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CARTAN SUBMERSIONS AND CARTAN FOLIATIONS[1]

BY

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1. Introduction

Apart from its intrinsic interest, the question of when a submersion is a fibration is an important one in foliation theory. Frequently one encounters a complicated foliation $\mathscr{F}$ of a manifold M which has the property that the lift $\tilde{\mathscr{F}}$ of $\mathscr{F}$ to the universal cover $\tilde{M}$ of M is defined by a submersion $f\colon \tilde{M} \to N$ where N is some smooth manifold (which we may assume is simply connected). If one knows that f is a fibration, then there is an action of $\pi_1(M)$ on N such that the leaf space $M/\mathscr{F}$ is identified with the orbit space of this action. From this one can obtain significant global information concerning the influence of the topology of M upon the structure of $\mathscr{F}$.

In Section 2 we address the question of when a submersion is a fibration. W. Ambrose [1] showed that a local isometry defined on a complete Riemannian manifold is a covering and N. Hicks [13] proved a similar result for local affine isomorphisms. The present author has obtained analogous results for local projective and conformal isomorphisms [4]. R. Hermann [12] showed that a Riemannian submersion defined on a complete Riemannian manifold is a locally trivial fiber bundle (thus generalizing the classical result of C. Ehresmann [9] that a submersion defined on a compact manifold is a locally trivial fiber bundle) and the present author has shown that an affine submersion defined on a complete affinely connected manifold is a Serre fibration [6]. All of the geometric structures occurring in the results quoted above (Riemannian, affine, conformal, and projective geometries) can be treated in a uniform fashion under the rubric of Cartan connections. We consider submersions between manifolds with Cartan connections and we give sufficient conditions for such maps to be fibrations.

Let M be a smooth manifold. Let G be a Lie group and $H \subset G$ a closed subgroup such that $\dim G/H = \dim M$, and let $\pi\colon P \to M$ be a smooth principal H-bundle. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively and for each $A \in \mathfrak{h}$ let A^* be the corresponding fundamental vertical vector field on P. A Cartan connection in P is a $\mathfrak{g}$-valued one-form ω on P

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satisfying:

(i) $\omega(A^*) = A$ for all $A \in \mathfrak{h}$,

(ii) $(R_a)^*\omega = ad(a^{-1})\omega$ for all $a \in H$ where R_a denotes the right translation by a acting on P and $ad(a^{-1})$ is the adjoint action of a^{-1} on $\mathfrak{g}$, and

(iii) ω_u: $T_u(P) \to \mathfrak{g}$ is an isomorphism for all $u \in P$.

The Cartan connection ω is complete if each vector field X on P such that $\omega(X)$ is constant is complete.

For example, in Riemannian geometry $G = \mathbf{R}^n \cdot O(n)$ is the group of isometries of $G/H = \mathbf{R}^n$, π: $P \to M$ is the orthonormal frame bundle of M, and $\omega = \theta + \bar{\omega}$ where $\bar{\omega}$ is the Riemannian connection in P and θ is the canonical $\mathbf{R}^n$-valued one-form on P. In affine geometry $G = \mathbf{R}^n \cdot GL(n, \mathbf{R})$ is the group of affine transformations of $G/H = \mathbf{R}^n$, π: $P \to M$ is the frame bundle of M, and $\omega = \theta + \bar{\omega}$ where $\bar{\omega}$ is the affine (linear) connection in question. In projective geometry $G = PSL(n, \mathbf{R})$ is the group of projective transformations of $G/H = \mathbf{R}P^n$, π: $P \to M$ is a reduction of the bundle $P^2(M)$ of 2-frames of M to $H \subset G^2(n)$ where $G^2(n)$ is the group of 2-frames at $0 \in \mathbf{R}^n$, and ω is the unique normal projective connection. In conformal geometry $G = O(n+1, 1)$ is the group of conformal transformations of $G/H = S^n$, π: $P \to M$ is a reduction of $P^2(M)$ to $H \subset G^2(n)$, and ω is the unique normal conformal connection.

Let M and M' be manifolds. Let G, G' be Lie groups and let H, H' be closed subgroups with $\dim G/H = \dim M$, $\dim G'/H' = \dim M'$. Let π: $P \to M$ be a principal H-bundle and let π': $P' \to M'$ be a principal H'-bundle. Let ω and ω' be Cartan connections in P and P' respectively.

DEFINITION. A Cartan map is a bundle homomorphism f: $M \to M'$, F: $P \to P'$, ϕ: $H \to H'$ and a homomorphism Φ: $G \to G'$ satisfying $F^*\omega' = \Phi_* \circ \omega$ where Φ_*: $\mathfrak{g} \to \mathfrak{g}'$ is the induced homomorphism between Lie algebras. If $G = G'$, $H = H'$, and $\Phi = \mathrm{Id}$, we say f is a local Cartan isomorphism.

THEOREM 1. *Let (f, F, Φ) be a Cartan map with Φ_*: $\mathfrak{g} \to \mathfrak{g}'$ onto. If ω is complete then f is a Serre fibration, F is a locally trivial fiber bundle, and ω' is complete.*

COROLLARY 1. *Let M and M' be connected manifolds of the same dimension each with a Cartan connection and let f: $M \to M'$ be a local Cartan isomorphism. If M is compete then f is a covering map.*

We remark that the results quoted above concerning local isometries and local affine, projective, and conformal isomorphisms are special cases of Corollary 1.

Suppose M and M' are manifolds with affine connections ∇ and ∇' respectively. A map f: $M \to M'$ is affine if whenever $X, Y \in X(M)$ are f-related to $X', Y' \in X(M')$ then $\nabla_X Y$ is f-related to $\nabla'_{X'} Y'$. As we shall see, any affine submersion is part of a Cartan map and so we obtain:

COROLLARY 2. *Let M and M' be affinely connected manifolds and let $f: M \to M'$ be an affine submersion. If M is complete then f is a Serre fibration.*

Recall that a homogeneous space G/H is weakly reductive if there is a subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and $\mathrm{ad}(H)\mathfrak{m} \subset \mathfrak{m}$. Suppose G/H is weakly reductive and let ω be a Cartan connection in the principal H-bundle $\pi: P \to M$. A curve $\sigma: (a, b) \to M$ is a geodesic of ω if $\sigma = \pi \circ \gamma$ where γ is an integral curve of a vector field X and P such that $\omega(X) \in \mathfrak{m}$ is constant. In [17] it is shown that ω is complete if and only if each geodesic is infinitely extendable. We remark that Riemannian and affine geometries are weakly reductive. Assume now that G/H and G'/H' are weakly reductive.

THEOREM 2. *Let $f: M \to M'$ be a submersion such that for each geodesic σ of ω, the curve $f \circ \sigma$ is a geodesic of ω'. If ω is complete then f is a Serre fibration.*

DEFINITION. A subbundle $Q \subset T(M)$ is totally geodesic if whenever a geodesic of ω is tangent to Q at one point, it is tangent to Q at all its points. A geodesic tangent to Q will be called a horizontal geodesic and we say that ω is horizontally complete if each horizontal geodesic is infinitely extendable.

THEOREM 3. *Let $f: M \to M'$ be a submersion. Let $E \subset T(M)$ be the kernel of f_* and let $Q \subset T(M)$ be a complementary totally geodesic subbundle such that for each horizontal geodesic σ of ω, the curve $f \circ \sigma$ is a geodesic of ω'. If ω is horizontally complete (e.g., if ω is complete), then f is a locally trivial fiber bundle.*

COROLLARY 3. *Let M and M' be affinely connected manifolds such that the holonomy group of M is completely reducible and let $f: M \to M'$ be an affine submersion. If M is complete then f is a locally trivial fiber bundle.*

We remark that R. Hermann's result on Riemannian submersions follows from Theorem 3 by taking $Q = E^{\perp}$ and N. Hicks' result on local affine isomorphisms is immediate by taking $Q = T(M)$.

In Section 3 we consider foliations whose transverse structure is modeled on a Cartan geometry. More precisely, any codimension q foliation $\mathscr{F}$ of a manifold M can be defined by an N-cocycle $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}_{\alpha,\beta \in A}$ where N is some q-dimensional manifold (not necessarily connected) and

(i) $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M,

(ii) $f_\alpha: U_\alpha \to N$ is a submersion whose level sets are the leaves of $\mathscr{F} \mid U_\alpha$,

(iii) $g_{\alpha\beta}: f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism satisfying $f_\alpha = g_{\alpha\beta} \circ f_\beta$ on $U_\alpha \cap U_\beta$.

Let G be a Lie group and let H be a closed subgroup of G such that $\dim G/H = \dim N = q$. Let $\pi\colon P \to N$ be a principal H-bundle and let ω be a Cartan connection in P. A diffeomorphism $f\colon U \to V$ between open subsets of N is a Cartan isomorphism if f is induced by a bundle automorphism $F\colon \pi^{-1}(U) \to \pi^{-1}(V)$ satisfying $F^*\omega = \omega$.

DEFINITION. We say $\mathscr{F}$ is a Cartan foliation if each $g_{\alpha\beta}$ is a Cartan isomorphism.

Among the examples of Cartan foliations are Riemannian foliation [27], foliations with transversely projectable connection [19], [20], [21], [22] (basic connection in the sense of [14]), projective foliations [24], [25], conformal foliations [24], [25], [28], Lie foliations [10], [11], and homogeneous foliations [3].

Using the concept of foliated bundle developed by Kamber and Tondeur [14], we define a notion of completeness of a Cartan foliation which generalizes the notion of completeness in the sense of B. Reinhart [27] of a Riemannian foliation (that is, there exists a complete bundle-like metric) and the notion of completeness in the sense of P. Molino [22] of a foliation with transversely projectable connection.

B. Reinhart [27] showed that if $\mathscr{F}$ is a complete Riemannian foliation then all the leaves of $\mathscr{F}$ have the same universal covering space and P. Molino [22] extended this result to foliations admitting a complete transversely projectable connection. A similar result holds for complete conformal foliations [5].

THEOREM 4. *Let $\mathscr{F}$ be a complete Cartan foliation. Then all the leaves of $\mathscr{F}$ have the same universal covering space.*

THEOREM 5. *Let $\mathscr{F}$ be a complete Cartan foliation of M modeled on (N, ω) where N is a connected analytic manifold and ω is a complete analytic Cartan connection. Let $\tilde{M}$ and $\tilde{N}$ denote the universal covers of M and N respectively and let $\tilde{\mathscr{F}}$ be the lift of $\mathscr{F}$ to $\tilde{M}$. Then there is a locally trivial fiber bundle $\tilde{M} \to \tilde{N}$ whose fibers are the leaves of $\tilde{\mathscr{F}}$.*

COROLLARY 4. *Let $\mathscr{F}$ be a complete homogeneous G/H-foliation of M. Then there is a locally trivial fiber bundle $\tilde{M} \to \widetilde{G/H}$ whose fibers are the leaves of $\tilde{\mathscr{F}}$.*

As a corollary to Theorem 5 we will obtain the structure theorem of G. Reeb [26] for codimension one foliations defined by a nonsingular closed one-form and, more generally, E. Fedida's structure theorem [11] for Lie foliations.

Finally, we consider structure equations for flat Riemannian, affine, conformal, and projective foliations. These are homogeneous G/H-foliations where:

(1) Flat Riemannian foliation: $G = \mathbf{R}^q \cdot O(q)$ is the group of isometries of $G/H = \mathbf{R}^q$.

(2) Flat affine foliation: $G = \mathbf{R}^q \cdot GL(q, \mathbf{R})$ is the group of affine transformations of $G/H = \mathbf{R}^q$.

(3) Flat conformal foliation: $G = O(q+1, 1)$ is the group of conformal transformations of $G/H = S^q$.

(4) Flat projective foliation: $G = PSL(q, \mathbf{R})$ is the group of projective transformations of $G/H = \mathbf{R}P^q$.

In [3] it is shown that a codimension one foliation of a manifold M is projective (homogeneous $PSL(2, \mathbf{R})/H = \mathbf{R}P^1$) if and only if it is defined by a smooth nowhere zero one-form ω on M satisfying $d\omega = \omega \wedge \omega_1$, $d\omega_1 = \frac{1}{2}\,\omega_2 \wedge \omega$, $d\omega_2 = \omega_1 \wedge \omega_2$ where ω_1 and ω_2 are one-forms on M. A similar result holds for flat Riemannian and affine foliations of arbitrary codimension. We extend these results to flat projective and conformal foliations of arbitrary codimension.

Specifically, let $m = \dim G$, $k = \dim H$, $q = m - k$. Let

$$\{\theta_1, \ldots, \theta_q, \theta_{q+1}, \ldots, \theta_m\}$$

be a basis of the space of left-invariant one-forms on G such that $\{\theta_{q+1}, \ldots, \theta_m\}$ is a basis of the left-invariant one-forms on H. Then

$$d\theta_i = \sum_{1 \le j < l \le m} C^i_{jl}\theta_j \wedge \theta_l$$

for $i = 1, \ldots, m$ where $C^i_{jl} \in \mathbf{R}$.

THEOREM 6. *Let $\mathscr{F}$ be a codimension q foliation of M with trivial normal bundle. Then $\mathscr{F}$ is a flat Riemannian, affine, conformal, or projective foliation if and only if there exist q linearly independent one-forms $\omega_1, \ldots, \omega_q$ on M defining $\mathscr{F}$ and one-forms $\omega_{q+1}, \ldots, \omega_m$ on M satisfying*

$$d\omega_i = \sum_{1 \le j < l \le m} C^i_{jl}\omega_j \wedge \omega_l, \qquad i = 1, \ldots, m.$$

2. Mappings between manifolds with Cartan connections

We now prove Theorem 1. Let M and M' be manifolds. Let G and G' be Lie groups with closed subgroups H and H' respectively such that $\dim G/H = \dim M$ and $\dim G'/H' = \dim M'$. Let $\pi\colon P \to M$ be a principal H-bundle, let $\pi'\colon P' \to M'$ be a principal H'-bundle, and let ω and ω' be Cartan connections in P and P' respectively with ω complete. Let $f\colon M \to M'$, $F\colon P \to P'$, $\Phi\colon G \to G'$ be a Cartan map with $\Phi_*\colon \mathfrak{g} \to \mathfrak{g}'$ onto. Let $u \in P$ and $Z' \in T_{F(u)}(P')$. Let $Y = \omega'(Z') \in \mathfrak{g}'$ and choose $X \in \mathfrak{g}$ such that $\Phi_*(X) = Y$. Letting Z be the unique vector in $T_u(P)$ satisfying $\omega(Z) = X$ we have that $\omega'(F_*(Z)) = (F^*\omega')(Z) = \Phi_*\omega(Z) = Y$ and so $F_*(Z) = Z'$ thus showing that F (and hence also f) is a submersion. Let $\mathscr{F}$ be the foliation of P defined by F and let $E = \text{kernel}(F_*) \subset T(P)$. Let $A_1, \ldots, A_r, B_1, \ldots, B_s$ be a basis of $\mathfrak{g}$ such that $A_1, \ldots, A_r$ is a basis of kernel (Φ_*). Let $X_1, \ldots, X_r, Y_1, \ldots, Y_s$ be smooth vector fields on P satisfying $\omega(X_i) = A_i$, $\omega(Y_j) = B_j$. Then $X_1, \ldots, X_r$ span E. Let $Q \subset T(P)$ be the subbundle spanned by $Y_1, \ldots, Y_s$. We may regard Q as the normal bundle of $\mathscr{F}$. Let Z_j be the unique vector field on P' satisfying $\omega'(Z_j) = \Phi_*(B_j)$. Then $\omega'(F_*(Y_j)) = \Phi_*\omega(Y_j) = \omega'(Z_j)$ and so Y_j is F-related to Z_j thus showing that $Y_1, \ldots, Y_s$ are parallel along the leaves of $\mathscr{F}$. Since ω is complete we have that $Y_1, \ldots, Y_s$ are complete and so $\mathscr{F}$ is a transversely complete foliation of P. Since the leaves of $\mathscr{F}$ are closed, the space of leaves $P/\mathscr{F}$ is a smooth Hausdorff manifold and the natural projection $q\colon P \to P/\mathscr{F}$ is a locally trivial fiber bundle [23]. Now F induces a local diffeomorphism $h\colon P/\mathscr{F} \to P'$ such that $F = h \circ q$ and $Y_1, \ldots Y_s$ project to complete vector fields on $P/\mathscr{F}$ which are h-related to $Z_1, \ldots, Z_s$. Hence by Theorem 2.1 below, h is a covering map and so F is a locally trivial fiber bundle. Since $f \circ \pi$ is a Serre fibration and π is a fiber bundle, it follows that f is a Serre fibration [6]. Clearly ω' is complete and so Theorem 1 is proved.

To obtain Corollary 1 note that Φ_* is the identity and so f is a Serre fibration by Theorem 1. Hence given any path σ in M' and $p_0 \in f^{-1}\{\sigma(0)\}$, there is a unique lift of σ to M starting at p_0 whence f is a covering projection. Corollary 2 is a consequence of Theorem 1 and the following example.

Example. Let M and M' be manifolds with linear connections ∇ and ∇', respectively and let $f\colon M \to M'$ be an affine submersion. Let $\pi\colon F(M) \to M$ be the bundle of frames of M, a principal $GL(n, \mathbf{R})$-bundle where $n = \dim M$. Let

$$E = \text{kernel}(f_*) \subset T(M).$$

We say that a frame u at $p \in M$ is adapted if $u = (u_1, \ldots u_k, v_1, \ldots, v_q)$ where $\{u_1, \ldots, u_k\}$ is a basis of E_p $(k + q = n)$. Let $\pi\colon P \to M$ be the bundle of adapted frames, a reduction of $F(M)$ to the group

$$H = \left\{ \begin{pmatrix} A & * \\ 0 & B \end{pmatrix} : A \in GL(k, \mathbf{R}) \right\}.$$

Let $u = (u_1, \ldots, u_k, v_1, \ldots, v_q) \in P$ and let $p = \pi(u)$. Since

$\{(f_*(v_1), \dots f_*(v_q))\}$ is a basis of $T_{f(p)}(M')$, we obtain a map $F\colon P \to P'$ such that $f \circ \pi = \pi' \circ F$ where $\pi'\colon P' \to M'$ is the frame bundle of M', a principal $H' = GL(q, \mathbf{R})$-bundle. Then (f, F, ϕ) is a homomorphism of principal bundles where $\phi\colon H \to H'$ is given by

$$\phi\begin{pmatrix} A & * \\ 0 & B \end{pmatrix} = B.$$

The connection in $F(M)$ corresponding to ∇ reduces to P. Let $\bar{\omega}$ be the associated connection form, an $\mathfrak{h}$-valued one-form on P. Let $\bar{\omega}'$ be the connection form on P' associated to ∇', an $\mathfrak{h}'$-valued one-form on P'. Let θ (respectively, θ') be the canonical $\mathbf{R}^n$ (respectively, $\mathbf{R}^q$)-valued one-form on P (respectively, P'). Let

$$G = \mathbf{R}^n \cdot H = \left\{\begin{pmatrix} A & & * & c_1 \\ & & & \vdots \\ 0 & & B & c_n \\ 0 & \cdots & 0 & 1 \end{pmatrix}\right\},$$

$$G' = \mathbf{R}^q \cdot H' = \left\{\begin{pmatrix} B & & & d_1 \\ & & & \vdots \\ & & & d_q \\ 0 & \cdots & 0 & 1 \end{pmatrix}\right\}.$$

Then $\omega = \theta + \bar{\omega}$ (respectively, $\omega' = \theta' + \bar{\omega}'$) is a $\mathfrak{g}$ (respectively, $\mathfrak{g}'$)-valued one-form on P (respectively, P') defining a Cartan connection in P (respectively, P'). Define $\Phi\colon G \to G'$ by

$$\begin{pmatrix} A & & * & c_1 \\ & & & \vdots \\ & & & c_k \\ & & & d_1 \\ & & & \vdots \\ 0 & & B & d_q \\ 0 & \cdots & 0 & 1 \end{pmatrix} \to \begin{pmatrix} B & & & d_1 \\ & & & \vdots \\ & & & d_q \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

then Φ extends ϕ. Let $\rho\colon \mathbf{R}^n \to \mathbf{R}^q$ be projection onto the last q-coordinates. An elementary argument shows that $F^*\bar{\omega}' = \phi_* \circ \bar{\omega}$ and $F^*\theta' = \rho \circ \theta$. Since $\Phi_*\colon \mathfrak{g} \to \mathfrak{g}'$ is just (ρ, ϕ_*), we have $F^*\omega' = \Phi_* \circ \omega$ and so (f, F, Φ) is a Cartan map. Note that Φ_* is onto. Also, ω is complete if and only if ∇ is complete [17].

Let M and M' be manifolds and let G/H and G'/H' be weakly reductive homogeneous spaces with $\dim G/H = \dim M$ and $\dim G'/H' = \dim M'$. Let $\pi: P \to M$ (respectively, $\pi': P' \to M'$) be a principal H (respectively, H')-bundle and let ω and ω' be Cartan connections in P and P' respectively with ω complete. Let $f: M \to M'$ be a submersion which sends geodesics of ω to geodesics of ω'. The geodesics of ω and ω' determine sprays X and X' on $T(M)$ and $T(M')$ [17] relative to which f is spray-preserving (that is, X and X' are f_*-related). There exist linear connections ∇ and ∇' on M and M' giving rise to the sprays X and X' [2] and with respect to these linear connections f is affine. Since ω is complete, X is complete (in fact, ω is complete if and only if X is complete [17]) and so ∇ is complete. Hence by Corollary 2, f is a Serre fibration which proves Theorem 2.

Let M be a smooth manifold and lèt $\pi: T(M) \to M$ be the tangent bundle of M. Let X be a vector field on $T(M)$. Then X is a spray if

$$\pi_* \circ X = \mathrm{Id}_{T(M)} \quad \text{and} \quad X_{cv} = c\mu_{c_*}(X_v) \quad \text{for } v \in T(M), c \in \mathbf{R}$$

where $\mu_c: T(M) \to T(M)$ is multiplication by c. For $v \in T(M)$, let σ_v be the integral curve of X through v and let $\alpha_v = \pi \circ \sigma_v$. Then X is a spray if and only if $\alpha_v' = \sigma_v$ and $\alpha_{cv}(t) = \alpha_v(ct)$ in which case the curves α_v are the geodesics of X and the exponential map at a point $p \in M$ given by $\exp(v) = \alpha_v(1)$ maps a neighborhood of 0 in $T_P(M)$ diffeomorphically onto a neighborhood of p in M. We say that a subbundle Q of $T(M)$ is totally geodesic if Q is a union of integral curves of X.

THEOREM 2.1. *Let M and M' be connected manifolds with sprays X and X' respectively and let f: $M \to M'$ be a submersion. Let $E \subset T(M)$ be the kernel of f_* and let $Q \subset T(M)$ be a complementary totally geodesic subbundle such that $X|Q$ is f_*-related to X'. If $X|Q$ is complete then f is onto, f: $M \to M'$ is a locally trivial fiber bundle, and X' is complete.*

Proof. Since f is a submersion, $f(M)$ is open in M'. To show f is onto it suffices to show that $\overline{f(M)}$ is also closed. Let $q \in \overline{f(M)}$. Let V be a neighborhood of q in M' and U a neighborhood of 0 in $T_q(M')$ such that $\exp: U \to V$ is a diffeomorphism. Let $z \in V \cap f(M)$. Let $u \in U$ be such that $\exp(u) = z$. Then $\alpha_u(0) = q$, $\dot\alpha_u(0) = u$, and $\alpha_u(1) = z$. Let $p \in f^{-1}\{z\}$ and let $v \in Q_p$ be the unique vector satisfying $f_*(v) = -\dot\alpha_u(1)$. Let $\rho(t) = \alpha_u(1-t)$. Then ρ is a geodesic in M' satisfying $\rho(0) = z$, $\dot\rho(0) = -\dot\alpha_u(1)$ and so $\rho = \alpha_{f_*(v)} = f \circ \alpha_v$ since $X|Q$ and X' are f_*-related. Now $\alpha_v(1)$ is defined (since $X|Q$ is complete) and $f(\alpha_v(1)) = q$. Then $q \in f(M)$ and so $f(M)$ is closed. Clearly X' is complete.

Let $q \in M'$. Let V be a neighborhood of q in M' and U a neighborhood of 0 in $T_q(M')$ such that $\exp: U \to V$ is a diffeomorphism. Let $L = f^{-1}\{q\}$.

Define Φ: $V \times L \to M$ as follows. Let $(z, p) \in V \times L$. Let $u \in U$ be the unique vector satisfying $\exp(u) = z$, let $v \in Q_p$ be the unique vector satisfying $f_*(v) = u$, and set $\Phi(z, p) = \exp(v)$. Clearly Φ: $V \times L \to f^{-1}(V)$ and $f \circ \Phi$ is projection onto the first factor. Define Ψ: $f^{-1}(V) \to V \times L$ as follows. Let $x \in f^{-1}(V)$. Let $y = f(x) \in V$. Then $y = \exp(u)$ where $u \in U$. Let τ be the unique geodesic in M' satisfying $\tau(0) = q$, $\dot{\tau}(0) = u$. Then $\tau(1) = y$. Let $v \in Q_x$ be the unique vector satisfying $f_*(v) = -\dot{\tau}(1)$. Let σ be the unique geodesic in M satisfying $\sigma(0) = x$, $\dot{\sigma}(0) = v$. Then $\sigma(1) \in L$. Indeed, if we let $\rho(t) = \tau(1 - t)$, then $f \circ \sigma$ and ρ are geodesics in M' satisfying the same initial condition whence $\rho = f \circ \sigma$ and so $f(\sigma(1)) = \rho(1) = q$. Set $\Psi(x) = (f(x), \sigma(1))$. Then Ψ is inverse to Φ (this uses that Q is totally geodesic) and so Φ is a diffeomorphism which completes the proof.

Theorem 3 now follows from Theorem 2.1. To obtain Corollary 3, let $E \subset T(M)$ be the kernel of f_*. Then E is a holonomy-invariant distribution on M. Since the holonomy group of ∇ is completely reducible, there exists a complementary holonomy-invariant distribution $Q \subset T(M)$. Since Q is totally geodesic, Corollary 3 now follows from Theorem 3.

COROLLARY 2.2 (R. Hermann [12]). *Let M and M' be connected Riemannian manifolds and let f: $M \to M'$ be a Riemannian submersion. If M is complete, then f is a locally trivial fiber bundle.*

Proof. Let ω and ω' be the Cartan connections in the orthonormal frame bundles of M and M' respectively arising from the Riemannian connections in these bundles. Let $Q = E^{\perp}$. Then Q is totally geodesic with respect to ω and for each geodesic σ of ω which is tangent to Q, the curve $f \circ \sigma$ is a geodesic of ω' [27]. If M is complete, then ω is horizontally complete and so by Theorem 3 f is a locally trivial fiber bundle.

3. Cartan foliations

Let N be a q-dimensional manifold. Let G be a Lie group and let H be a closed subgroup of G such that $\dim G/H = q$. Let π: $P \to N$ be a principal H-bundle and let ω be a Cartan connection in P. Let $\mathcal{F}$ be a codimension q foliation of the manifold M. We say $\mathcal{F}$ is a Cartan foliation modeled on (N, ω) if there is an N-cocycle $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}$ defining $\mathcal{F}$ such that each $g_{\alpha\beta}$ is a Cartan isomorphism.

Examples. (1) Riemannian foliation [27]. Here N is a Riemannian manifold, each $g_{\alpha\beta}$ is an isometry, $G = \mathbf{R}^q \cdot O(q)$, $H = O(q)$, $G/H = \mathbf{R}^q$, π: $P \to N$ is the orthonormal frame bundle of N, and $\omega = \theta + \bar{\omega}$ where θ is the canonical $\mathbf{R}^q$-valued one-form on P and $\bar{\omega}$ is the connection form ($o(q)$-valued) of the Riemannian connection in P.

(2) Foliation with transversely projectable connection [19], [20], [21], [22] (basic connection in the sense of [14]). Here N is a manifold with a linear connection ∇, each $g_{\alpha\beta}$ is an affine transformation, $G = \mathbf{R}^q \cdot GL(q, \mathbf{R})$, $H = GL(q, \mathbf{R})$, $G/H = \mathbf{R}^q$, $\pi: P \to N$ is the frame bundle of N, and $\omega = \theta + \overline{\omega}$ where θ is the canonical $\mathbf{R}^q$-valued one-form on P and $\overline{\omega}$ is the connection form ($gl(q, \mathbf{R})$-valued) corresponding to ∇.

(3) Projective foliation [24], [25]. Here N is a manifold with a torsion-free linear connection ∇, each $g_{\alpha\beta}$ is a projective transformation (i.e., $g_{\alpha\beta}$ sends geodesics of ∇ to geodesics of ∇ disregarding parameterizations), $G = PSL(q, \mathbf{R})$, and $G/H = \mathbf{R}P^q$. The projective equivalence class of ∇ determines a projective structure on N, i.e., a reduction of the bundle $P^2(N) \to N$ of 2-frames over N (a principal $G^2(q)$-bundle where $G^2(q)$ is the group of 2-frames at $0 \in \mathbf{R}^q$) to $H \subset G^2(q)$. Then $\pi: P \to N$ is this reduced bundle and ω is the normal projective connection in P [18].

(4) Conformal foliation [24], [25], [28]. Here N is a Riemannian manifold, each $g_{\alpha\beta}$ is a conformal transformation, $G = O(q + 1, 1)$, and $G/H = S^q$. The conformal equivalence class of the metric on N determines a conformal structure on N, i.e., a reduction of $P^2(N)$ to $H \subset G^2(q)$. Then $\pi: P \to N$ is this reduced bundle and ω is the normal conformal connection in P [18].

(5) Lie foliation [10], [11]. Here N is a Lie group, each $g_{\alpha\beta}$ is the restriction of a left translation, $G = N$, $H = \{e\}$, $P = G$, $\pi: P \to N$ is the identity map, and ω is the Maurer-Cartan form on G.

(6) Homogeneous foliations [3]. Here N is a homogeneous space G/H, and $g_{\alpha\beta}$ is the restriction of a G-translation of G/H, $\pi: P \to N$ is the principal H-bundle $G \to G/H$, and ω is the Maurer-Cartan form on G.

Let us recall briefly the notion of foliated bundle [7], [14], [15], [20], [21]. Let $(M, \mathscr{F})$ be a foliated manifold and let $E \subset T(M)$ be the tangent bundle of $\mathscr{F}$. Let H be a Lie group. A principal H-bundle $\overline{\pi}: \overline{P} \to M$ is called a foliated bundle if there is an H-invariant foliation $\tilde{\mathscr{F}}$ of $\overline{P}$ satisfying

$$\tilde{E}_u \cap V_u = 0 \quad \text{and} \quad \pi_{*u}(\tilde{E}_u) = E_{\pi(u)}$$

for all $u \in P$ where $\tilde{E}$ is the tangent bundle of $\tilde{\mathscr{F}}$ and $V = \text{kernel}\ (\overline{\pi}_*)$.

PROPOSITION 3.1. *Let $(M, \mathscr{F})$ be a Cartan foliation modeled on (N, ω). There is a canonically defined foliated principal H-bundle $\overline{\pi}: \overline{P} \to M$ and a $\mathfrak{g}$-valued one-form $\tilde{\omega}$ on $\overline{P}$ satisfying*:

(i) $\tilde{\omega}(A^*) = A$ *for all* $A \in \mathfrak{h}$;

(ii) $(R_a^*)\tilde{\omega} = \text{ad}(a^{-1})\tilde{\omega}$ *for all* $a \in H$;

(iii) $\tilde{\omega}_u: T_u(\overline{P}) \to \mathfrak{g}$ *is onto and* $\tilde{\omega}_u(\tilde{E}_u) = 0$ *for all* $u \in P$;

(iv) $L_X\tilde{\omega} = 0$ *for all* $X \in \Gamma(\tilde{E})$ *where* $\Gamma(\tilde{E})$ *denotes the smooth sections of* $\tilde{E}$ *and* L_X *is the Lie derivative.*

Proof. Let $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}$ be an N-cocycle defining $\mathscr{F}$. Define $\bar{P}$ so that $\bar{P}|U_\alpha = f_\alpha^*(P|f_\alpha(U_\alpha))$. This is well-defined since each $g_{\alpha\beta}$ is a local automorphism of π: $P \to N$. Then $\tilde{\omega}$ is defined by locally pulling back ω to $\bar{P}$. These local pull-backs agree on overlaps since the lifts of the $g_{\alpha\beta}$'s to P preserve ω.

For example, in the case of a Riemannian foliation the metric on N induces a metric on the normal bundle Q of $\mathscr{F}$, the bundle $\bar{\pi}$: $\bar{P} \to M$ is the orthonormal frame bundle of Q, and $\tilde{\omega} = \theta + \bar{\omega}$ where θ is the canonical $\mathbf{R}^q$-valued one-form on $\bar{P}$ and $\bar{\omega}$ is the unique torsion-free basic connection in $\bar{P}$. In the case of a foliation with a transversely projectable connection $\bar{\omega}$, the bundle $\bar{\pi}$: $\bar{P} \to M$ is the frame bundle of Q and $\tilde{\omega} = \theta + \bar{\omega}$. In the case of a projective (respectively, conformal) foliation $\bar{\pi}$: $\bar{P} \to M$ is the projective (respectively, conformal) normal bundle of $\mathscr{F}$ and $\bar{\omega}$ is the "pull-back" of ω. In the case of a Lie or homogeneous foliation $\bar{\pi}$: $\bar{P} \to M$ is the principal H-bundle constructed in the proof of Theorem 2 of [3].

Let $\tilde{Q} = T(\bar{P})/\tilde{E}$ be the normal bundle of $\tilde{\mathscr{F}}$. Then for all $u \in P$, $\tilde{\omega}_u$: $T_u(\bar{P}) \to \mathfrak{g}$ induces an isomorphism $\tilde{\omega}_u$: $\tilde{Q}_u \to \mathfrak{g}$.

DEFINITION. A section $\tilde{Y}$ of $\tilde{Q}$ is said to be complete if there exists a complete vector field Y on $\bar{P}$ such that $\tau(Y) = \tilde{Y}$ where τ: $T(\bar{P}) \to \tilde{Q}$ is the natural projection. We say $\tilde{\omega}$ is complete if each section $\tilde{Y}$ of $\tilde{Q}$ such that $\tilde{\omega}(\tilde{Y})$ is constant is complete in which case we say that $\mathscr{F}$ is a complete Cartan foliation.

Example. If M and H are compact, then $\mathscr{F}$ is complete. In particular, a Lie foliation of a compact manifold is a complete Cartan foliation.

Example. Let $\bar{\omega}$ be a transversely projectable connection for $\mathscr{F}$ and let $\tilde{\omega} = \theta + \bar{\omega}$. Then $\mathscr{F}$ is a complete Cartan foliation (i.e., $\tilde{\omega}$ is complete) if and only if $\bar{\omega}$ is complete in the sense of [22].

Example. Let $\mathscr{F}$ be a complete Riemannian foliation in the sense of B. Reinhart [27]. The following proposition shows that $\mathscr{F}$ is a complete Cartan foliation.

PROPOSITION 3.2. *Let $\mathscr{F}$ be a foliation of a manifold M and let g be a Riemannian metric on M which is bundle-like with respect to $\mathscr{F}$. If g is horizontally complete (e.g., if g is complete), then $\mathscr{F}$ is a complete Cartan foliation.*

Proof. We already know that $\mathscr{F}$ is a Cartan foliation. We must show that $\tilde{\omega} = \theta + \bar{\omega}$ defined on the orthonormal frame bundle $\bar{\pi}$: $O(Q) \to M$ of Q is complete where θ is the canonical $\mathbf{R}^q$-valued one-form on $O(Q)$ and $\bar{\omega}$ is the Riemannian Bott connection in $O(Q)$. Let $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}$ be an N-cocycle defining $\mathscr{F}$ where N is a Riemannian manifold and each f_α is a Riemannian submersion. Let $H' \subset T(O(N))$ be the horizontal distribution corresponding to the Riemannian connection on N and for each $h \in \mathbf{R}^q$ let $B'(h)$ be the corresponding standard horizontal vector field on $O(N)$.

Now $\{\bar{\pi}^{-1}(U_\alpha), f_{\alpha_*}, g_{\alpha\beta_*})\}$ is an $O(N)$-cocycle on $O(Q)$ and hence defines a foliation $\tilde{\mathscr{F}}$ of $O(Q)$ with $\dim \tilde{\mathscr{F}} = \dim \mathscr{F}$. Let $H \subset T(O(Q))$ be the horizontal distribution corresponding to $\bar{\omega}$. Then $\tilde{E} \subset H$ where $\tilde{E}$ is the tangent bundle of $\tilde{\mathscr{F}}$. We may regard each $u \in O(Q)$ as the vector space isomorphism $u\colon \mathbf{R}^q \to Q_{\bar{\pi}(u)}$ which sends the standard basis $\{e_1, \ldots, e_q\}$ of $\mathbf{R}^q$ to the frame u of $Q_{\bar{\pi}(u)}$. Let $Q' = H/\tilde{E}$, a q-plane bundle over $O(Q)$. Note that $\bar{\pi}\colon O(Q) \to M$ induces $\bar{\pi}_*\colon Q' \to Q$, an isomorphism on fibers. Let $h \in \mathbf{R}^q$. For $u \in O(Q)$, let $B(h)_u \in Q'_u$ be the unique element such that $\bar{\pi}_{*u}(B(h)_u) = u(h)$. Then $B(h)$ is a section of Q'. Note that $Q' \subset \tilde{Q} = T(O(Q))/\tilde{E} =$ normal bundle of $\tilde{\mathscr{F}}$.

To prove the proposition we must show that $B(h)$ is complete for all $h \in \mathbf{R}^q$. Let $h \in \mathbf{R}^q$. We must produce a complete vector field $C(h)$ on $O(Q)$ such that $\tau(C(h)) = B(h)$ where $\tau\colon T(O(Q)) \to \tilde{Q}$ is the natural projection. Note, necessarily $C(h) \in \Gamma(H)$. For each $u \in O(Q)$, $\bar{\pi}_{*u}\colon H_u \to T_{\bar{\pi}(u)}(M)$ is an isomorphism. Let $(E_u^\perp)^\sim \subset H_u$ be the subspace corresponding to $E^\perp_{\bar{\pi}(u)} \subset T_{\bar{\pi}(u)}(M)$. Then $(E^\perp)^\sim$ is a q-plane bundle over $O(Q)$ which is isomorphic to Q'. Let $C(h)_u$ be the unique vector in $(E_u^\perp)^\sim$ corresponding to $B(h)_u \in Q'_u$. Then $C(h)$ is a vector field on $O(Q)$ satisfying $\tau(C(h)) = B(h)$. An elementary argument shows that $C(h)$ is f_{α_*}-related to $B'(h)$. Let σ be an integral curve of $C(h)$. Then $\bar{\pi} \circ \sigma$ is an $E^\perp$-curve in M which projects under f_α to a geodesic in N. Hence $\bar{\pi} \circ \sigma$ is a (horizontal) geodesic in M and so is infinitely extendable. Hence σ is infinitely extendable and so $C(h)$ is complete.

We now prove Theorem 4. Since the leaves of $\tilde{\mathscr{F}}$ are coverings of the leaves of $\mathscr{F}$, it suffices to show that all the leaves of $\tilde{\mathscr{F}}$ are diffeomorphic. Let Y be any smooth vector field on $\bar{P}$ such that $\tilde{\omega}(Y)$ is constant. Then for any $X \in \Gamma(\tilde{E})$ we have

$$-\tfrac{1}{2}\tilde{\omega}[X, Y] = d\tilde{\omega}(X, Y) = (L_X\tilde{\omega})(Y) - d(\tilde{\omega}(X))(Y) = 0$$

and so $[X, Y] \in \Gamma(\tilde{E})$. Let $X_1, \ldots, X_r$ be a basis of $\mathfrak{g}$. Let $\tilde{Y}_1, \ldots, \tilde{Y}_r \in \Gamma(\tilde{Q})$ be the unique sections satisfying $\tilde{\omega}(\tilde{Y}_i) = X_i$, $i = 1, \ldots, r$. Let $Y_1, \ldots, Y_r$ be complete vector fields on $\bar{P}$ satisfying $\tau(Y_i) = \tilde{Y}_i$, $i = 1, \ldots, r$. For $i = 1, \ldots, r$ let ϕ_t^i, $t \in \mathbf{R}$, be the flow generated by Y_i. Since $[X, Y_i] \in \Gamma(\tilde{E})$ for all $X \in \Gamma(\tilde{E})$, it follows that ϕ_t^i preserves $\tilde{\mathscr{F}}$ and the group generated by the diffeomorphisms ϕ_t^i acts transitively on the set of leaves of each connected component of $\bar{P}$ [23]. Finally, we get from one component of $\bar{P}$ to another by a suitable element of H.

COROLLARY 3.3 (B. Reinhart [27]). *Let $\mathscr{F}$ be a Riemannian foliation of a manifold M with a complete bundle-like metric. Then all the leaves of $\mathscr{F}$ have the same universal covering space.*

Proof. By Proposition 3.2, $\mathscr{F}$ is a complete Cartan foliation and so the result follows from Theorem 4.

COROLLARY 3.4 (P. Molino [22]). *Let $\mathscr{F}$ be a foliation admitting a complete transversely projectable connection. Then all the leaves of $\mathscr{F}$ have the same universal covering space.*

Proof. $\mathscr{F}$ is a complete Cartan foliation.

To prove Theorem 5 we will need the following two lemmas. The proof of the first is an elementary argument which we omit.

LEMMA 3.5. *Let $\mathscr{F}$ be a Cartan foliation of M. Let $\tilde{M}$ be the universal cover of M and let $\tilde{\mathscr{F}}$ be the Cartan foliation of $\tilde{M}$ obtained by lifting $\mathscr{F}$. If $\mathscr{F}$ is complete, then so is $\tilde{\mathscr{F}}$.*

LEMMA 3.6. *Let N be a connected manifold with $\dim N = \dim G/H$, let $\pi\colon P \to N$ be a principal H-bundle, and let ω be a Cartan connection in P. Let M be a connected manifold and let $f\colon M \to N$ be a submersion. Let $\bar{\pi}\colon \bar{P} \to M$ be the pull-back of P under f and let $F\colon \bar{P} \to P$ be the map such that $\pi \circ F = f \circ \bar{\pi}$. Let $\tilde{\omega} = F^*\omega$. If $\tilde{\omega}$ is complete, then f is a locally trivial fiber bundle.*

Proof. Let $\mathscr{F}$ and $\tilde{\mathscr{F}}$ be the foliations of M and $\bar{P}$ respectively defined by the submersions f and F respectively. Let $X_1, \ldots, X_q, X_{q+1}, \ldots, X_r$ be a basis of $\mathfrak{g}$ such that $X_{q+1}, \ldots, X_r$ is a basis of $\mathfrak{h}$. For each $i = 1, \ldots, q$ let $\tilde{Y}_i$ be the section of the normal bundle of $\tilde{\mathscr{F}}$ satisfying $\tilde{\omega}(\tilde{Y}_i) = X_i$ and let Y_i be a complete vector field on $\bar{P}$ which projects to $\tilde{Y}_i$. For each $i = q+1, \ldots, r$ let Y_i be the fundamental vector field on $\bar{P}$ corresponding to X_i. For each $i = 1, \ldots, r$ let ϕ_t^i, $t \in \mathbf{R}$, be the flow generated by Y_i. Since $\tilde{\omega}(Y_i)$ is constant, the diffeomorphisms ϕ_t^i send leaves to leaves. Let $y_0 \in \bar{P}$ and let $\tilde{L}$ be the leaf of $\tilde{\mathscr{F}}$ through y_0. Define $\Phi\colon \mathbf{R}^r \times \tilde{L} \to \bar{P}$ by

$$\Phi(t_{q+1}, \ldots, t_r, t_1, \ldots, t_q, y) = \phi_{t_{q+1}}^{q+1} \circ \cdots \circ \phi_{t_r}^r \circ \phi_{t_1}^1 \circ \cdots \circ \phi_{t_q}^q(y).$$

Since the leaves of $\tilde{\mathscr{F}}$ are closed, there is a neighborhood V of 0 in $\mathbf{R}^r$ such that $\Phi\colon V \times \tilde{L} \to U$ is a diffeomorphism where U is an open saturated set in $\bar{P}$ [23]. We may assume V is of the form $V_1 \times V_2$ where V_1 is a neighborhood of 0 in $\mathbf{R}^{r-q}$ and V_2 is a neighborhood of 0 in $\mathbf{R}^q$. Let $L = \bar{\pi}(\tilde{L})$. Since $Y_{q+1}, \ldots, Y_r$ are vertical, Φ induces a smooth map $\Psi\colon V_2 \times L \to M$ such that $\bar{\pi} \circ \Phi = \Psi \circ (\rho \times \bar{\pi})$ where $\rho\colon V \to V_2$ is projection onto the second factor. By shrinking V_2 if necessary, we may assume that Ψ is a diffeomorphism. Thus $\bar{\pi}(U)$ is an open saturated set in M and Ψ maps the foliation of $V_2 \times L$ by leaves of the form $\{t\} \times L$, $t \in V_2$, diffeomorphically to $\mathcal{F}$. Let C be a compact neighborhood of 0 in $\mathbf{R}^q$, $C \subset V_2$. Then $\Psi(C \times L)$ is a closed saturated neighborhood of L in M and so the leaf space $M/\mathcal{F}$ is regular. Since the leaves of $\mathcal{F}$ are closed, $M/\mathcal{F}$ is Hausdorff. Thus $M/\mathcal{F}$ is a smooth Hausdorff manifold and the natural projection $M \to M/\mathcal{F}$ is a locally trivial fiber bundle. The principal H-bundle $\bar{\pi}$: $\bar{P} \to M$ induces a principal H-bundle π': $\bar{P}/\tilde{\mathcal{F}} \to M/\mathcal{F}$ and $\tilde{\omega}$ induces a complete Cartan connection ω' in this bundle such that the map $\bar{f}$: $M/\mathcal{F} \to N$ induced by f is a local Cartan isomorphism. By Corollary 1, $\bar{f}$ is a covering map and hence f is a locally trivial fiber bundle.

Now assume the hypotheses of Theorem 5. We may assume that $\mathcal{F}$ is defined by an $\tilde{N}$-cocycle $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}$ where each $g_{\alpha\beta}$ is a Cartan isomorphism of the lift of ω to $\tilde{N}$. Without loss of generality, we may assume that $U_\alpha \cap U_\beta$ is connected whenever it is nonempty. Since the lift of ω to $\tilde{N}$ is a complete analytic Cartan connection on a simply connected analytic manifold, $g_{\alpha\beta}$ can be uniquely extended to a Cartan isomorphism of $\tilde{N}$ [8], [16]. A monodromy argument yields a submersion f: $\tilde{M} \to \tilde{N}$ constant along the leaves of $\tilde{\mathcal{F}}$. By Lemma 3.5 $\tilde{\mathcal{F}}$ is a complete Cartan foliation and hence by Lemma 3.6 f is a locally trivial fiber bundle.

Definition. A homogeneous foliation is said to be complete if it is complete as a Cartan foliation.

To prove Corollary 4 note that a complete homogeneous foliation is a complete Cartan foliation modeled on (N, ω) where $N = G/H$ and ω is the Maurer-Cartan form on G. Since G/H is an analytic manifold and ω is a complete analytic Cartan connection, the result follows from Theorem 5.

From Corollary 4 we obtain Reeb's structure theorem for codimension one foliations defined by a closed one-form.

Corollary 3.7 (G. Reeb [26]). *Let $\mathcal{F}$ be a codimension one foliation of a compact manifold M defined by a nonsingular closed one-form. Then the universal cover of M is a product $L \times \mathbf{R}$ and the lifted foliation is the product foliation.*

Proof. $\mathcal{F}$ is a complete homogeneous foliation modeled on G/H where $G = \mathbf{R}$, $H = 0$. By Corollary 4, the leaves of the lifted foliation are the fibers of a locally trivially fiber bundle $\tilde{M} \to \mathbf{R}$. Since $\mathbf{R}$ is contractible, we have $\tilde{M} \cong L \times \mathbf{R}$.

More generally, since a Lie foliation of a compact manifold is a complete homogeneous foliation, from Corollary 4 we obtain Fedida's structure theorem for Lie foliations.

Corollary 3.8 (E. Fédida [11]). *Let G be a Lie group and let $\mathcal{F}$ be a Lie G-foliation of a compact manifold M. Then the universal cover of M fibers over the universal cover of G, the fibers being the leaves of the lifted foliation.*

We now prove Theorem 6. The proof for the flat Riemannian and affine cases is found in [3]. To prove the result in the flat conformal and projective cases we first remark that the "if" part of the theorem is established by an elementary argument which we omit. To prove the "only if" part, assume $\mathscr{F}$ is a codimension q flat conformal or projective foliation with trivial normal bundle. We first handle the conformal case. Thus $G = O(q+1,1)$, $G/H = S^q$, and

$$H = \left\{ \begin{pmatrix} a^{-1} & 0 & 0 \\ v & A & 0 \\ b & \xi & a \end{pmatrix} \in O(q+1,1) \right\}$$

where $A \in O(q)$, $a \in \mathbf{R}$, ξ is a row q-vector. Let π_2: $P^2(G/H) \to G/H$ be the bundle of 2-frames of G/H, a principal $G^2(q)$-bundle where $G^2(q)$ is the group of 2-frames at $0 \in \mathbf{R}^q$. There are maps $G \to P^2(G/H)$ and $H \to G^2(q)$ such that the principal H-bundle p: $G \to G/H$ is a reduction of the principal $G^2(q)$-bundle π_2: $P^2(G/H) \to G/H$ to H [18]. If $g \in G$ then g: $G/H \to G/H$ induces

$$g^{(2)}\colon P^2(G/H) \to P^2(G/H)$$

which preserves G and $g^{(2)}|G = L_g$, left translation by g. Let

$$\bar{\pi}\colon P^2(M,\mathscr{F}) \to M$$

be the bundle of transverse 2-frames over M, a principal $G^2(q)$-bundle. Let $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}$ be a G/H-cocycle defining $\mathscr{F}$. Since each

$$g_{\alpha\beta}^{(2)}\colon P^2(G/H)|f_\beta(U_\alpha \cap U_\beta) \to P^2(G/H)|f_\alpha(U_\alpha \cap U_\beta)$$

preserves the subbundle G, it follows that $P^2(M,\mathscr{F})$ admits a reduction to a foliated principal H-bundle $\bar{P} \subset P^2(M,\mathscr{F})$ such that $\{\bar{\pi}^{-1}(U_\alpha), f_\alpha^{(2)}, g_{\alpha\beta}^{(2)} = L_{g_{\alpha\beta}})\}$ is a G-cocycle on $\bar{P}$. Let $\mathscr{F}_0$ be the foliation of G defined by the submersion p: $G \to G/H$. Then $\mathscr{F}_0$ is defined by $\theta_1, \dots, \theta_q$ satisfying

$$d\theta_i = \sum_{1 \le j < l \le m} C_{jl}^i \theta_j \wedge \theta_l, \quad i = 1, \dots, m.$$

For each α, $f_\alpha^{(2)}$: $\bar{P}|U_\alpha \to G$ satisfies $\pi_2 \circ f_\alpha^{(2)} = f_\alpha \circ \bar{\pi}$, $f_\alpha^{(2)} \pitchfork \mathscr{F}_0$, and

$$f_\alpha^{(2)^{-1}}\left(\mathscr{F}_0 | f_\alpha^{(2)}\left(\bar{\pi}^{-1}(U_\alpha)\right)\right) = \bar{\pi}^{-1}(\mathscr{F})|\bar{\pi}^{-1}(U_\alpha).$$

Hence $\bar{\pi}^{-1}(\mathscr{F})|\bar{\pi}^{-1}(U_\alpha)$ is defined by $\tilde{\omega}_1^\alpha, \dots, \tilde{\omega}_q^\alpha$ satisfying

$$d\tilde{\omega}_i^\alpha = \sum C_{jl}^i \tilde{\omega}_j^\alpha \wedge \tilde{\omega}_l^\alpha$$

where $\tilde{\omega}_i^\alpha = f_\alpha^{(2)*}\theta_i$ for $i = 1, \dots, m$. Since $f_\alpha^{(2)} = L_{g_{\alpha\beta}} \circ f_\beta^{(2)}$ and $\theta_1, \dots, \theta_m$ are left-invariant, it follows that $\tilde{\omega}_i^\alpha = \tilde{\omega}_i^\beta$ for $i = 1, \dots, m$ whenever $U_\alpha \cap U_\beta \neq \varnothing$.

Thus $\bar{\pi}^{-1}(\mathscr{F})$ is defined by one-forms $\tilde{\omega}_1, \ldots, \tilde{\omega}_q$ on $\bar{P}$ satisfying

$$d\tilde{\omega}_i = \sum C^i_{jl}\tilde{\omega}_j \wedge \tilde{\omega}_l, \quad i = 1, \ldots, m.$$

We claim that the bundle $\bar{\pi}\colon \bar{P} \to M$ is trivial. Assuming the claim, let $s\colon M \to \bar{P}$ be a section. Then $s \pitchfork \bar{\pi}^{-1}(\mathscr{F})$ and $s^{-1}(\bar{\pi}^{-1}(\mathscr{F})) = \mathscr{F}$. Let $\omega_i = s^*\tilde{\omega}_i$, $i = 1, \ldots, m$. Then $\mathscr{F}$ is defined by $\omega_1, \ldots, \omega_q$ satisfying

$$d\omega_i = \sum C^i_{jl}\omega_j \wedge \omega_l, \quad i = 1, \ldots, m$$

as desired. To prove the claim, let

$$K = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ {}^t\xi & I_q & 0 \\ b & \xi & 1 \end{pmatrix} \in O(q+1,1) \colon \xi \in (\mathbf{R}^q)^* \right\} \subset H.$$

Then $\bar{P}_0 = \bar{P}/K$ is a foliated reduction of $P^1(M, \mathscr{F}) = F(Q)$ (the frame bundle of Q) to $H/K = CO(q)$. (Note that K is the first prolongation of $CO(q)$ and $\bar{P} \to \bar{P}_0$ is the first prolongation of the transverse $CO(q)$ structure $\bar{P}_0 \to M$). Since $K \cong (\mathbf{R}^q)^*$ is contractible, it follows that the bundle $\bar{P} \to \bar{P}_0$ is trivial. Since $CO(q)/O(q)$ is contractible, $\bar{P}_0 \to M$ admits a reduction $\bar{P}' \to M$ to $O(q)$. Since $F(Q) \to M$ is trivial, it follows that $\bar{P}' \to M$ is trivial and hence $\bar{P}_0 \to M$ is trivial. Thus $\bar{\pi}\colon \bar{P} \to M$ is trivial proving the claim.

In the projective case $G = PSL(q, \mathbf{R}) = SL(q+1, \mathbf{R})/\text{center}$, $G/H = \mathbf{R}P^q$, and

$$H = \left\{ \begin{pmatrix} A & 0 \\ \xi & a \end{pmatrix} \in SL(q+1, \mathbf{R}) \right\} \Big/ \text{center}$$

where $A \in GL(q, \mathbf{R})$, and ξ is a row q-vector. The proof is identical to the argument used in the conformal case except that to establish the claim that the bundle $\bar{\pi}\colon \bar{P} \to M$ is trivial, we let

$$K = \left\{ \begin{pmatrix} I_q & 0 \\ \xi & 1 \end{pmatrix} \colon \xi \in (\mathbf{R}^q)^* \right\} \subset H.$$

Then $\bar{P}/K = F(Q)$ the frame bundle of Q, a principal $H/K = GL(q, \mathbf{R})$-bundle. Since K is contractible, the bundle $\bar{P} \to F(Q)$ is trivial and so $\bar{\pi}\colon \bar{P} \to M$ is trivial.

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