $K_0(\mathscr{C}_S) \to K_0(\mathscr{C}_R)$ with reasonable properties whenever we have an inclusion f: $R \to S$ of normal Cohen Macaulay semi-local rings of the same dimension with S finite over R. Indeed, under these conditions, there is a "natural" map f^*: $K_0(\mathscr{C}_R) \to K_0(\mathscr{C}_S)$ (shown to me by Mohan Kumar) induced by the functor $\mathscr{C}_R \to \mathscr{C}_S$:

$$M \to \mathrm{Ext}^n_S(\mathrm{Ext}^n_R(M, S), S)$$

Another way of getting a map f^*: $K_0(\mathscr{C}_R) \to K_0(\mathscr{C}_S)$ is to use a result of Foxby [7] that $K_0(\mathscr{C}_S) = K_0(\mathscr{C}'_S)$ where $\mathscr{C}'_S$ is the category of bounded complexes of free S-modules with finite length homology. Now if $M \in \mathscr{C}_{R'}$

$$0 \to R^{m_n} \to R^{m_{n-1}} \to \cdots \to R^{m_0} \to M \to 0$$

is a free R-resolution, and E is the complex of free S-modules obtained by tensoring with S (after omitting M), then f^*: $K_0(\mathscr{C}_R) \to K_0(\mathscr{C}'_S)$ is given by $[M] \mapsto [E] \in K_0(\mathscr{C}'_S) = K_0(\mathscr{C}_S)$. The two definitions suggested for f^* agree on the subgroup of $K_0(\mathscr{C}_R)$ generated by classes $R/(f_1, \dots, f_n)$ for regular sequences $f_1, \dots, f_n \in R$, and hence agree in dimension 2; however in view of the results of Dutta, Hochster and Maclaughlin [6], this is not clear to me in general. We would expect the transfer map to satisfy $f_* \circ f^*[M] = d[M]$ for all $M \in \mathscr{C}_R$, where $d = \deg f$ (i.e., the R-rank of S). We are unable to define such a map (except in the trivial case when S is flat). However such a transfer exists on the level of 0-cycles for proper maps f: $X \to Y$ such that f is generically finite and surjective, and such that $f^{-1}(Y_{\mathrm{sing}})$, the inverse image of the singular locus, has codimension at least 2 (this follows from results of Levine [14]).

Finally we make some remarks about characteristic p. Theorem 1 of [19] shows that $K_0(\mathscr{C}_R) = \mathbf{Z} + (N\text{-torsion})$ if R has a rational double point of a type which occurs on a rational surface (e.g., the ones which lift to characteristic 0). Here $p|N$; perhaps an extension of the methods used here can give the result without the torsion. In [4], Coombes and the author showed that $K_0(\mathscr{C}_R) = \mathbf{Z} + (N\text{-torsion})$ for R of the type

$$k[[x, y, z]]/(z^{p^n} - f(x, y)),$$

k algebraically closed of characteristic $p > 0$. In fact we can use a transfer argument (in the "wrong" direction) to prove (without K-theory!):

THEOREM 3. *Let R be a complete local ring of equicharacteristic $p > 0$ with algebraically closed residue field k. Let $S = R[z]/(z^{p^n} - y)$ where $y \notin R^*$. Then the natural maps $f^*: K_0(\mathscr{C}_R) \to K_0(\mathscr{C}_S)$, $f_*: K_0(\mathscr{C}_S) \to K_0(\mathscr{C}_R)$ have the property that both composites $f_* \circ f^*$ and $f^* \circ f_*$ are multiplication by p^n. In particular, if*

$$S = k[[x_1, \ldots, x_r, z]]/(z^{p^n} - f(x_1, \ldots, x_r))$$

then

$$K_0(\mathscr{C}_S) = \mathbf{Z} + (p^n\text{-torsion}).$$

After the preparation of this manuscript, I received the preprint [21] of Levine, proving results about $K_0(\mathscr{C}_R)$ which imply Theorem 1 for arbitrary quotient singularities. This is done by proving a new localisation theorem in algebraic K-theory, generalizing a result of Quillen. The work in the present paper was done independently of Levine's, and the methods are quite different.

1. Proof of Theorem 1

We begin by making the reduction to the case $k = \mathbf{C}$. Indeed, we have:

LEMMA (1.1). *Let $(R, \mathfrak{m})$ be a local ring essentially of finite type over an algebraically closed field k, and let L be an extension of k. Let $S = (R \otimes_k L)_{\mathfrak{m}'}$ where $\mathfrak{m}' = \mathfrak{m}(R \otimes_k L)$. Then $K_0(\mathscr{C}_R) \to K_0(\mathscr{C}_S)$ is injective.*

Proof. L is the direct limit of its finitely generated k-subalgebras, so that S is the direct limit of its subrings of the form $R_{A,f} = (R \otimes_k A)\,[1/f]$ where $A \subset L$ is a finite type k-algebra, and $f \in (R \otimes_k A) - \mathfrak{m}(R \otimes_k A)$. Consider the map

$$R \otimes_k A \to (R \otimes_k A)/\mathfrak{m}(R \otimes_k A) \simeq A.$$

Clearly f has non-zero image, so that there exists a maximal ideal $\mathfrak{m}'$ of A not containing this image. Then

$$f \notin \mathfrak{m}(R \otimes_k A) + \mathfrak{m}'(R \otimes_k A)$$

so that f maps to a unit under

$$\varphi\colon R \otimes_k A \to (R \otimes A)/\mathfrak{m}' \simeq R.$$

This $\varphi\colon R \otimes A \to R$ factors through $R_{A,f}$; i.e., $R \to R_{A,f}$ has a splitting. Let $\mathscr{C}_{A,f}$ denote the category of finite $R_{A,f}$-modules of finite projective dimension annihilated by a power of $\mathfrak{m}$. Then

$$K_0(\mathscr{C}_S) = \varinjlim_{A,f} K_0(\mathscr{C}_{A,f}).$$

Since $K_0(\mathscr{C}_R) \to K_0(\mathscr{C}_{A,f})$ is injective for all A, f, the lemma follows.

LEMMA (1.2). *Let R be the semi-local ring of a finite set of points $\{P_1, \ldots, P_n\}$ on a quasi-projective surface $X/\mathbf{C}$. Assume that each P_i is smooth, or is a rational double point which is not of type E_8. Then there exists a regular semi-local ring S which is finite over R, and étale over the punctured spectrum* $\operatorname{Spec} R - \{P_1, \ldots, P_n\}$.

Proof. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the maximal ideals of R, and let $\hat{R}$ denote the completion with respect to the radical $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$. Then $\hat{R} \simeq \oplus \hat{R}_{\mathfrak{m}_i}$, and each $\hat{R}_{\mathfrak{m}_i}$ is the ring of invariants of a finite group $G_i \subset SL_2(\mathbf{C})$ acting on $\mathbf{C}[[x, y]]$ (see [5], for example). We work by induction on $\sup |G_i|$. The ideal class group of $\hat{R}_{\mathfrak{m}_i}$ is the Pontryagin dual of $G_i^{ab} = G_i/[G_i, G_i]$; since we have excluded the case of E_8, this is non-trivial whenever G_i is non-trivial, i.e., whenever P_i is singular (indeed, the G_i are all solvable). The formula for the class group of $\hat{R}_{\mathfrak{m}_i}$, i.e., the Picard group of the punctured spectrum of $\hat{R}_{\mathfrak{m}_i}$, is immediate from the fact that G_i is the algebraic fundamental group of the punctured spectrum (by construction), and the Kummer sequence [16, III, Prop. 4.11].

The first step is to construct a semi-local ring T, étale over R, such that for any maximal ideal $\mathfrak{m}$ of T, the natural map of ideal class groups $\mathrm{Cl}(T) \to \mathrm{Cl}(T_{\mathfrak{m}})$ is non-zero whenever $T_{\mathfrak{m}}$ is not regular (equivalently, $\mathrm{Cl}\, T_{\mathfrak{m}} \neq 0$). If $\mathfrak{m}'_1, \ldots, \mathfrak{m}'_r$ are the maximal ideals of T,

$$\mathrm{Cl}\, T = \operatorname{Pic}(\operatorname{Spec} T - \{\mathfrak{m}'_1, \ldots, \mathfrak{m}'_r\})$$

is a certain finite abelian group G_T. There is an étale Galois (finite) cover of $\operatorname{Spec} T - \{\mathfrak{m}'_1, \ldots, \mathfrak{m}'_r\}$ with group G_T (not unique) - indeed, if

$$G_T = \mathbf{Z}/_{(a_1)} \oplus \cdots \oplus \mathbf{Z}/_{(a_k)},$$

and $L_1, \dots, L_k$ are the corresponding line bundles, choose isomorphisms

$$L_i^{\otimes a_i} \simeq \mathcal{O}_U, \quad U = \operatorname{Spec} T - \{\mathfrak{m}_1, \dots, \mathfrak{m}_r\},$$

giving a sheaf of algebras

$$\bigoplus_{i=1}^{k} \bigoplus_{j=0}^{a_i - 1} L_i^{\otimes j}$$

over U whose spectrum gives the required cover. The normalisation R_1 of R in the function field of this cover of U has the following property: each maximal ideal of R_1 is either regular or has a rational double point which is not of type E_8. Further, if $\mathfrak{m}$ dominates $\mathfrak{m}_i$ in R, and $\mathfrak{m}'$ in T, and we write $\hat{R}_{\mathfrak{m}_i} = \mathbf{C}[[x, y]]^{G_i}$, then $(\hat{R}_1)_{\mathfrak{m}} \simeq \mathbf{C}[[x, y]]^H$ where H is the kernel of the natural composite $G_i \twoheadrightarrow G_i^{ab} = (\operatorname{Cl} \hat{T}_{\mathfrak{m}'})^* \twoheadrightarrow (\operatorname{Cl} T_{\mathfrak{m}'})^*$, where $*$ denotes the Pontryagin dual, i.e., $H \subsetneqq G_i$. Thus $\sup |G_i|$ has been reduced by passing to R_1, and we are done by induction.

Thus, the whole point is to construct T. Now $R = \mathcal{O}_{\{P_1, \dots, P_n\}, X}$ where we may assume X is affine. We claim that, by shrinking X if necessary, we can find a curve $C \subset X$ such that:

(i) C passes through all the P_i, and is smooth away from the P_i.

(ii) C is smooth at P_i if P_i is smooth.

(iii) C is an analytically reducible Cartier divisor at P_i with a double point, if P_i is singular.

(iv) If P_i has an A_m-singularity for some m, then C has an ordinary double point at P_i.

Under the above conditions, we claim that each analytic branch of C at P_i is smooth and represents a non-trivial element of $\operatorname{Cl}(\hat{R}_{\mathfrak{m}_i})$ if P_i is singular. Assuming the existence of C satisfying (i)–(iv), and the claim above, we construct T as follows. Let $D \to C$ be the normalisation. The generic projection π: $X \to \mathbf{A}^1$ will have the property that the induced map $D \to \mathbf{A}^1$ is étale and finite over $\pi(P_1), \dots, \pi(P_n)$. Let $Y = X \times_{\mathbf{A}^1} D$ be the fiber product. The map $D \to C$ gives a section $D_0 \subset Y$ of $Y \to D$; if Q is an inverse image in Y of P_i, and D_0 passes through Q, then $[D_0] \neq 0$ in $\operatorname{Cl}(\mathcal{O}_{Q,Y})$ if Q (i.e., P_i) is singular. Let $E \to \mathbf{A}^1$ be the Galois closure of the morphism $D \to \mathbf{A}^1$; again $E \to \mathbf{A}^1$ is étale and finite over $\pi(P_1), \dots, \pi(P_n)$. Let $Z = X \times_{\mathbf{A}^1} E$, let Ψ: $Z \to X$ be the projection, and let $\Psi^{-1}\{P_1, \dots, P_n\} = \{Q_1, \dots, Q_s\}$. Take $T = \mathcal{O}_{\{Q_1, \dots, Q_s\}, Z}$. The morphism $Z \to E$ has a section E_0 induced by D_0, at least in a neighbourhood of the inverse images of $\pi(P_1), \dots, \pi(P_n)$. The translates of E_0 under the Galois group now give sections through all the Q_j. Further, if $g \in \operatorname{Gal}(E/\mathbf{A}^1)$ and $Q_i \in g(E_0)$ is a singular point of $\mathbf{Z}$, then $g(E_0) \neq 0$ in $\operatorname{Cl}(\mathcal{O}_{Q_i, Z})$.

Thus, we have only to construct C and verify that the analytic branches of C are non-trivial in the class groups of the non-regular points. Suppose we can

TABLE I

Type	Analytic equation	f_i
Regular	—	regular parameter
A_m	$z^{m+1} = xy$	z
D_{m+2} $(m \geq 2)$	$z^2 = xy^2 - x^{m+1}$	y (if m is odd) $y - x^{m/2}(1 + x)$ (if m is even)
E_6	$z^2 = y^3 + x^4$	y
E_7	$z^2 = x^3y + y^3$	$y - x^3$

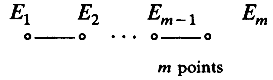
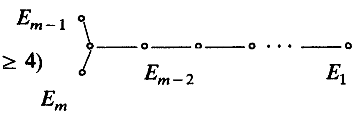
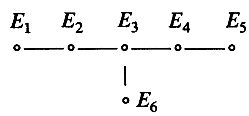
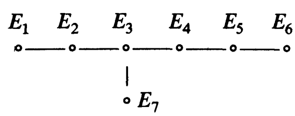
find $f_i \in \hat{\mathfrak{m}}_i$ such that $\hat{R}_{\mathfrak{m}_i}/(f_i)$ has multiplicity 2 and is analytically reducible. Then there exists N_i such that if $g_i \in \hat{R}_{\mathfrak{m}_i}$ and $g_i \equiv f_i \pmod{\hat{\mathfrak{m}}_i^{N_i}}$, then $\hat{R}_{\mathfrak{m}_i}/(g_i)$ also has multiplicity 2 and is reducible. Since $R \to \oplus(\hat{R}_{\mathfrak{m}_i}/\hat{\mathfrak{m}}_i^{N_i})$ is surjective by the Chinese Remainder Theorem, we can find $f \in R$ such that $f \equiv f_i \pmod{\hat{\mathfrak{m}}_i^{N_i}}$. Similarly if P_i is a rational double point of type A_m for some m, and f_i is chosen to have an ordinary double point, we can find a sufficiently close approximation $f \in R$ which will also have an ordinary double point at P_i (in addition to prescribed behaviour elsewhere). We choose the f_i as in Table I. Thus, we can find $f \in R$ such that $f \in \cap \mathfrak{m}_i$, $f \in R_{\mathfrak{m}_i}$ is a regular parameter if $R_{\mathfrak{m}_i}$ is regular, $\hat{R}_{\mathfrak{m}_i}/(f)$ has multiplicity 2 and is reducible if $R_{\mathfrak{m}_i}$ is not regular, and finally $\hat{R}_{\mathfrak{m}_i}/(f) \simeq \mathbf{C}[[x, y]]/(xy)$ if $R_{\mathfrak{m}_i}$ has an A_m-singularity for some m. By shrinking X if necessary, we can take C to be the divisor of zeroes of f; clearly C satisfies (i)–(iv).

Now let P_i be a singular point. Since C is analytically reducible at P_i with a double point, the two branches of C are smooth (but perhaps tangent). Let $V_i \to \operatorname{Spec} \hat{R}_{\mathfrak{m}_i}$ be the minimal resolution of singularities, $E_{i1}, \ldots, E_{im}$ the exceptional curves. Then each E_{ij} is a smooth rational curve, $E_{ij}^2 = -2$, and the total transform of P_i in V_i is the fundamental cycle $\Sigma_j n_j E_{ij}$ where n_j are certain (positive) multiplicities, according to Table II (see [5]).

Let C_i', C_i'' be the two components of the strict transform of C in V_i. We claim C_i', meets the exceptional divisor at 1 point, lying on exactly one E_{ij}; further this E_{ij} occurs with multiplicity 1 in the fundamental cycle, and the intersection is transverse. An easy computation (see appendix) shows that the corresponding branch of C at P_i is non-trivial in $\mathrm{Cl}(\hat{R}_{\mathfrak{m}_i})$, at least in the cases D_m, E_6, E_7; the other branch of C is also non-trivial in $\mathrm{Cl}(\hat{R}_{\mathfrak{m}_i})$ since $[C] = 0$ (C is Cartier by choice). To check the claim about C_i', we look at the tangent map (i.e., derivative) of $V_i \to \operatorname{Spec} \hat{R}_{\mathfrak{m}_i}$; this must have rank ≥ 1 at the closed point Q of C_i. In algebraic terms, if $\mathfrak{m}_0$ is the maximal ideal of $\mathcal{O}_{Q, V_i}$, and $I \subset \mathcal{O}_{Q, V_i}$ the ideal of C_i', $J = \mathfrak{m}_i \cap I \subset \hat{R}_{\mathfrak{m}_i}$ the ideal of the branch of C, then we have a diagram

$$\begin{array}{ccc} \mathfrak{m}_0/\mathfrak{m}_0^2 & \twoheadrightarrow (\mathfrak{m}_0/\mathfrak{m}_0^2 + I) & \cong \mathbf{C} \\ \uparrow & \uparrow \wr\!\wr\!\wr & \\ \mathfrak{m}_i/\mathfrak{m}_i^2 & \twoheadrightarrow (\mathfrak{m}_i/\mathfrak{m}_i^2 + J) & \cong \mathbf{C} \end{array}$$

TABLE II

Type	Dual Graph	Fundamental Cycle
A_m		$E_1 + \cdots + E_m$
$D_m (m \geq 4)$		$E_1 + 2E_2 + \cdots + 2E_{m-2} + E_{m-1} + E_m$
E_6		$E_1 + 2E_2 + 3E_3 + 2E_4 + E_5 + 2E_6$
E_7		$2E_1 + 3E_2 + 4E_3 + 3E_4 + 2E_4 + 2E_5 + E_6 + 2E_7$

where $\mathfrak{m}_0/\mathfrak{m}_0^2 + I$, $\mathfrak{m}_i/\mathfrak{m}_i^2 + J$ are 1-dimensional since C_i' and its image are smooth. Thus $\mathfrak{m}_i \not\subset \mathfrak{m}_0^2$, i.e., exactly one component E_{ij} of the exceptional divisor, occurring with multiplicity 1 in the fundamental cycle, passes through Q; further C_i and E_{ij} meet transversally at Q, as $\mathfrak{m}_i/\mathfrak{m}_i^2 \twoheadrightarrow \mathfrak{m}_0/\mathfrak{m}_0^2 + I$. This argument also suffices for A_1 and A_2 since C has an ordinary double point. If P_i is of type A_m, $m \geq 3$, the tangent cone to X at P_i is a union of 2 planes, so that the blow up of Spec $\hat{R}_{\mathfrak{m}_i}$ at P_i has 2 exceptional curves which are smooth and rational, and meet at a single point which is an A_{m-2} singularity of the surface. Since C has distinct tangents, both components C_i', C_i'' of the strict transform cannot pass through the A_{m-2}-singularity, so that at least one of them (say, C_i') meets an end component of the fundamental cycle in the resolution V_i. This component then gives a generator of $\mathrm{Cl}(\hat{R}_{\mathfrak{m}_i})$; since C is Cartier, the other component also gives a generator of $\mathrm{Cl}(\hat{R}_{\mathfrak{m}_i}) = \mathbf{Z}/(m+1)$. This finishes the proof of the lemma.

Before stating the next lemma, we need to recall some facts from algebraic K-theory; proof for statements without giving a reference can be found in Quillen [18]. For any noetherian ring R, consider the statement that the complex

$$\begin{aligned} 0 \to K_i(R) \to \oplus_{\mathfrak{P} \in (\mathrm{Sp}\, R)^0} K_i(k(\mathfrak{P})) &\to \oplus_{\mathfrak{P} \in (\mathrm{Sp}\, R)^1} K_{i-1}(k(\mathfrak{P})) \\ \to \cdots \oplus_{\mathfrak{P} \in (\mathrm{Sp}\, R)^i} K_0(k(\mathfrak{P})) &\to 0 \end{aligned}$$

is exact for each i, where $(\mathrm{Sp}\, R)^j$ denotes the set of prime ideals of height j, and $k(\mathfrak{P})$ is the residue field of $\mathfrak{P}$. The maps $K_i(R) \to K_i(k(\mathfrak{P}))$ for ht $\mathfrak{P} = 0$ are the natural ones, while the other maps come from boundary maps in certain localisation sequences. Gersten conjectured that this complex is exact for any regular local ring; Quillen proved it for any regular semi-local ring which is essentially of finite type over a field. Let X be a Noetherian regular scheme, such that $\mathcal{O}_{P,X}$ satisfies Gersten's conjecture for every $P \in X$. Let $\mathcal{K}_{i,X}$ denote the sheaf for the Zariski topology associated to the presheaf

$$U \mapsto K_i(\Gamma(U, \mathcal{O}_X)).$$

Then there is a flasque resolution

$$0 \to \mathcal{K}_{i,X} \to \oplus_{x \in X^0}(i_x)_* K_i(k(x)) \to \oplus_{x \in X^1}(i_x)_* K_{i-1}(k(x)) \to \cdots \to \oplus_{x \in X^i}(i_x)_* K_0(k(x)) \to 0$$

where X^j is the set of codimension j points of X, and for $x \in X, (i_x)_* K_j(k(x))$ is the direct image on X of the constant sheaf $K_j(k(x))$ on the closure of x. In particular, if $\dim X = 2$ and X is irreducible, we have a resolution

$$0 \to \mathcal{K}_{2,X} \to i_* K_2(k(X)) \xrightarrow{T} \otimes_{z \in X^1}(i_z)_* k(z)^* \xrightarrow{\partial} \oplus_{P \in X^2}(i_P)_* \mathbf{Z} \to 0.$$

Here $K_2(k(X)) = k(X)^* \oplus k(X)^* / \langle \{ \{a, 1-a\} | a \in k(X)^* - \{1\} \} \rangle$ by a result of Matsumoto and T is the sum of tame symbols T_z (see [8]) defined by

$$\{a, b\} \overset{T_z}{\mapsto} \varphi\left\{(-1)^{v(a)v(b)} \frac{a^{v(b)}}{b^{v(a)}}\right\},$$

where v is the valuation of $k(X)^*$ corresponding to $z \in X^1$, and

$$\varphi \colon \mathcal{O}^*_{z,X} \to k(z)^*$$

is the natural map. The map ∂ is the sum of the divisor maps

$$\partial_z \colon k(z)^* \to \oplus_{P \in X} \mathbf{Z}.$$

We have a subgroup $\oplus_{z \in X^1} k^* \subset \oplus_{z \in X^1} k(z)^*$ contained in the kernel of ∂. This gives a map

$$\oplus_{z \in X^1} k^* = \operatorname{Div}(X) \otimes_{\mathbf{Z}} k^* \to H^1(X, \mathcal{K}_{2,X})$$

where $\operatorname{Div}(X)$ is the group of divisors (the free abelian group on X^1). By considering elements of the form $T\{\alpha, f\}$ for $\alpha \in k^*$, $f \in k(X)^*$, we see that there is an induced map

$$\operatorname{Pic} X \otimes_{\mathbf{Z}} k^* \to H^1(X, \mathcal{K}_{2,X}).$$

LEMMA (1.3). *Let R be the local ring of a rational double point on a surface $X/\mathbf{C}$, and let $f\colon Y \to \operatorname{Spec} R$ be the minimal resolution of singularities. Suppose R is not of type E_8; let $E_1, \dots, E_n$ be the exceptional curves. Then the natural composite map*

$$\oplus \mathbf{C}^*[E_i] \to \operatorname{Pic} Y \otimes_{\mathbf{Z}} \mathbf{C}^* \to H^1(Y, \mathscr{K}_{2,Y})$$

is surjective.

Proof. Let $S \supset R$ be the semi-local ring constructed in Lemma (1.2). Let Z be a resolution of singularities of $Y \times_{\operatorname{Spec} R} \operatorname{Spec} S$. Since S is r semi-local, $H^1(S, \mathscr{K}_{2,S}) = 0$ by the result of Quillen. Since $H^1(Z, \mathscr{K}_2)$ is H^1 of the complex

$$0 \to K_2(F) \to \oplus_{z \in Z^1} \mathbf{C}(z)^* \to \oplus_{P \in Z^2} \mathbf{Z} \to 0$$

where $F = \mathbf{C}(Z)$ is the function field, and the graph of the exceptional divisor of $g\colon Z \to \operatorname{Spec} S$ has no loops (this morphism is a composition of blowings up at smooth points), an easy argument shows that if $E_1, \dots, E_m$ are the exceptional curves of g, then the composite

$$\oplus [E_i]\mathbf{C}^* \to \operatorname{Pic} Z \otimes_{\mathbf{Z}} \mathbf{C}^* \to H^1(Z, \mathscr{K}_{2,Z})$$

is surjective (in fact it is an isomorphism). The morphism $f\colon Z \to Y$ is proper, and finite away from a finite set of points of Y. Hence there are homomorphisms $f^*\colon H^1(Y, \mathscr{K}_{2,Y}) \to H^1(Z, \mathscr{K}_{2,Z})$ and $f_*\colon H^1(Z, \mathscr{K}_{2,Z}) \to H^1(Y, \mathscr{K}_{2,Y})$, such that $f_* \circ f^* =$ multiplication by $(\deg f)$. We also have a commutative diagram

$$\begin{tikzcd} \operatorname{Pic} Z \otimes_{\mathbf{Z}} \mathbf{C}^* \arrow[r] \arrow[d, "f_* \otimes 1"'] & H^1(Z, \mathscr{K}_2) \arrow[d, "f_*"] \\ \operatorname{Pic} Y \otimes_{\mathbf{Z}} \mathbf{C}^* \arrow[r] & H^1(Y, \mathscr{K}_{2,Y}) \end{tikzcd}$$

where f_* on the left is the transfer map $\operatorname{Pic} Z \to \operatorname{Pic} Y$. Thus, the cokernel of

$$\oplus [E_i]\mathbf{C}^* \to H^1(Y, \mathscr{K}_{2,Y})$$

is annihilated by $n = \deg f$. So to finish the proof, it suffices to show that $H^1(Y, \mathscr{K}_{2,Y})$ is divisible.

From results of Mercurjev-Suslin [15] and Bloch-Ogus [2], we have an isomorphism

$$\mathscr{K}_{2,Y} \otimes \mathbf{Z}/N \simeq \mathscr{H}_Y^2(\mu_N^{\otimes 2})$$

where $\mathscr{H}_Y^i(\mu_N^{\otimes j})$ is the Zariski sheaf associated to the presheaf $U \to H^i_{\text{ét}}(U, \mu_N^{\otimes j})$.

Further, there is a spectral sequence (which we can think of as the Leray spectral sequence associated to the morphism of sites $Y_{\text{ét}} \to Y_{\text{Zar}}$):

$$E_2^{p,q} = H^p\big(Y, \mathscr{H}_Y^q(\mu_N^{\otimes r})\big) \Rightarrow H_{\text{ét}}^{p+q}\big(Y, \mu_N^{\otimes r}\big).$$

This spectral sequence has $E_2^{p,q} = 0$ if $p > q$ (since $\mathscr{H}_Y^q$ has a flasque resolution of length $\leq q$ analogous to the Gersten-Quillen resolution for $\mathscr{K}_q$) and if $q > 2$ (since $\mathscr{H}_Y^q = 0$ if $q > 2$, as $\dim Y = 2$). Thus we get an isomorphism

$$H^1(Y, \mathscr{K}_{2,Y} \otimes \mathbf{Z}/N) \simeq H^1\big(Y, \mathscr{H}_Y^2(\mu_N^{\otimes 2})\big) \simeq H_{\text{ét}}^3\big(Y, \mu_N^{\otimes 2}\big).$$

As in Bloch [1], Chapter V, one checks that there is an exact sequence

$$0 \to H^1(Y, \mathscr{K}_{2,Y}) \otimes \mathbf{Z}/N \to H^1(Y, \mathscr{K}_{2,Y} \otimes \mathbf{Z}/N) \to {}_N CH^2(Y) \to 0.$$

So it suffices to prove $H_{\text{ét}}^3(Y, \mu_N^{\otimes 2}) = 0$. Let W be a smooth projective surface containing Y, and let W_0 be the corresponding singular surface. For any affine neighbourhood U of the singular point of W_0, let $\tilde{U} \subset W$ be the inverse image. Since étale cohomology commutes with filtered inverse limits of schemes (Milne [16], Chapter III, Lemma (1.6))

$$H_{\text{ét}}^3\big(Y, \mu_N^{\otimes 2}\big) \simeq \varinjlim H_{\text{ét}}^3\big(\tilde{U}, \mu_N^{\otimes 2}\big).$$

But $H_{\text{ét}}^3(\tilde{U}, \mu_N^{\otimes 2}) \cong H_{\text{an}}^3(\tilde{U}, \mathbf{Z}/N)$ where H_{an}^i denotes singular cohomology. So it suffices to check $H_{\text{an}}^3(\tilde{U}, \mathbf{Z}/N) = 0$ for U as above. Now $H_{\text{ét}}^3(U, \mu_N^{\otimes 2}) \cong H_{\text{an}}^3(U, \mathbf{Z}/N) = 0$ since U is affine of dimension 2 (Milne [16], Ch. VI, Thm (7.2)). If $f\colon \tilde{U} \to U$ then the Leray spectral sequence gives an exact sequence

$$\to H_{\text{an}}^i(U, \mathbf{Z}/N) \to H_{\text{an}}^i(\tilde{U}, \mathbf{Z}/N) \to \Gamma\big(U, R^i f_* \mathbf{Z}/N\big) \to$$

Since $R^3 f_* \mathbf{Z}/N = 0$, as the exceptional divisor E has cohomological dimension 2, we see that $H_{\text{an}}^3(\tilde{U}, \mathbf{Z}/N) = 0$.

PROPOSITION (1.4). *Let R be the local ring of a rational double point on surface $X/\mathbf{C}$. Suppose R does not have an E_8-singularity, and R is a unique factorisation domain. Then $K_0(\mathscr{C}_R) = \mathbf{Z}$.*

Proof. This is similar to arguments in [19, I] and [4]. Let

$$A = \operatorname{coker}\big(T\colon K_2(\mathbf{C}(Y)) \to \oplus \mathbf{C}(\mathfrak{P})^*\big),$$

where $\mathfrak{P}$ runs over the height 1 primes of R, and T is the sum of the corresponding tame symbols. There is a natural map (at least when R is a UFD) $\Psi\colon A \to K_0(\mathscr{C}_R)$, defined as follows:

If $g \in R - \mathfrak{P}$, $\bar{g}$ its image in $R/\mathfrak{P}$ and $\mathfrak{P} = fR$, then let

$$\varphi(\bar{g}) = [R/(f, g)] \in K_0(\mathscr{C}_R);$$

extending by additivity gives a map $\varphi\colon \oplus \mathbf{C}(\mathfrak{P})^* \to K_0(\mathscr{C}_R)$. To check that this descends to A, we have to show $\varphi(T\{f, g\}) = 0$ for all $f, g \in \mathbf{C}(X)^*$. We easily reduce to two cases: f, g distinct prime elements of R; and f prime, g a unit in R. If f, g are primes,

$$T(\{f, g\}) = \frac{1}{\bar{g}} + \bar{f} \in \mathbf{C}(fR)^* \oplus \mathbf{C}(gR)^* \subset \oplus \mathbf{C}(\mathfrak{P})^*.$$

(where an overbar denotes the image in the appropriate residue field). Thus

$$\varphi(T\{f, g\}) = -[R/(f, g)] + [R/(f, g)] = 0.$$

Similarly if $g \in R^*$ and f is prime,

$$T(\{f, g\}) = \frac{1}{\bar{g}} \in \mathbf{C}(fR)^* \subset \oplus \mathbf{C}(\mathfrak{P})^*.$$

Hence $\varphi(T\{f, g\}) = -[R/(f, g)] = 0$ since $(f, g) = R$. Thus there is a well defined map $A \to K_0(\mathscr{C}_R)$. By a result of Hochster (proved independently by Mohan Kumar—see [4]) $K_0(\mathscr{C}_R)$ is generated by the classes $[R/(f, g)]$ for all regular sequences $f, g \in R$. Hence $\Psi\colon A \to K_0(\mathscr{C}_R)$ is surjective.

We now use Lemma (1.3) to show that $A = \mathbf{Z}$. Let $Y \to \operatorname{Spec} R$ be a resolution of singularities. Consider the complexes $(\alpha), (\beta), (\gamma)$ defined by

$$(\alpha) \qquad 0 \to K_2(\mathbf{C}(Y)) \xrightarrow{T} \oplus_{x \in Y^1} \mathbf{C}(x)^* \xrightarrow{\partial} \oplus_{x \in Y^2} \mathbf{Z} \to 0,$$

$$(\beta) \qquad 0 \to 0 \to \oplus_{\text{exceptional curves}} \mathbf{C}(x)^* \xrightarrow{\partial} \oplus_{x \in Y^2} \mathbf{Z} \to 0,$$

$$(\gamma) \qquad 0 \to K_2(C(Y)) \xrightarrow{T} \oplus_{\operatorname{ht} \mathfrak{P} = 1} \mathbf{C}(\mathfrak{P})^* \to 0 \to 0.$$

We have an exact sequence $0 \to (\beta) \to (\alpha) \to (\gamma) \to 0$, and $H^1(\gamma) = A$ fits into a sequence

$$H^1(\beta) \to H^1(\alpha) \to A \to H^2(\beta) \to H^2(\alpha).$$

We claim $H^2(\alpha) = 0$, $H^1(\alpha) = H^1(Y, \mathscr{K}_{2,Y})$. This is clear because (α) is the complex of global sections associated to the Gersten-Quillen resolution of $\mathscr{K}_{2,Y}$. Next, $H^2(\beta) = \mathbf{Z}$. For, a zero cycle of degree 0 on Y is a sum of zero cycles δ_i, each of which has degree 0 and is supported on exactly one exceptional curve. Since the exceptional curves are smooth rational curves, every such δ_i is the divisor of a rational function on the corresponding exceptional curve. Finally, $H^1(\beta) = \oplus \mathbf{C}^* \cdot [E_i]$ where E_i are the exceptional curves, since the exceptional divisor contains no loops. But then Lemma (1.3) says precisely that $H^1(\beta) \twoheadrightarrow H^1(\gamma)$. Hence $A = \mathbf{Z}$, and the proposition is proved.

2. Construction of rational double points which are unique factorisation domains

Our examples will be constructed on elliptic surfaces. We begin by recalling some results of Kodaira. Let $\mathfrak{h}$ denote the upper half plane $\{\operatorname{Im} z > 0\} \subset \mathbf{C}$. The group $PSL_2(\mathbf{Z})$ acts on $\mathfrak{h}$ via Mobius transformations given by the formula

$$\sigma(z) = \frac{az + b}{cz + d} \quad \text{for } \sigma \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix} (\bmod \pm 1).$$

We have the universal elliptic surface π: $X \to \mathfrak{h}$ defined by $X = (\mathfrak{h} \times \mathbf{C})/\mathbf{Z}^{\oplus 2}$ where the action of $\mathbf{Z}^{\oplus 2}$ is given by

$$(m, n) \cdot (\tau, z) = (\tau, z + m\tau + n).$$

The group $SL_2(\mathbf{Z})$ acts on $\mathfrak{h} \times \mathbf{C}$ by the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right).$$

One checks that since $ad - bc = 1$, this descends to an action on X, so that $X \to \mathfrak{h}$ is $SL_2(\mathbf{Z})$-equivariant. Let j: $\mathfrak{h} \to \mathbf{C}$ be the quotient map modulo $PSL_2(\mathbf{Z})$ (the modular function); then one knows that

$$\pi^{-1}(h_1) \cong \pi^{-1}(h_2) \Leftrightarrow j(h_1) = j(h_2).$$

We can adjoin cusps to $\mathfrak{h}$ (in bijection with the points of $\mathbf{P}^1_{\mathbf{Q}}$), given by the rational points on the real axis together with ∞, and extend the $PSL_2(\mathbf{Z})$ action via its usual action in $\mathbf{P}^1_{\mathbf{Q}}$. If $\mathfrak{h}^* = \mathfrak{h} \cup$ (Cusps), we can extend j to a map $\mathfrak{h}^* \to \mathbf{P}^1$, such that for suitable local coordinates near the cusps, j is holomorphic. If we put the elliptic curve $X_\tau = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$ in Weierstrass form

$$y^2 = 4x^3 - g_2 x - g_3,$$

with

$$g_2 = 60 \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^4}, \quad g_3 = 140 \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^6},$$

and $\Delta = g_2^3 - 27g_3^2$, then we take

$$j = \frac{g_2^3}{g_2^3 - 27g_3^2}.$$

With this normalisation, $j(i) = 1$, $j(\omega) = 0$ where $i = \sqrt{-1}$, $\omega = \exp(2\pi i/3)$. The isotropy groups of i, ω in $PSL_2(\mathbf{Z})$ are isomorphic to $\mathbf{Z}/2$ and $\mathbf{Z}/3$ respectively, while the isotropy group of ∞ is infinite cyclic (the image of the group of upper triangular matrices in $SL_2(\mathbf{Z})$). Let

$$\mathfrak{h}^0 = \mathfrak{h}^* - j^{-1}\{0,1,\infty\}, \quad X^0 = \pi^{-1}(\mathfrak{h}^0).$$

Then $j\colon \mathfrak{h}^0 \to \mathbf{P}^1 - \{0,1,\infty\}$ is a covering space with group $PSL_2(\mathbf{Z})$.

Consider a curve C together with a (non-constant) morphism $f\colon C \to \mathbf{P}^1$. Let Δ be the universal cover of $C' = C - f^{-1}\{0,1,\infty\}$. Since Δ is simply connected, we have a diagram of covering spaces

$$\begin{tikzcd} \Delta \arrow[r] \arrow[d] & \mathfrak{h}^0 \arrow[d, "j"] \\ C' \arrow[r] & \mathbf{P}^1 - \{0,1,\infty\} \end{tikzcd}$$

Further we have a map of covering groups, i.e., a projective representation $\rho_0\colon \pi_1(C') \to PSL_2(\mathbf{Z})$. Since $\pi_1(C')$ is a free group, we can lift this projective representation to a representation

$$\rho\colon \pi_1(C') \to SL_2(\mathbf{Z});$$

the choices of such lifts are in bijection with characters of order 2 of $\pi_1(C')$ (since $\{\pm 1\} \subset SL_2(\mathbf{Z})$ is the center) i.e., with elements of $H^1(C', \mathbf{Z}/2)$ (from now onwards, cohomology will be singular cohomology unless specified otherwise). A lifting ρ as above gives a $\pi_1(C')$ action on X^0 making $\pi\colon X^0 \to \mathfrak{h}^0$ $\pi_1(C')$-equivariant. Hence we have an action of $\pi_1(C')$ on the fiber product $Y^0 = X^0 \times_{\mathfrak{h}^0} \Delta$ such that $Y^0 \to \Delta$ is a $\pi_1(C')$-map. Since $\pi_1(C')$ acts freely on Δ, it also acts freely on Y^0, and we obtain a quotient elliptic fibration φ: $Z^0 \to C'$. By construction Z^0 is a smooth surface such that:

(i) All fibers of φ are smooth elliptic curves with j-invariant $\neq 0, 1$.

(ii) φ has a section.

(iii) The map $C' \to \mathbf{P}^1 - \{0,1,\infty\}$ given by $x \to j(\varphi^{-1}(x))$ is just f: $C' \to \mathbf{P}^1 - \{0,1,\infty\}$.

Kodaira shows [11, pp. 1330] that the above construction yields all elliptic surfaces over C' with the above properties. Further, he proves that we can compactify Z^0 to a smooth surface Z in a unique way such that φ extends to a morphism $Z \to \mathbf{P}^1$ which is relatively minimal, and he determines the "singular" fibers in terms of the local monodromy, i.e., the representation ρ. If $P \in C - C'$, the local monodromy is determined up to sign by the projective representation ρ_0, i.e., the map f. Thus if e_P is the ramification index of f at P, we can describe Kodaira's classification of the possible singular fibers Z_P as follows:

(i) $j(P) = 0$, $e_P \equiv 0 \pmod 3$, or $j(P) = 1$, $e_P \equiv 0 \pmod 2$. Then Z_P is either smooth elliptic, or of type I_0^*.

(ii) $j(P) = 0$, $e_P \equiv 1 \pmod 3$. Then Z_P is either of type II (rational curve with a cusp) or type IV*.

(iii) $j(P) = 0$, $e_P \equiv 2 \pmod 3$. Then Z_P is either of type IV (three smooth rational curves meeting at a point) or type II*.

(iv) $j(P) = 1$, $e_p \equiv 1 \pmod 2$. Then Z_P is of type III (two smooth rational curves meeting at a point and tangent there) or type III*.

(v) $j(P) = \infty$. Then if $n = e_P$, Z_P is of type I_n (an n-sided polygon; all components are smooth rational curves occurring with multiplicity 1) or type I_n^*.

The singular fibers of types II*, III*, IV*, I_n^* $(n \geq 0)$ consist of a tree of smooth rational curves with only normal crossings, represented by the following graphs (the integer adjacent to an edge gives the multiplicity of the component in the singular fiber).

Kodaira shows that Z-(multiple components) is a group scheme over C which is the Neron model of the generic fiber Z_η over the function field $C(\eta)$ of C; in particular the torsion group $(\mathrm{Pic}^0 Z_\eta)_{\mathrm{tors}} \cong (Z_\eta)_{\mathrm{tors}}$ is a subgroup of Z_P-(multiple components) for any $P \in C$. The group structure on the singular fiber is as follows:

(i) I_0^*: $\mathbf{C} \times \mathbf{Z}/2 \times \mathbf{Z}/2$
(ii) I_n: $\mathbf{C}^* \times \mathbf{Z}/n$
(iii) I_n^*: $\mathbf{C} \times \mathbf{Z}/4$ or $\mathbf{C} \times \mathbf{Z}/2 \times \mathbf{Z}/2$
(iv) II, II*: $\mathbf{C}$
(v) III, III*: $\mathbf{C} \times \mathbf{Z}/2$
(vi) IV, IV*: $\mathbf{C} \times \mathbf{Z}/3$.

The connection with rational double points rests on the following observation. Suppose Z_P is of one of the type I_n $(n \geq 2)$, I_n^* $(n \geq 0)$, II*, III* or IV*. Let F_1 be any component of Z_P of multiplicity 1, and write $Z_P = F_1 + F_2$ as divisors. Then F_2 is the fundamental cycle of a rational double point, and every rational double point occurs in this way. The correspondence is I_n $(n \geq 2) \leftrightarrow A_{n-1}$, I_n^* $(n \geq 0) \leftrightarrow D_{n+4}$, $\mathrm{II}^* \leftrightarrow E_8$, $\mathrm{III}^* \leftrightarrow E_7$, $\mathrm{IV}^* \leftrightarrow E_6$.

Suppose $Z \to C$ has the property that $\mathrm{Pic}^0 Z_\eta = 0$. Then the Neron Severi group $NS(Z)$ is generated by the 0-section, and classes of components of the singular fibers. Hence for any P such that Z_P is of one of the above types, if

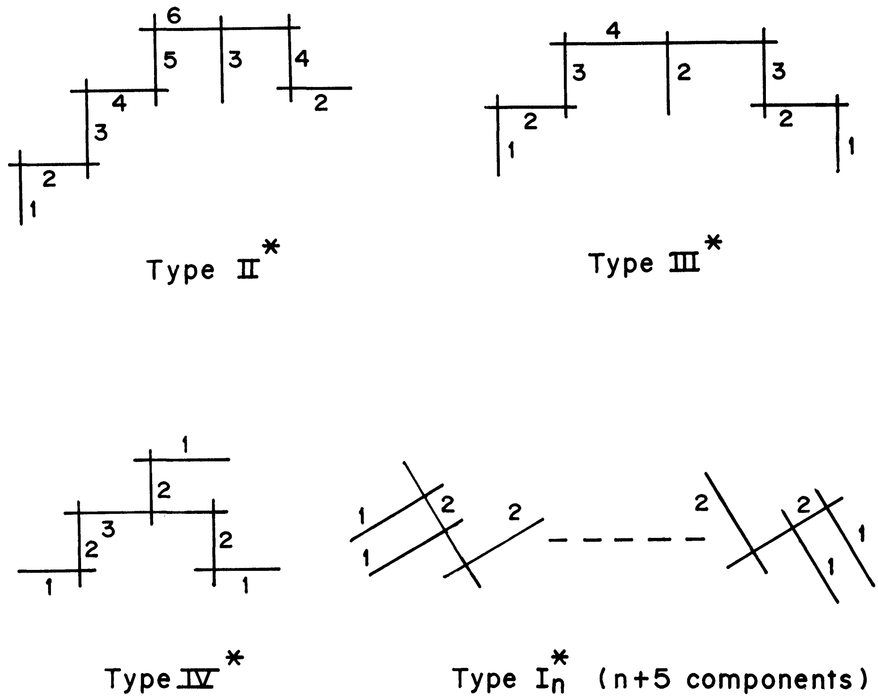


FIG. A

we take F_1 to be the component meeting the 0-section, and if $E_1, \ldots, E_n$ are the components of F_2, then the image of Ψ: $NS(Z) \to \mathbf{Z}^{\oplus n}$ given by intersection with the various E_i, is generated by the vectors $\Psi(E_i)$. By a standard argument, this is equivalent to the statement that the rational double point obtained by contracting F_2 is a UFD (see appendix). Thus, we need a method of constructing elliptic surfaces $Z \to C$ with $\mathrm{Pic}^0 Z_\eta = 0$.

For any singular fiber F, let $\nu(F)$ be one less than the number of components of F; thus the components of F contribute $\nu(F)$ to the rank of the Neron Severi group $NS(Z)$. We will mainly consider elliptic surfaces $Z \to C$ such that

$$(*) \qquad \operatorname{rank} NS(Z) = h^{1,1}(Z) = 2 + \sum_{\substack{\text{singular}\\ \text{fibers}}} \nu(F).$$

Thus $NS(Z) \otimes \mathbf{Q}$ has a basis consisting of the zero section and components of the singular fibers; equivalently $\mathrm{Pic}^0 Z_\eta$ is torsion. We use the following result of Mangala Nori [17]:

THEOREM. *Let φ: $Z \to C$ be an elliptic fibration as above, and F: $C \to \mathbf{P}^1$ the morphism induced by the j-function. Then Z satisfies $(*)$ iff the following conditions hold*:

(i) $C - f^{-1}\{0, 1, \infty\} \to \mathbf{P}^1 - \{0, 1, \infty)$ *is unramified*

(ii) *The ramification index of any point lying over* 0 *or* 1 *is bounded respectively by* 3 *or* 2.

(iii) *The fibers over* 0 *are either smooth, or of types* II*, IV*

(iv) *The fibers over* 1 *are smooth or of type* III*

(*In particular there is no restriction on the fiber over* ∞).

For our purposes, it suffices to know that a surface satisfying (i)–(iv) also satisfies $(*)$. For completeness, we give the proof.

Suppose $\deg f = n$, and there are r, s and t singular fibers of types II*, III*, and IV* respectively. Further suppose the fibers over ∞ are of types $\mathrm{I}_{e_1}, \ldots, \mathrm{I}_{e_a}, \mathrm{I}^*_{e_{a+1}}, \ldots, \mathrm{I}^*_{e_{a+b}}$. The ramification of f: $C \to \mathbf{P}^1$ must then be as follows: $f^{-1}(0)$ consists of t unramified points, r points with ramification index 2, and $(n - t - 2r)/3$ points with ramification index 3; $f^{-1}(1)$ consists of s unramified points and $(n - s)/2$ points with double ramification; and $f^{-1}(\infty)$ consists of $a + b$ points with ramification indices $e_1, \ldots, e_{a+b}$. From the Riemann-Hurwitz formula the genus g of C (which equals the irregularity q of Z, since $Z \to C$ is a non-constant elliptic family) satisfies

$$\begin{aligned} 2g - 2 &= -2n + r + \frac{(n - 2r - t)}{3}2 + \frac{n - s}{2} + \sum(e_i - 1) \\ &= \frac{n}{6} - \frac{r}{3} - \frac{2t}{3} - \frac{s}{2} - (a + b). \end{aligned}$$

Since $K_Z^2 = 0$, Noether's formula yields $C_2(Z) = 12\chi(\mathcal{O}_Z)$. But

$$\begin{aligned} C_2(Z) &= \sum_{\substack{\text{singular} \\ \text{fibers}}} C_2(F) \quad \text{(for an elliptic surface)} \\ &= 10r + 9s + 8t + \sum_{i \le a} e_i + \sum_{i > a} (e_i + 6) \\ &= 10r + 9s + 8t + n + 6b. \end{aligned}$$

If p_g is the geometric genus of Z,

$$C_2(Z) = 2 - 4q + 2p_g + h^{1,1} = 2\chi(\mathcal{O}_Z) - 2g + h^{1,1} \quad (\text{since } q = g),$$

$$\begin{aligned} h^{1,1} &= 2g + \frac{5}{6}C_2(Z) \\ &= 2 + \frac{n}{6} - \frac{r}{3} - \frac{2t}{3} - \frac{s}{2} - (a + b) + \frac{5}{6}(10r + 9s + 8t + n + 6b) \\ &= 2 + 8r + 7s + 6t + n + 4b - a \\ &= 2 + \sum_{\substack{\text{singular} \\ \text{fibers}}} \nu(F), \text{ as desired.} \end{aligned}$$

We also note here that for a surface Z satisfying (i)–(iv), the parity of the number b of I_n^* fibers at ∞ is determined by the formula for $C_2(Z)$, and the requirement that $12|C_2$. In fact this is the only restriction on the fibers at ∞ in the following sense: given any non-negative number b of the correct parity and at most equal to the cardinality of $f^{-1}(\infty)$, we can find a (unique) surface $Z \to C$ with I_n^* fibers at any prescribed b points of $f^{-1}(\infty)$, such that Z also has the prescribed fibers over 0 and 1, i.e., Z satifies $(*)$. Indeed, the various Z's (for the distinct choices of b and of b points of $f^{-1}((\infty))$ can be obtained from any one by twisting the representation $\rho\colon \pi_1(C') \to SL_2(\mathbf{Z})$ by a suitable element of $H^1(C', \mathbf{Z}/2)$.

We illustrate this in the first example below, shown to me by Mangala Nori. Let $f\colon C \to \mathbf{P}^1$ be the identity map, i.e., $C \cong \mathfrak{h}^*/PSL_2(\mathbf{Z})$. Then the possible singular fibers are (i) type IV* or type II over 0 (ii) type III* or type III over 1 (iii) type I_1^* or type I_1 over ∞. First choose an arbitrary lift

$$\rho\colon \pi_1\left(\mathbf{P}^1 - \{0,1,\infty\}\right) \to SL_2(\mathbf{Z})$$

of the natural surjection

$$\rho_0\colon \pi_1\left(\mathbf{P}^1 - \{0,1,\infty\}\right) \twoheadrightarrow PSL_2(Z).$$

Then twist by a character of π_1 which is non-trivial over 0 iff we have a type II fiber, and which is non-trivial over 1 iff we have a type III fiber (these requirements unique specify a character of $\pi_1(\mathbf{P}^1 - \{0,1,\infty\})$ which is the free group on 2 generators). Thus we can find a unique surface $Z \to C$ with a IV* and a III* fiber. Parity reasons force the fiber over ∞ to be of type I_1^*. Since the torsion subgroups of the III* and IV* fibers are $\mathbf{Z}/2$ and $\mathbf{Z}/3$ respectively, $\mathrm{Pic}^0 Z_\eta$ is torsion free, i.e. $\mathrm{Pic}^0 Z_\eta = 0$. Hence we get examples of E_6, E_7 and D_5 singularities which are UFD's.

For $n \geq 3$, let $\Gamma(n) = \ker(PSL_2(\mathbf{Z}) \to PSL_2(\mathbf{Z}/n))$, and let $G \subset PSL_2(\mathbf{Z})$ be the inverse image of the cyclic subgroup $(\mathbf{Z}/3) \subset PSL_2(\mathbf{Z}/n)$ generated by

$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Take $C = \mathfrak{h}^*/G = X(n)/\mathbf{Z}/3$, where $X(n) = \mathfrak{h}^*/\Gamma(n)$ is the modular curve. $X(n)$ has $(1/n)|PSL_2(\mathbf{Z}/n)|$ points in the fiber over $\infty \in \mathbf{P}^1$, since the stabilizer of one of the points (the image of $\infty \in \mathfrak{h}^*$) is the group of upper triangular matrices, which has order n (note that $X(n) \to \mathbf{P}^1$ is Galois with group $PSL_2(\mathbf{Z}/n)$). If $n = \prod p_i^{\alpha_i}$ is the prime factorisation of n, then

$$SL_2(\mathbf{Z}/n) = \prod SL_2(\mathbf{Z}/p_i^{\alpha_i})$$

by the Chinese Remainder theorem, since $SL_2(\mathbf{Z}/p^\alpha)$ is generated by elementary matrices. Hence

$$\frac{1}{n}|PSL_2(\mathbf{Z}/n)| = \frac{1}{2}\prod \frac{1}{p_i^{\alpha_i}}|SL_2(\mathbf{Z}/p_i^{\alpha_i})|.$$

Clearly ker $(SL_2(\mathbf{Z}/p^{\alpha+1}) \to SL_2(\mathbf{Z}/p^{\alpha}))$ has order p^3 if $\alpha \geq 1$, so that

$$|SL_2(\mathbf{Z}/p^{\alpha})| = p^{3(\alpha-1)}\frac{(p^2-1)(p^2-p)}{p-1} = p^{3\alpha-2}(p^2-1)$$

Hence

$$\frac{1}{n}|PSL_2(\mathbf{Z}/n)| = \frac{1}{2}\prod p^{2(\alpha_i-1)}(p_i^2-1) \geq 4 \text{ if } n \geq 3$$

(and ≥ 12 if $n \geq 5$). Thus if $n \geq 3$, $f^{-1}(\infty)$ has cardinality at least 2 (at least 4, if $n \geq 5$). Further, G and $\Gamma(n)$ have the same intersection in $PSL_2(\mathbf{Z})$ with the upper triangular group. Hence if $P \in C$ is the image of $\infty \in \mathfrak{h}^*$, $e_P = n = e_Q$ where Q is the image of $\infty \in \mathfrak{h}^*$ in $X(n)$. Thus if $n \geq 3$, we can find a surface $Z \to C$ satisfying $(*)$ with an I_n^* fiber over ∞, and at least one IV^* fiber over 0 (this is because the group $\mathbf{Z}/3$ in $PSL_2(\mathbf{Z}/n)$ is precisely the stabilizer of one of the points over 0 in $X(n)$). Since the torsion subgroup of the IV^* and I_n^* fibers have relatively prime orders, this gives examples of D_{n+4} singularities which are UFD's, for $n \geq 3$. If $n \geq 5$, we can similarly find $Z \to C$ satisfying $(*)$ with an I_n fiber at P and an I_b^* fiber (for some b) at some other point of $F^{-1}(\infty)$. Again the torsion subgroups of the IV^* and I_b^* fibers have relatively prime order, so that the resulting A_{n-1} singularity $(n \geq 5)$ is a UFD. The only remaining cases are D_4, D_6, A_1, A_2, A_3 (corresponding to $\mathrm{I}_0^*, \mathrm{I}_2^*, \mathrm{I}_2, \mathrm{I}_3, \mathrm{I}_4$).

We analyse the cases $n = 3, 4$ of the above construction a little more closely. As computed above, $|PSL_2(\mathbf{Z}/3)| = 12$, and the ramification picture for $X(3) \to \mathbf{P}^1$ is as follows: each fiber over $0, \infty$ consists of 4 points with ramification index 3, and the fiber over 1 has 6 points with ramification index 2. If $C = X(3)/\mathbf{Z}/3$ where $\mathbf{Z}/3$ acts via

$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix},$$

we see that there are exactly two fixed points for the $\mathbf{Z}/3$ action, one each over $0, \infty$ (one can check this directly in terms of the $\mathbf{Z}/3$ action on the cosets of stabilizers, or by Riemann-Hurwitz, since $X(3) \cong \mathbf{P}^1$). Thus the surfaces $Z \to C$ satisfying $(*)$ have a IV^* fiber over 0, and two singular fibers over ∞, of types I_1 and I_3, or of types I_1^* and I_3^* (this is from parity); all other fibers are smooth elliptic. Let $R \in C$ be the point of $f^{-1}(0)$ with IV^* fiber, and let P, Q be the points at ∞ with the I_3 and I_1 (or I_3^* and I_1^*) fibers respectively. Consider the double cover $D \to C$ branched at Q, R. If $Z \to C$ has the I_1 and I_3 fibers, $\tilde{Z} \to D$ the minimal resolution of $Z \times_C D$, then $\tilde{Z}$ has 4 singular fibers of types $\mathrm{IV}, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_2$. To get a surface satisfying $(*)$, we can twist by a character of $\pi_1(D)$ non-trivial on the loop about the inverse image of R, and on the loop about a suitable point at ∞. Thus we can get surfaces with 4 fibers of types $\mathrm{II}^*, \mathrm{I}_3, \mathrm{I}_3, \mathrm{I}_2^*$ and of type $\mathrm{II}^*, \mathrm{I}_3, \mathrm{I}_3^*, \mathrm{I}_2$. These yield examples of D_6, A_1, A_2 singularities which are UFD's (the II^* fiber has no torsion, so $\mathrm{Pic}^0 Z_\eta = 0$ here).

If $n = 4$, $X(4)$ has 6 points over $\infty \in \mathbf{P}^1$, so that C has at least 2 points over ∞, one of which is the point P with $e_p = 4$. Hence we can find $Z \to C$ with I_4 fiber over ∞ and a IV^* fiber, whose torsion subgroups have relatively prime order. This gives an A_3 singularity which is a UFD.

The D_4 singularity does not occur on surfaces satisfying $(*)$, so we need a special construction. This example is due to Madhav Nori. Let E be the elliptic curve with automorphism group $\mathbf{Z}/6$, i.e., $E = \mathbf{C}/\mathbf{Z} + \mathbf{Z}\omega$, $\omega = e^{2\pi i/3}$ a primitive cube root of unity. Let $\sigma\colon E \to E$ be the automorphism induced by multiplication by $(-\omega)$ on $\mathbf{C}$. Let $X = (E \times E)/(\mathbf{Z}/6)$ where the generator of $\mathbf{Z}/6$ acts by $(e_1, e_2) \mapsto (\sigma(e_1), \sigma^5(e_2))$. Then X is singular; let $Y \to X$ be the minimal resolution, so that $f\colon Y \to \mathbf{P}^1 = E/(\mathbf{Z}/6)$ (induced by projection) is a minimal elliptic surface. The only section of this map f is the 0-section, since there are no maps $E \to E$ equivariant with respect to the two $\mathbf{Z}/6$ actions (where the generators act respectively as σ and as σ^5) except the map $E \to \{0\} \subset E$. Hence the generic fiber Y_η satisfies $\mathrm{Pic}\, Y_\eta = 0$. If $P \in E$ is a point of order 2, its isotropy group is $\mathbf{Z}/2$. Thus if $Q \in \mathbf{P}^1$ is the image, the fiber Y_Q is an I_0^* fiber (indeed the fiber over Q of X is a smooth rational curve passing through 4 ordinary double points of the surface). In the usual way this yields a D_4 singularity which is a UFD.

3. Theorem 1 for E_8-singularities

Here we directly show that

$$K_0(\mathscr{C}_R) = \mathbf{Z} \quad \text{for } R = \mathbf{C}[x, y, z]_{(x,y,z)}/(z^2 + x^3 + y^5).$$

As in the proof of Prop. (1.4) in §1, since R is a UFD, we have only to prove the following claim: if $Y \to \operatorname{Spec} R$ is the minimal resolution of singularities, $E_1, \ldots, E_8$ the exceptional curves, then

$$\oplus \mathbf{C}^*[E_i] \to H^1(Y, \mathscr{K}_{2,Y})$$

is surjective. Let X be the affine surface in $A^3_{\mathbf{C}}$ defined by $z^2 + x^3 + y^5 = 0$, and let $Z \supset X$ be a projective surface such that $Z - X$ consists of smooth points. If $\pi\colon \tilde{Z} \to Z$ is a resolution of singularities, then the map on Chow groups

$$\pi^*\colon F_0K_0(Z) \to F_0K_0(\tilde{Z})$$

is an isomorphism—in fact $F_0K_0(Z) = \mathbf{Z}$ since $Z - X$ consist only of rational curves, and $F_0K_0(X) = 0$ ($\mathbf{C}[x, y, z]/(z^2 + x^3 + y^5)$ is the ring of invariants for the action of the binary icosahedral group of order 120 on the polynomial ring $\mathbf{C}[u, v]$—see [5]). Now $\tilde{Z}$ is a smooth rational surface (since a unirational surface over $\mathbf{C}$ is rational [10, V (6.2.1)]), so that

$$H^1(\tilde{Z}, \mathscr{K}_{2,\tilde{Z}}) \cong \operatorname{Pic} \tilde{Z} \otimes_{\mathbf{Z}} \mathbf{C}^*$$

(see Bloch [1], Chapter VII, Prop. (7.9)). The Leray spectral sequence for π gives

$$H^1(\tilde{Z}, \mathscr{K}_{2,\tilde{Z}}) \to \Gamma(Z, R^1\pi_*\mathscr{K}_{2,\tilde{Z}}) \to H^2(Z, \pi_*\mathscr{K}_{2,\tilde{Z}}) \to H^2(\tilde{Z}, \mathscr{K}_{2,\tilde{Z}})$$

By Bloch's formula (see Quillen [18], for example)

$$H^2(\tilde{Z}, \mathscr{K}_{2,\tilde{Z}}) = F_0K_0(\tilde{Z}),$$

while $H^2(Z, \mathscr{K}_{2,Z}) = H^2(Z, \pi_*\mathscr{K}_{2,\tilde{Z}}) = F_0K_0(Z)$ by a result of Collino [3] (note that $\mathscr{K}_{2,Z} \to \pi_*\mathscr{K}_{2,\tilde{Z}}$ induces an isomorphism on H^2 since the kernel and cokernel are supported at a point). Hence

$$H^2(Z, \pi_*\mathscr{K}_{2,\tilde{Z}}) \cong H^2(\tilde{Z}, \mathscr{K}_{2,\tilde{Z}})$$

is an isomorphism, and

$$\operatorname{Pic} \tilde{Z} \otimes \mathbf{C}^* \to \Gamma(Z, R^1\pi_*\mathscr{K}_{2,\tilde{Z}}) = H^1(Y, \mathscr{K}_{2,Y})$$

is onto. This map clearly factors through $\operatorname{Pic} Y \otimes_{\mathbf{Z}} \mathbf{C}^* = \oplus \mathbf{C}^*[E_i]$ (since R is a UFD).

Remark. One can also use the fact that R is the ring of invariants of $\mathbf{C}[u, v]_{(u,v)}$ for an action of the binary icosahedral group to prove Lemma (1.3) directly for R. The argument given here uses the divisibility of $F_0K_0(X)$ instead of the Mercurjev-Suslin theorem.

4. Proofs of Theorems 2 and 3

Let X/k be a normal quasi-projective surface over an algebraically closed field k of characteristic 0, and suppose X has only quotient singularities. Let R be the semi-local ring of the singular locus. The dualising module ω_R of R gives a torsion element of the ideal class group $\mathrm{Cl}(R)$ (indeed $\mathrm{Cl}(R)$ is finite). Suppose $[\omega_R]$ has order n; then there is an étale $\mathbf{Z}/n$-cover of $U = \operatorname{Spec} R$-(closed points) corresponding to a choice of an isomorphism $\omega_U^{\otimes n} \simeq \mathcal{O}_U$. Let S be the integral closure of R in the function field of this cover; then S has

Gorenstein quotient singularities, i.e., rational double points (see [20], p. 140). Let Y_0 be the normalisation of X in the quotient field of S, and let $Y \to Y_0$ be a resolution of the singularities which map to smooth points of X. Then $f\colon Y \to X$ is proper, and finite over a neighbourhood $V \subset X$ of the singular locus X_{sing}. By Collino's results [3] there is a transfer map $f_*\colon F_0K_0(Y) \to F_0K_0(X)$ such that $f_* \circ f^*$ is multiplication by n. We now use:

LEMMA (4.1). *Let Z be a normal quasi-projective surface, $P \in Z$ a singular point, $\pi\colon Y \to Z$ a resolution of the singularity at P. Then*

$$\ker(F_0K_0(Z) \to F_0K_0(Y)) \subset \operatorname{image}(K_0(\mathscr{C}_R) \to F_0K_0(Z))$$

where $R = \mathcal{O}_{P,Z}$ (and $K_0(\mathscr{C}_R) \to F_0K_0(Z)$ is the natural map obtained by treating a module of finite length and finite projective dimension as a skyscraper sheaf).

From this lemma, if $Y' \to Y$ is any resolution of singularities, $F_0K_0(Y) \to F_0K_0(Y')$ is bijective since Y has only rational double points. If $\tilde{X} \to X$ is the minimal resolution, we can find a resolution $Y' \to Y$ which dominates X. Hence

$$\ker(F_0K_0(X) \to F_0K_0(\tilde{X}))$$

is n-torsion. However this kernel is divisible by Lemma 11 of [13]. This proves Theorem 2. Lemma (4.1) is proved in [19, I]; we reproduce the proof here for completeness.

Proof of Lemma (4.1). Let δ be a zero cycle in the smooth locus of Z such that $\pi^*[\delta] = 0$ in $F_0K_0(Y)$. Then we can find curves $C_1, \dots, C_n$ on Z and rational functions $f_1, \dots, f_m$ such that:

(i) $C_i - \{P\}$ does not meet the singular locus of Z.

(ii) If $\tilde{C}_i$ is the strict transform of C_i in Y, then

$$\pi^*\delta = \Sigma(f_i)_{\tilde{C}_i} + \delta_0$$

as cycles, where δ_0 is a cycle of degree 0 supported on the exceptional divisor over $P \in Z$.

Write $(f_i)_{C_i} = \delta_i + n_i(P)$ where $P \notin \operatorname{supp} \delta_i$. Then $\delta = \Sigma\delta_i$ as cycles, and $\Sigma n_i = 0$. Clearly it suffices to show that each δ_i lies in the image of $K_0(\mathscr{C}_R)$. By Bertini's theorem we can find Weil divisors $D_i \subset Z$ such that:

(i) $D_i - \{P\}$ does not meet the singular locus.

(ii) $D_i + C_i$ is a reduced Cartier divisor on Z.

Let g_i be the rational function on $C_i + D_i$ which restricts to f_i on C_i and to the constant function 1 on D_i. Let $S = \{P\} \cup$ (singular locus of $C_i + D_i$), and let

R_i be the semi-local ring of S on $C_i + D_i$. Write $g_i = h_i/k_i$ where $h_i, k_i \in R_i$ are non-zero divisors (clearly g_i lies in the total quotient ring of R_i). If $E_i = (C_i + D_i) - S_i$ then

$$[R_i/(h_i)] + (h_i)_{E_i}, [R_i/(k_i)] + (k_i)_{E_i}$$

both represent 0 in $\mathrm{Pic}(C_i + D_i)$ and hence in $F_0K_0(Z)$. But then in $F_0K_0(Z)$ we have

$$\begin{aligned} 0 &= [R_i/(h_i)] + (h_i)_{E_i} - [R_i/(k_i)] - (k_i)_{E_i} \\ &= [\delta_i] + [R/(\varphi_i, h_i)] - [R/(\varphi_i, k_i)] \end{aligned}$$

where φ_i generates the (principal) ideal of $C_i + D_i$ in R. This proves the lemma.

We recall the statement of Theorem 3.

THEOREM 3. *Let R be a complete local ring of equicharacteristic $p > 0$ with residue field k, and let $S = R[z]/(z^{p^n} - f)$ for some non-unit $f \in R$. Let f^*: $K_0(\mathscr{C}_R) \to K_0(\mathscr{C}_S)$, f_*: $K_0(\mathscr{C}_S) \to K_0(\mathscr{C}_R)$ be the natural maps (note that f is flat). Then $f^* \circ f_*$, $f_* \circ f^*$ are both multiplication by p^n. In particular $K_0(\mathscr{C}_S) = \mathbf{Z} \oplus p^n$-torsion if R is regular.*

Proof. This is obvious for $f_* \circ f^*$. The other composite is induced by the functor $\mathscr{C}_S \to \mathscr{C}_S$ given by

$$M \mapsto M \otimes_S (S \otimes_R S)$$

where the S-module structure comes from that on $S \otimes_R S$ induced by $S \to S \otimes_R S$, $s \mapsto 1 \otimes s$. Let $I \subset S \otimes_R S$ be the kernel of multiplication $S \otimes_R S \to S$. The key point is that $I^{p^n} = 0$, and $I^j/I^{j+1} \cong S$ for $0 < i < p^n$, where the left side has the above S-structure (in fact, the "left" and "right" S-structures on an $S \otimes_R S$-module agree precisely when it is annihilated by I). Thus $M \otimes_S (S \otimes_R S)$ is equivalent in $K_0(\mathscr{C}_S)$ to p^n copies of M.

Appendix

We collect here some well-known facts about the class groups of rational double points which we need above. Proofs of any unproven assertions can be found in [22].

Let R be the local ring of a rational double point on a surface over $\mathbf{C}$. Let $X = \mathrm{Spec}\, R$, $P \in X$ the closed point, π: $Y \to X$ the minimal resolution of singularities, $E_1, \ldots, E_n$ the components of $\pi^{-1}(P)$. Let $\hat{R}$ be the completion of R, $\hat{X} = \mathrm{Spec}\, \hat{R}$, $\hat{Y} = Y \times_X \hat{X}$ the minimal resolution of $\hat{X}$. Let E_i be the inverse image of E_i in $\hat{Y}_i$. Let $U = Y - \bigcup E_i$, $\hat{U} = \hat{Y} - \bigcup \hat{E}_i$. Then

$$\begin{aligned} \mathrm{Cl}(R) &\cong \mathrm{Pic}(X - \{P\}) \\ &\cong \mathrm{Pic}\, U \\ &\cong \mathrm{Pic}\, Y / (\text{subgroup generated by the } \mathcal{O}_Y(E_i)), \end{aligned}$$

and similarly

$$\mathrm{Cl}(\hat{R}) \cong \mathrm{Pic}\, \hat{U} \cong \mathrm{Pic}\, \hat{Y} / (\text{subgroup generated by the } \mathcal{O}_{\hat{Y}}(\hat{E}_i)).$$

There are natural maps $\mathrm{Pic}\, Y \to \mathbf{Z}^n$, $\mathrm{Pic}\, \hat{Y} \to \mathbf{Z}^n$ given by restriction of line bundles to the E_i (respectively the $\hat{E}_i$), and using $E_i \cong \hat{E}_i \cong \mathbf{P}^1$, $\mathrm{Pic}\, \mathbf{P}^1 = \mathbf{Z}$, giving a diagram

$$\begin{tikzcd} \mathrm{Pic}\, Y \arrow[rr] \arrow[dr] & & \mathbf{Z}^n \\ & \mathrm{Pic}\, \hat{Y} \arrow[ur] & \end{tikzcd}$$

For any divisor D on Y, the intersection number $(D \cdot E_i)$ is defined to be the degree of the line bundle $\mathcal{O}_Y(D) \otimes \mathcal{O}_{E_i}$ on E_i. The map $\mathrm{Pic}\, Y \to \mathbf{Z}^n$ can be described as $\mathcal{O}_Y(D) \to (\ldots, (D \cdot E_i), \ldots)$. The intersection matrix $((E_i \cdot E_j))_{1 \leq i, j \leq n}$ is negative definite; hence the composite

$$\bigoplus_{i=1}^{n} \mathbf{Z}[\mathcal{O}_Y(E_i)] \to \mathrm{Pic}\, Y \to \mathbf{Z}^n$$

is injective, and the image has finite index. A similar result holds for $\hat{Y}$. Thus if G is the cokernel of the endomorphism of $\mathbf{Z}^n$ given by the matrix $((E_i \cdot E_j))$, G is a finite abelian group, and we have a diagram

$$\begin{tikzcd} \mathrm{Cl}\, R \arrow[rr] \arrow[dr] & & G \\ & \mathrm{Cl}\, \hat{R} \arrow[ur] & \end{tikzcd}$$

Now $\mathrm{Cl}\, R \hookrightarrow \mathrm{Cl}\, \hat{R}$; since R has a rational singularity, $\mathrm{Cl}\, \hat{R} \cong G$. Thus $\mathrm{Cl}\, R \hookrightarrow G$. Given an ideal $I \subset R$ pure of height 1, it corresponds to a Weil divisor $C \subset X$. If $D \subset Y$ is the strict transform, then $[I] \subset \mathrm{Cl}\, R$ corresponds to the element of G determined by the vector $(\ldots, (D \cdot E_i), \ldots)$. Similar reasoning applies to R.

We deduce two facts used above. First, if R has a rational double point of type D_n, E_6 or E_7, and if E_j is a component of the exceptional divisor with multiplicity 1 in the fundamental cycle then a curve on $\hat{Y}$ meeting $\hat{E}_j$ transversally at 1 point and otherwise disjoint from the exceptional divisor corresponds to the vector $(0, \ldots, 0, 1, 0 \ldots) \in \mathbf{Z}^n$ with 1 in the j^{th} place and 0 elsewhere. Using the known intersection matrix (computed easily from Table II) one checks that this element of G is always non-zero. Similarly, if E_1, E_{n+1} are the extreme components of the exceptional divisor in the minimal resolution of A_n, then a curve on Y meeting exactly one of $\hat{E}_1$, $\hat{E}_{n+1}$ transversally at 1 point, and otherwise disjoint from the exceptional divisor, gives a generator of the class group $\mathbf{Z}/(n+1)\mathbf{Z} \cong G$.

Secondly, we have a criterion for R to be a UFD, namely that the image of the intersection map $\operatorname{Pic} Y \to \mathbf{Z}^n$ is generated by the images of the $\mathcal{O}_Y(E_i)$. Thus if Z is smooth projective surface containing Y (in the obvious sense), and we can find generators for the Neron-Severi group $NS(Z)$ consisting of the exceptional curves $E_1, \ldots, E_n$ and certain other divisors whose supports are disjoint from $\bigcup E_i$, then R is a UFD. Conversely if R is a UFD, such a system of generators can be found for $NS(Z)$. We note that if the geometric genus $P_g(Z) = 0$, the intersection pairing on $NS(Z)$ is unimodular. Since the intersection form of the E_i is an orthogonal direct summand of $NS(Z)$ (if R is a UFD), this forces $G = 1$, i.e., R has an E_8-singularity. Thus the only rational double point which can occur as a UFD on a surface with $P_g = 0$ (eg. a rational surface) is E_8. This explains why our examples of UFD's are somewhat complicated. The above remark is due to R.V. Gurjar.

REFERENCES

1. S. Bloch, *Lectures on algebraic cycles*, Duke Univ. Math. Ser. IV, Durham, North Carolina, 1976.
2. S. Bloch and A. Ogus, *Gersten's conjecture and the homology of schemes*, Ann. Sci. Ecole Norm. Sup. (4), vol. 7 (1974), pp. 181–201.
3. A. Collino, *Quillen's $\mathcal{K}$-theory and algebraic cycles on almost non-singular varieties*, Illinois J. Math, vol. 25 (1981), pp. 654–666.
4. K.R. Coombes and V. Srinivas, *A remark on K_1 of an algebraic surface*, Math. Ann., vol. 265 (1983), pp. 335–342.
5. M. Demazure, H. Pinkham and B. Teissier, *Seminaire sur le singularites des surfaces*, Centre de Math., Ecole Polytechnique, Palaiseau, 1976–77.
6. S. Dutta, M. Hochster and J.E. McLaughlin, preprint, 1982.
7. H.-B. Foxby, *K-theory for complexes with finite length homology*, preprint.
8. D. Grayson, *Localisation for flat modules in algebraic K-theory*, J. Algebra, vol. 61 (1979), pp. 463–496.
9. P. Griffiths and J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience, New York, 1978.
10. R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math., No. 52, Springer-Verlag, New York, 1977.
11. K. Kodaira, *Collected works III*, Princeton Univ. Press, Princeton, N.J., 1975.
12. M. Levine, *On K_0 of a surface with cyclic quotient singularities*, preprint, 1983.
13. M. Levine and V. Srinivas, *Zero cycles on certain singular elliptic surfaces*, Composito Math., vol. 52 (1984), pp. 179–196.
14. M. Levine, *A geometric theory of the Chow ring of a singular variety*, preprint, 1983.
15. A.S. Mercurjev and A.A. Suslin, *K-cohomology of Severi-Brauer varieties and norm residue homomorphism*, Izv. Acad. Sci. USSR., Ser. Math., vol. 46 (1982), pp. 1011–1046.
16. J. Milne, *Etale cohomology*, Princeton Math. Ser. 33, Princeton, 1980.

17. M. NORI, *On certain elliptic surfaces with maximal Picard number*, preprint, 1984.
18. D. QUILLEN, *Higher algebraic K-theory*, Lecture Notes in Math., No. 341, Springer-Verlag, New York, 1973.
19. V. SRINIVAS, *Zero cycles on a singular surface, I, II*, J. Reine Ang. Math., to appear.
20. A. DURFEE, *Fifteen characterisations of rational double points*, Enseignement Math., vol. 25 (1979), pp. 131–163.
21. M. LEVINE, *On the K-groups of surface singularities*, preprint, 1984.
22. J. LIPMAN, *Rational singularities with applications to algebraic surfaces and unique factorisation*, Publ. Math. IHES, vol. 36 (1969), pp. 195–279.

TATA INSTITUTE OF FUNDAMENTAL RESEARCH
BOMBAY, INDIA