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EVIDENCE FOR A CONJECTURE OF ELLINGSRUD AND STRØMME ON THE CHOW RING OF $\mathrm{HILB}_d(\mathbf{P}^2)$

BY

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Introduction

According to a theorem of Fogarty the Hilbert scheme $\mathrm{Hilb}_d(X)$, which parametrizes finite subschemes of length d in a non singular surface X, is a non singular variety, of dimension $2d$. Recently Ellingsrud and Strømme (see [5]) have computed the homology groups of $\mathrm{Hilb}_d(\mathbf{P}^2)$. They have proved:

(1) $\mathrm{Hilb}_d(\mathbf{P}^2)$ has no odd homology and the homology groups are all free.

(2) The cycle map induces an isomorphism from the Chow group $A^k(\mathrm{Hilb}_d(\mathbf{P}^2))$ to $H^{2k}(\mathrm{Hilb}_d(\mathbf{P}^2), \mathbf{Z})$.

(3) The ranks of $A^k(\mathrm{Hilb}_d(\mathbf{P}^2))$ can be computed by means of certain functions related to the partition function.

We recall the table in Fig. 1 for the ranks of $A^k(\mathrm{Hilb}_d(\mathbf{P}^2))$.

Ellingsrud and Strømme [5] propose the following conjecture.

Let $p: I \to \mathrm{Hilb}_d(\mathbf{P}^2)$ be the universal family, let $q: I \to \mathbf{P}^2$ be the natural projection, let $\mathscr{L}$ be a line bundle on $\mathbf{P}^2$ then $E(\mathscr{L}) := p_* q^* \mathscr{L}$ is a vector bundle of rank d on $\mathrm{Hilb}_d(\mathbf{P}^2)$,

Conjecture (Ellingsrud and Strømme). The Chern classes of the bundles $E(\mathcal{O}(m))$, $m = 0, 1, 2$, generate the homology *ring* of $\mathrm{Hilb}_d(\mathbf{P}^2)$.

It is easy to see that conjecture is true for $\mathrm{Hilb}_2(\mathbf{P}^2)$. We have been informed that the conjecture has been verified by the two authors for $\mathrm{Hilb}_3(\mathbf{P}^2)$; see [6]. They have considered the birational image H of $\mathrm{Hilb}_3(\mathbf{P}^2)$ inside the Grassmannian which parametrizes nets of conics in the plane and they have computed the Chow ring of H using methods from the theory of principal Gln-bundles.

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d \ k	0	1	2	3
1	1	1		
2	1	2	3	
3	1	2	5	6
4	1	2	6	10
5	1	2	6	12
6	1	2	6	13
$n > 6$	1	2	6	13

FIG. 1

Here we shall prove:

THEOREM. (1) *The monomials of weight* 1 *in the Chern classes of* $E(\mathcal{O}(m))$, $m = 0, 1, 2$, *generate* $A^1(\mathrm{Hilb}_d(\mathbf{P}^2))$, $d \geq 3$.

(2) *The monomials of weight* 2 *in the Chern classes of* $E(\mathcal{O}(m))$, $m = 0, 1, 2$, *generate* $A^2(\mathrm{Hilb}_d(\mathbf{P}^2))$, $d \geq 3$.

(3) *The monomials of weight* 3 *in the Chern classes of* $E(\mathcal{O}(m))$, $m = 0, 1, 2$, *generate* $A^3(\mathrm{Hilb}_d(\mathbf{P}^2))$, $d \geq 3$.

Part (1) in the theorem is contained implicitly in [3], where a description of a basis for $\mathrm{Pic}(\mathrm{Hilb}_d(\mathbf{P}^2))$ is given.

Our approach is quite direct. First we describe b_1 curves, b_2 surfaces, b_3 threefolds which are candidates for the elements of a basis for $A_1(\mathrm{Hilb}_d(\mathbf{P}^2))$, $A_2(\mathrm{Hilb}_d(\mathbf{P}^2))$, $A_3(\mathrm{Hilb}_d(\mathbf{P}^2))$. Next we compute the degree over these varieties of the monomials in the Chern classes of $E(\mathcal{O}(m))$. For $i = 1, 2, 3$, we have a matrix M_i of intersection degrees. We find inside M_i minors of rank b_i such that the associated determinants generate the ideal (1) in $\mathbf{Z}$. This implies that in $A_i(\mathrm{Hilb}_d(\mathbf{P}^2))$ there are b_i independent elements which have a unimodular matrix of intersection with b_i elements in the lattice generated by the monomials of weight i in the Chern classes of $E(\mathcal{O}(m))$. From Poincaré duality it follows that the monomials of weight $1, 2, 3$, generate $A^1(\mathrm{Hilb}_d(\mathbf{P}^2))$, $A^2(\mathrm{Hilb}_d(\mathbf{P}^2))$, $A^3(\mathrm{Hilb}_d(\mathbf{P}^2))$ and that the proposed generators generate indeed $A_1(\mathrm{Hilb}_d(\mathbf{P}^2))$, $A_2(\mathrm{Hilb}_d(\mathbf{P}^2))$, $A_3(\mathrm{Hilb}_d(\mathbf{P}^2))$.

In order to compute certain degrees of intersection we need to study (a) the threefold in $\mathrm{Hilb}_3(\mathbf{P}^2)$ which parametrizes not reduced subschemes of length 3 with support a single point P which moves in a line L and (b) the threefold in

$\mathrm{Hilb}_4(\mathbf{P}^2)$ which parametrizes not reduced subschemes of length 4 with support a fixed point P_0. We have thus been induced to study certain varieties F, S, T which are the desingularization of the locus in $\mathrm{Hilb}_2(\mathbf{P}^2)$, $\mathrm{Hilb}_3(\mathbf{P}^2)$, $\mathrm{Hilb}_4(\mathbf{P}^2)$, respectively which parametrizes closed subschemes of length 2, 3, 4 which are supported on a single point moving in $\mathbf{P}^2$. This is the content of Part 1.

In Part 2 we prove the theorem for A^1 and A^2; in Part 3 we prove the results on A^3.

We work over the field of complex numbers because we use some elementary facts from [2] on the classification of subschemes of $\mathbf{P}^2$ of length 3 and 4; but our arguments seem to be characteristic free.

We shall use the word bundle in two ways, to denote a locally free sheaf or as an abbreviation of projectivized bundle. We follow the Grothendieck convention that if $\mathscr{E}$ is a locally free sheaf then $\mathbf{P}(\mathscr{E})$ is the projectivized bundle of quotient lines of $\mathscr{E}$, i.e, $\mathbf{P}(\mathscr{E}) = \mathrm{Proj}(\oplus \mathrm{Sym}\, \mathscr{E})$. The standard properties of Chern classes, cf. Chapter 3 of [7], will be applied without explicit comment.

We shall write often E_m for the bundle $E(\mathscr{O}(m))$ on Hilb_d; on the other hand when we deal with the pull back of E_m to some variety W we shall use a notation like $\mathscr{E}(W, m)$ or $\mathscr{E}(W, \mathscr{O}^2_{\mathbf{P}}(m))$ in order to avoid confusion with other objects denoted E on W.

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Part 1

(1.1) In this part we construct the families of second and third order data on a non singular surface, compute their Chow rings and indicate how one can find the Chern classes of the secant bundles on these families. We do things in greater generality than we actually need for the applications in Part 2, because we think that our construction should be useful in some other situation.

(1.2) Definition. Datum of order $(n-1)$ on a non singular surface X is a set Z of n closed finite subschemes of X, $Z = \{Z_1, Z_2, \ldots, Z_n\}$, such that: (a) length $Z_m = m$, (b) Z_m is a closed subscheme of Z_{m+1} and they have the same support.

We call support of a datum Z the point Z_1. If Z_n is a closed subscheme of a non singular curve we say that Z_n is linear, and we say that Z is linear if Z_n is linear. Every datum of order 1 is of course linear.

A datum of order 0 is a point of X, so X itself is the family of 0-order data (the Latin grammar says that the plural of datum is data). A datum of order 1 is a point with associated a tangent direction, hence the family F of first order data on X is a $\mathbf{P}^1$ bundle over X. Note that F is the projectivized bundle $\mathbf{P}(\Omega^1_X)$ over X; cf. [9, V, B].

(1.3) A linear datum Z of order $(d-1)$ is uniquely determined by Z_d, so that the set of linear data of order $(d-1)$ is identified with a subset, say $U(d)$, of $\mathrm{Hilb}_d(\mathbf{P}^2)$. We denote by $V(d)$ the subvariety of $\mathrm{Hilb}_d(X)$ which is the closure of $U(d)$. The *family of second order data S* and the *family of third order data T* which we consider are desingularizations of $V(3)$ and $V(4)$ respectively and they contain $U(3)$ and $U(4)$ respectively as open subsets. More precisely a datum of order $(n-1)$ determines a point in $X \times \mathrm{Hilb}_2 \times \cdots \times \mathrm{Hilb}_n$; we denote by $D(n-1)$ the closed subvariety of $X \times \mathrm{Hilb}_2 \times \cdots \times \mathrm{Hilb}_n$ which is supported on the set of data of order $(n-1)$. It turns out $D(0) = X$, $D(1) = F$, $D(2) = S$, but $D(3) = T \cup R$, where R does not contain linear data. S is a $\mathbf{P}^1$ bundle over F and T is a $\mathbf{P}^1$ bundle over S under the natural projections.

We have noted that $F = \mathbf{P}(\Omega^1_X)$ and we produce below bundles of rank 2, $\mathscr{S}$ on F and $\mathscr{T}$ on S such that $S = \mathbf{P}(\mathscr{S})$ and $T = \mathbf{P}(\mathscr{T})$. Since the Chern classes of $\mathscr{S}$ and $\mathscr{T}$ can be computed, the Chow rings of S and T are known when the Chow ring and the canonical classes of X are known.

If $X = \mathbf{P}^2$ then the variety S is a classical object of Study; we show in fact that it coincides with the variety of curvilinear elements of order 2 studied in [13], [12], [11]. Semple in his paper constructs a certain $\mathbf{P}^1$ bundle over S and he claims that points of this variety correspond to the curvilinear elements of order 3 in the plane. We check that our variety T is not the variety constructed by Semple; this is somewhat expected and simply says that non linear data of order 3 and non linear curvilinear elements of order 3 are different objects.

One motivation we had for our construction came from reading the paper of Roberts and Speiser [11], where they compute the intersection ring of the variety of curvilinear elements of order 2 when X is $\mathbf{P}^2$. Their method seems to depend on the special geometry of $\mathbf{P}^2$. We refer to this paper for a discussion of interesting applications of the structure of the intersection ring $A(S)$ to problems in enumerative geometry of contacts.

(1.4) *Local description.* Let R denote $\mathbf{C}(x, y)$, the analytic local ring of the origin in $\mathbf{A}^2$, let $\mathbf{m}$ denote the maximal ideal in R and let $R^m = R/\mathbf{m}^{m+1}$ be the truncated polynomial ring. If we fix an isomorphism between R and the completion of the local ring of X at P, then the ideals in R of colength n

determine closed subschemes of X of finite length n supported at P and conversely. The classification of data of order n with support P is then equivalent to the classification of sequences of nested ideals $(I_1, I_2, \ldots, I_{n+1})$ with $I_1 = \mathbf{m}$, colength $I_k = k$, and $I_i \supset I_{i+1}$. If $m > n$ the classification of sequences of nested ideals as before is the same inside R and R^m.

The analytic classification of ideals of R of colength ≤ 4 is elementary, cf. [2]. The ideals of colength 2 are all of type $I = (f) + \mathbf{m}^2$, where f is a local parameter, i.e. $f \notin \mathbf{m}^2$. The ideals of colength 3 divide into two types: (a) $(f) + \mathbf{m}^3$, (b) $\mathbf{m}^2$. The ideals of colength 4 are of three types: (a) $(f) + \mathbf{m}^4$, (b) $(f)\mathbf{m} + \mathbf{m}^3$, (c) $(f^2, g^2) + \mathbf{m}^3$ where f and g are independent local parameters. We sometime refer to $\mathrm{Spec}(R/\mathbf{m}^2)$ as to "the big point" at P. Note that in our list the ideals of type (a) are exactly the ideals of the linear subschemes supported at P.

In the following we construct varieties A, B, C, which are the fibres over P of the families F, S, T discussed in the introduction. In other words A is the family of ideals of colength 2 and B is the family of sequences of nested ideals of the form $(\mathbf{m}, I_2, I_3)$. The variety C parametrizes also sequences of nested ideals of the form $(\mathbf{m}, I_2, I_3, I_4)$, and it is in fact the irreducible component in the family of such sequences which contains as a open set the locus of the sequences where I_3 is of type (a). Given an ideal of colength 4 there is a point of C where it appears as I_4.

(1.4.1) *Remark.* The family of nested ideals of type $(\mathbf{m}, I_2, \mathbf{m}^2, I_4)$, where I_4 is of type (b) or (c) is a 3 dimensional variety; it is the fibre over P of the second component R of $D(3)$; cf. (1.3) above.

(1.4.2) The variety A is the $\mathbf{P}^1$ which parametrizes the lines in the vector space $V := (\mathbf{m}/\mathbf{m}^2)$. A point a of A determines a local parameter f_a in V, up to a constant, and conversely. On A there is the tautological sequence

$$0 \to Y_2 \to V_A \to \mathcal{O}_A(1) \to 0.$$

The fibre of Y_2 at a point a is the line (f_a) in V. The vector space R^2 lifts to a bundle R^2_A on A; similarly so does $(\mathbf{m}^2/\mathbf{m}^3)$. The bundle R^2_A is in fact a sheaf of rings, $Y_2 \otimes V_A$ is a sheaf of ideals and a subbundle of rank 2 in R^2_A. The fibre of $Y_2 \otimes V_A$ at a point a is the ideal $(f_a) \cdot \mathbf{m}$. We define

$$Y_2^+ := Y_2 \oplus \left(\mathbf{m}^2/\mathbf{m}^3\right)_A;$$

Y_2^+ is a sheaf of ideals in R^2 and a bundle of rank 4. We denote by B the grassmannian bundle of lines in $Y_2^+/Y_2 \otimes V_A$.

Clearly B is a $\mathbf{P}^1$ bundle over A. If there is no confusion we denote in the same ways bundles on A and their lifting to B. By construction there is on B a tautological bundle Y_3 of rank 3 with $Y_2 \otimes V_A \subset Y_3 \subset Y_2^+$.

The fibre of the bundle Y_3, at a point b of B which maps to a, is a vector space $Y_3(b)$ which satisfies the inclusion $(f_a)\mathbf{m} \subset Y_3(b) \subset (f_a) + \mathbf{m}^2 \subset R^2$; hence $\mathbf{m}Y_3(b) \subset \mathbf{m}(f_a) \subset Y_3(b)$. Therefore $Y_3(b)$ is an ideal in R^2, of colength 3.

The ideals of colength 3 in R^2 are in 1-1 correspondence with the ideals of the same kind in R, and we classify them in the same way. Looking at the list we find that a linear ideal occurs exactly once as $Y_3(b)$, while $\mathbf{m}^2$ occurs once in each fibre of $B \to A$. In fact $\mathbf{m}^2$ fits in the inclusions $(f_a)\mathbf{m} \subset \mathbf{m}^2 \subset (f_a) + \mathbf{m}^2$; hence it gives a section from A to B.

Next we construct C by a similar procedure. On B there is a bundle

$$Y_3^+ := Y_3 \oplus (\mathbf{m}^3/\mathbf{m}^4)_B;$$

it can be seen as the universal sheaf of ideals of R^3 of colength 3. We also consider

$$W := \mathrm{Image}\big(Y_2 \otimes (\mathbf{m}^2/\mathbf{m}^3)_B \oplus Y_2^{\otimes 2} \oplus \mathcal{O}_A(1) \otimes Y_3\big) \quad \text{in } R_B^3,$$

where $Y_2^{\otimes 2} \subset (\mathbf{m}^2/\mathbf{m}^3)_B$ and $\mathcal{O}_A(1) \otimes Y_3 \subset (\mathbf{m}^2/\mathbf{m}^3)_B \oplus (\mathbf{m}^3/\mathbf{m}^4)_B$.

It is easy to see that W is a bundle of subvector spaces of corank 5 in R_B^3. Note that $W \subset Y_3^+$. We define C to be the grassmannian of lines in Y_3^+/W. Again C is a $\mathbf{P}^1$-bundle over B. As before there is on C a tautological bundle Y_4 which fits in the inclusions $W_C \subset Y_4 \subset (Y_3^+)_C$. By construction the fibre of Y_4 at a point c which projects to b in B is a vector space which fits in the inclusions

$$W(b) = \big((f_a)\mathbf{m}^2 + (f_a^2) + \mathbf{m}Y_3(b)\big) \subset Y_4(c) \subset \big(Y_3(b) \oplus \mathbf{m}^3/\mathbf{m}^4\big) \subset R^3;$$

conversely any such vector space determines uniquely a point c. Further such a vector space is an ideal because $Y_4(c)\mathbf{m} \subset Y_3(b)\mathbf{m} \subset Y_4(c)$.

On C we have the inclusion of sheaf of ideals

$$Y_4 \subset (Y_3^+)_C \subset (Y_2^+) \oplus \mathbf{m}^3/\mathbf{m}^4 \subset R_C^3;$$

the fibres at a point $c \in C$ give

$$Y_4(c) \subset Y_3^+(b) \subset (f_a) + \mathbf{m}^2 \subset R^3,$$

which is the nested sequence of ideals associated to a datum of order 3. On the other hand a datum of order 3 determines a unique point of C if it is not of the type in remark (1.4.1). The simplest way to verify this assertion is to look at the list of ideals of colength 4. A linear ideal may appear as $Y_4(c)$ exactly for one point c. The ideals of type (b) provide a section of

$C \to B$, indeed for all b

$$W(b) \subset (f_a)\mathbf{m} + \mathbf{m}^3/\mathbf{m}^4 \subset Y_3(b) + \mathbf{m}^3/\mathbf{m}^4 \subset R^4.$$

(1.5) *Global description.* We produce globally on X the construction given locally at P in (1.4). The notations are independent; also we change the point of view a little, focusing on the quotient rings instead of the ideals.

Let $\mathscr{P}^n$ be the bundle of principal parts associated to $\mathcal{O}_X$; cf. [9, IV, A]. The fibre at P of $\mathscr{P}^n$ is R^n, the truncated polynomial ring of (1.4). There are several exact sequences involving $\mathscr{P}^n$ which we will use without comment, e.g.,

$$0 \to \mathrm{Sym}^3\Omega^1 \to \mathscr{P}^3 \to \mathscr{P}^2 \to 0.$$

If $Y \to X$ is a morphism we shall abuse notations, denoting Ω^1, $\mathscr{P}^n$, etc., the pull back to Y of those bundles on X. Similarly given a map $Z \to Y$ we shall denote in the same way a sheaf on Y and its pullback to Z, if no confusion arises.

We define $F := \mathbf{P}(\Omega^1_X)$; it is a $\mathbf{P}^1$ bundle on X. F is the family of first order data; cf. [9, V, B]. On F one has the tautological sequence

$$0 \to \mathscr{L} \to \Omega^1 \to \mathcal{O}_F(1) \to 0, \tag{1.5.1}$$

hence also

$$0 \to \mathscr{L} \to \mathscr{P}^1 \to \mathscr{Q} \to 0,$$

where $\mathscr{Q}$ is a bundle of rank 2. The fibre of $\mathscr{Q}$ at a point of F is the structure ring of the associated scheme of length 2.

Next we introduce the global form of the sheaf $Y_2 \otimes V_A$ used in (1.4), i.e., the bundle $\mathscr{L} \otimes \Omega^1$. There is a diagram of bundles on F:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{L}\otimes\Omega^1 & \to & \mathrm{Sym}^2\Omega^1_X & \to & \mathcal{O}_F(2) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{L}\otimes\Omega^1 & \to & \mathscr{P}^2 & \to & \mathscr{C} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathscr{P}^1 & = & \mathscr{P}^1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{1.5.2}$$

where $\mathscr{C}$ is defined by exactness, rank $\mathscr{C} = 4$. There is a surjection $\mathscr{C} \to \mathscr{Q} \to 0$; let $\mathscr{K}$ be the kernel, a bundle of rank 2 on F. Note the exact sequence

$$0 \to \mathcal{O}_F(2) \to \mathscr{K} \to \mathscr{L} \to 0. \tag{1.5.3}$$

(1.5.4) Let $\mathcal{S} = \mathcal{K}^\vee$, the dual line bundle; we define $S := \mathbf{P}(\mathcal{S})$. In order to see that S is indeed the variety of second order data we have only to check that the fibre of S over a point P of X is the variety denoted B in (1.4). This is straightforward.

The tautological inclusion $\mathcal{O}_S(-1) \to K$ composes with $\mathcal{K} \to \mathcal{C}$, we define $\mathcal{D}$ by exactness in

$$(1.5.5) \qquad 0 \to \mathcal{O}_S(-1) \to \mathcal{C} \to \mathcal{D} \to 0;$$

$\mathcal{D}$ is the "universal" bundle of rank 3 on S. The fibre of $\mathcal{D}$ at a point s which maps to z in F is by definition a vector space D_s with $\mathcal{C}_z \Rightarrow \mathcal{D}_s$, $\mathcal{D}_s \Rightarrow \mathcal{Q}_z$. Note that D_s is the structure ring of a scheme of length 3.

From (1.5.3) one has a surjection $\mathcal{S} \to \mathcal{Q}_F(-2)$, hence a section $\sigma : F \to S$. Let $\Sigma = \sigma(F)$, using the defining property $\sigma^*(\mathcal{O}_S(1)) = \mathcal{O}_F(-2)$, it is easy to check that

$$\mathcal{O}_S(\Sigma) \approx \mathcal{O}_S(1) \otimes \mathcal{L}.$$

The global section Σ corresponds to the map $\mathcal{O}_S(-1) \to \mathcal{L}$ induced from $\mathcal{O}_S(-1) \to \mathcal{K}$. For later use we note the following diagram:

$$(1.5.6) \qquad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & 0 & \to \mathcal{O}_S(-1) & \to & \mathcal{L} & & \\ & \downarrow & \downarrow & & \downarrow & & \\ 0 \to & \mathcal{O}_F(2) & \to \mathcal{C} & \to & \mathcal{P}^1 & \to 0. & \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{D} & \to & \mathcal{Q} & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

(1.5.7) The construction of T is similar, in order to compute Chern classes we proceed in two steps to produce a bundle $\mathcal{T}$ on S with $T = \mathbf{P}(\mathcal{T})$.

First we shall construct a bundle $\mathcal{H}$ of rank 6 with a natural surjection $\mathcal{H} \to \mathcal{D}$; the fibre of $\mathcal{H}$ at a point s as above represents the quotient

$$R^3/\left(f_a \mathbf{m}^2 + f_a^2\right),$$

notations as in (1.4). Next we shall produce $\mathcal{M}$, a quotient bundle of $\mathcal{H}$ of rank 5 with a surjection π: $\mathcal{M} \to \mathcal{D} \to 0$. $\mathcal{M}$ is the global form of the bundle R_B^3/W, which appears in (1.4). We define $\mathcal{T} = (\operatorname{Kern} \pi)^\vee$ and $T = \mathbf{P}(\mathcal{T})$.

To begin, we consider $\mathcal{H}$ a bundle of rank 6 on S which is the quotient bundle of $\mathcal{P}^3$ whose fibre at a point s as above represents the quotient

$R^3/(f_a\mathbf{m}^2 + f_a^2)$. There is an exact diagram:

$$
\text{(1.5.8)} \qquad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{L}\otimes \mathrm{Sym}^2\Omega^1 \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{L}^{\otimes 2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Sym}^3\Omega^1 \arrow[r] \arrow[d] & \mathcal{P}^3 \arrow[r] \arrow[d] & \mathcal{P}^2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_F(3) \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & \mathcal{I} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

The diagram is obtained by defining $\mathcal{E}$ as the kernel of the map $\mathcal{P}^3 \to \mathcal{H}$ and $\mathcal{I}$ is by definition the quotient of $\mathcal{L}^{\otimes 2} \to \mathcal{P}^2$. Then the middle column surjects onto the right column and the left column is the exact sequence of the kernels. At $a \in F$ the kernel of $\mathcal{E} \to \mathcal{L}^{\otimes 2}$ is a subbundle of $\mathrm{Sym}^3\Omega^1$ which locally represents the ideal $f_a\mathbf{m}^2$; hence it is $\mathcal{L}\otimes \mathrm{Sym}^2\Omega^1$. By computing Chern classes we find that the kernel of $\mathcal{H} \to \mathcal{I}$ is the line bundle $\mathcal{O}_F(3)$.

Next we consider $\mathcal{M}$, the rank 5 bundle on S whose fibre at a point is the quotient

$$R^3/\big(f_a\mathbf{m}^2 + f_a^2 + \mathbf{m}Y_3(b)\big).$$

$\mathcal{M}$ is a quotient of $\mathcal{H}$, we denote by $\mathcal{G}$ the kernel of $\mathcal{H} \to \mathcal{M}$. $\mathcal{G}$ is a line bundle which we determine in a moment. There is an exact diagram

$$
\text{(1.5.9)} \qquad
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \mathcal{G} \arrow[r, "\rho"] \arrow[d] & \mathcal{L}\otimes \mathcal{O}_F(1) \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_F(3) \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & \mathcal{I} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{J} \arrow[r] & \mathcal{M} \arrow[r] \arrow[d] & \mathcal{C} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
$$

Here $\mathcal{J}$ is by definition the kernel of $\mathcal{M} \to \mathcal{C}$; it is a line bundle. The right column in the diagram comes from the known surjection $\mathcal{J} \to \mathcal{C}$. Direct inspection, based on the local considerations of (1.4), shows that the map

$$\rho : \mathcal{G} \to \mathcal{L}\otimes \mathcal{O}_F(1)$$

is an isomorphism away from Σ and that over Σ the map ρ vanishes simply.

Therefore there is an isomorphism of line bundles:

$$\mathscr{G}^{-1} \otimes \mathscr{L} \otimes \mathcal{O}_F(1) \approx \mathcal{O}(\Sigma);$$

we have computed above that $\mathcal{O}_S(\Sigma) \approx \mathcal{O}_S(1) \otimes \mathscr{L}$; hence

$$\mathscr{G} \approx \mathcal{O}_S(-1) \otimes \mathcal{O}_F(1). \tag{1.5.10}$$

We compute the line bundle $\mathscr{J}$; using the snake lemma we find the exact sequence

$$0 \to \mathcal{O}_F(3) \to \mathscr{J} \to \mathcal{O}_\Sigma \otimes \mathcal{O}_F(1) \to 0, \tag{1.5.11}$$

i.e., $\mathscr{J} = \mathcal{O}_F(3) \otimes \mathcal{O}(\Sigma) = \mathcal{O}_F(3) \otimes \mathscr{L} \otimes \mathcal{O}_S(1)$.

Composing $\mathscr{M} \to \mathscr{C}$ with $\mathscr{C} \to \mathscr{D}$, we have a surjection π: $\mathscr{M} \to \mathscr{D}$; we define $\mathscr{Y}$ to be the kernel of π. From the preceding diagrams we find the exact sequence

$$0 \to \mathcal{O}_F(3) \otimes \mathscr{L} \otimes \mathcal{O}_S(1) \to \mathscr{Y} \to \mathcal{O}_S(-1) \to 0. \tag{1.5.12}$$

(1.5.13) We define $\mathscr{T} = \mathscr{Y}^\vee$, $T = \mathbf{P}(\mathscr{T})$. As we said T is the family of third order data on X. The "universal" bundle of rank 4 on T is the bundle $\mathscr{N}$ which appears in the diagram (1.5.14) below.

Using the tautological inclusion $\mathcal{O}_T(-1) \to \mathscr{Y}$ we have a map $\mathcal{O}_T(-1) \to \mathscr{M}$ which fits in the exact diagram below. Recall that we remarked in (1.4) that the ideals of colength 4 and type (b) determine a section φ: $S \to T$, φ is the section associated with the surjection $\mathscr{M} \to \mathscr{C}$. More precisely there is a morphism $\mathcal{O}_T(-1) \to \mathcal{O}_S(-1)$ obtained by composing $\mathcal{O}_T(-1) \to \mathscr{M}$ with $\mathscr{M} \to \mathscr{C}$ (in fact by definition $\mathcal{O}_T(-1) \to \mathscr{D}$ is the zero map). The related global section of $\mathcal{O}_T(1) \otimes \mathcal{O}_S(-1)$ is $\varphi(S)$, in other words $\mathcal{O}_T(\varphi(S)) \approx \mathcal{O}_T(1) \otimes \mathcal{O}_S(-1)$. We have also here a diagram analogous to (1.5.6):

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & 0 & & \to \mathcal{O}_T(-1) & \to & \mathcal{O}_S(-1) & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathcal{O}_F(3) \otimes \mathscr{L} \otimes \mathcal{O}_S(1) & \to & \mathscr{M} & \to & \mathscr{C} & \to 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & \mathscr{N} & \to & \mathscr{D} & \to 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array} \tag{1.5.14}$$

where $\mathscr{N}$ is defined by exactness.

(1.6) *Semple Bundles.* We keep the notations used before in (1.5) and let $f\colon F \to X$ be the natural projection from the projectivized bundle $\mathbf{P}(\Omega^1_X)$ to the smooth surface X.

We are going to define a $\mathbf{P}^1$ bundle over F which we shall denote $F(2)$; since the costruction is iterative we find convenient to let $F(0) = X$, $F(1) = F$, $f(1) = f$, and $f(2)\colon F(2) \to F(1)$ be the natural projection.

By definition the sheaf of relative differentials for $f(1)$ is the cokernel R_1 in the following exact sequence of bundles on $F(1)$:

$$0 \to f(1)^*\Omega^1_{F(0)} \to \Omega^1_{F(1)} \to R_1 \to 0. \tag{1.6.1}$$

From this sequence, by pushout via $f(1)^*\Omega^1_{F(0)} \to \mathcal{O}_{F(1)}(1) \to 0$, we obtain the following bundle $\mathcal{G}_1$, which is an extension of R_1 by $\mathcal{O}_{F(1)}(1)$,

$$\begin{array}{ccccccccc} 0 & \to & f(1)^*\Omega^1_{F(0)} & \to & \Omega^1_{F(1)} & \to & R_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \to & \mathcal{O}_{F(1)}(1) & \to & \mathcal{G}_1 & \to & R_1 & \to & 0. \end{array} \tag{1.6.2}$$

We define $F(2) = \mathbf{P}(\mathcal{G}_1)$. On $F(2)$ we have the bundle R_2 of relative differentials defined as before and we obtain a rank 2 bundle $\mathcal{G}_2$ by iterating the same construction we used for $\mathcal{G}_1$. More precisely we use the surjection

$$f(2)^*\Omega^1_{F(1)} \to \mathcal{O}_{F(2)}(1) \to 0,$$

which comes by composition of

$$f(2)^*\Omega^1_{F(1)} \to f(2)^*(\mathcal{G}_1) \to 0$$

with

$$f(2)^*(\mathcal{G}_1) \to \mathcal{O}_{F(2)}(1) \to 0.$$

So we have

$$\begin{array}{ccccccccc} 0 & \to & f(2)^*\Omega^1_{F(1)} & \to & \Omega^1_{F(2)} & \to & R_2 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \to & \mathcal{O}_{F(2)}(1) & \to & \mathcal{G}_2 & \to & R_2 & \to & 0. \end{array} \tag{1.6.3}$$

(1.6.4) By the iterative procedure indicated above we may define more generally $F(m+1) = \mathbf{P}(\mathcal{G}_m)$, and let $f(m+1)\colon F(m+1) \to F(m)$ be the natural projection.

The projective bundles $F(m)$ are the modern interpretation of a construction proposed by Semple [12] in the case when $X = \mathbf{P}^2$. More precisely given the map $f(m)\colon F(m) \to F(m-1)$ Semple deals with the projectivization of the rank 2 bundle of the "focal planes" supported by $F(m)$. We recall that by definition the focal plane at a point $p \in F(m)$ is the plane in $T_{F(m),p}$, the tangent space of $F(m)$ at p, which is the preimage under the differential map

$$f(m)_*: T_{F(m),p} \to T_{F(m-1),f(p)}$$

of the tautological line l in $T_{F(m-1),f(p)}$. By tautological line we mean the dual of the quotient line which is the fibre of $\mathcal{O}_{F(m)}(1)$ at p (here we use the duality between tangent and cotangent space and think of the surjection

$$\left(f(m)_*\left(\Omega^1_{F(m-1)}\right)_p \to \mathcal{O}_{F(m)}(1)_p\right).$$

Since we use the Grothendieck definition of projectivized bundles, then our bundle $\mathbf{P}(\mathcal{G}_m)$ is indeed the variety proposed by Semple.

Semple shows that $F(2)$ is the variety of "curvilinear elements of order 2" in the plane (for the definition of this classical concept we refer to [2] again). He also asserts [12, p. 35, line 4 above] that the points of $F(3)$ correspond evidently to the curvilinear elements of order 3 in the plane. On the other hand Semple does not claim to have proved that his variety $F(3)$ is indeed the model of the family of the said curvilinear elements. We do not deal here with the question whether $F(3)$ is the correct model of the family of the curvilinear elements of order 3 in the plane; but we remark the following:

(1.6.5) PROPOSITION. (1) *$F(2)$ and S are isomorphic $\mathbf{P}^1$ bundles over $F(1)$, for any surface X.*
(2) *$F(3)$ and T are different $\mathbf{P}^1$ bundles over $S = F(2)$.*

Proof. We prove (1) when $X = \mathbf{P}^2$ first. In this case the projection $S \to F(1)$ has two disjoint section. One is the section σ with image Σ, which we have described above in (1.5.4); the other section, which we will call linear with image Λ, is given by associating to a datum of order 2 the unique datum of order 3 in the plane which contains the said datum and is a closed subscheme of a line. Since S has two disjoint sections, the bundle $\mathcal{S}$ to which it is associated splits. Further since the section Σ is associated with the quotient line bundle $\mathcal{O}_F(-2)$ (see the discussion before (1.5.6)), this line bundle is one of the summands and therefore

$$\mathcal{S} \approx \mathcal{L}^{-1} \oplus \mathcal{O}_F(-2). \tag{1.6.6}$$

We prove that $\mathcal{G}_1$ also splits by using the same argument. Associated with the surjection $\mathcal{G}_1 \to R_1 \to 0$ we have a section, σ' say, $\sigma': F(1) \to F(2)$. A second section comes from the geometry of $F(1)$, which is the incidence correspondence point-line in the plane. Through any point $x \in F(1)$ there is associated a distinguished line l_x. The second section, say λ', is obtained by associating to x the tangent direction in the focal plane which is determined by l_x. Since l_x and the fibre of $f(1)$ are not tangent then they determine different directions in the focal plane hence the two sections are disjoint. Therefore

$$\mathcal{G}_1 \approx \mathcal{O}_{F(1)}(1) \oplus R_1 \tag{1.6.7}$$

Using the Euler sequence on $F(1)$ (cf. [7, B 5.8]),

$$0 \to R_1(1) \to f(1)^*\Omega^1_{F(0)} \to \mathcal{O}_{F(1)}(1) \to 0, \tag{1.6.8}$$

we see from (1.5.1) that $\mathcal{L} = R_1(1)$ and we conclude

$$\mathcal{S} \otimes \mathcal{L} \otimes \mathcal{O}_{F(1)}(1) \approx \mathcal{G}_1. \tag{1.6.9}$$

Therefore $F(2)$ and S are isomorphic projective bundles over F because the associated vector bundles $\mathcal{G}_1$ and $\mathcal{S}$ become isomorphic after tensoring with a line bundle.

In order to prove the isomorphism of $F(2)$ and S for any surface X we shall use some explicit computation due to Semple [12, p. 33].

Let $\alpha : \mathbf{C} \to V$ and $\beta : \mathbf{C} \to V$ be two irreducible smooth analytic branches centered at a point $p \in X$ and contained in the analytic neighbourhood V of p. By associating to $t \in \mathbf{C}$ the tangent direction at $\alpha(t)$ and $\beta(t)$ one has liftings α' and β' to F. If α and β are tangent at p then $\alpha'(0) = \beta'(0)$. Now Semple proves that α' and β' have the same tangent direction at $\alpha'(0) = \beta'(0)$ on F if and only if the branches α and β support the same second order datum at p. We note that by definition the tangent direction at $\alpha'(0)$ lies in the "focal plane". Also, when α is singular and represents an ordinary cusp at p, then $\alpha'(0)$ is still defined as the limit of the $\alpha'(t)$ and the curve α' is tangent at the point $\alpha'(0)$ to the fibre of $F \to X$.

In this way we have the description over the fibre at p of the isomorphism from $F(2)$ to S which we have produced globally above in the case $X = \mathbf{P}^2$. The point now is that we may cover X by open analytic neighborhoods V which are isomorphic with open sets in $\mathbf{P}^2$. Then the restrictions of $F(2)$ and S to V are isomorphic by what we proved above and the computation of Semple shows that they are naturally isomorphic; hence there is a global isomorphism from $F(2)$ to S.

To prove point (2) we remark that $F(3)$ and T are isomorphic $\mathbf{P}^1$-bundles over S if and only if there is a line bundle $\mathcal{X}$ say such that $\mathcal{X} \otimes \mathcal{G}_2 \approx \mathcal{T}$. Now

$$c_1(\mathcal{X} \otimes \mathcal{G}_2) = 2c_1(\mathcal{X}) + c_1(\mathcal{G}_2),$$

so verifying that T and $F(3)$ are not isomorphic will be enough to prove that $c_1(\mathcal{G}_2) - c_1(\mathcal{T})$ is not divisible by 2 in the Picard group of S. From sequence

(1.5.12) we have $c_1(\mathscr{T}) = -c_1(\mathscr{O}_F(3) \otimes \mathscr{L})$. From the sequences above we compute

$$c_1(\mathscr{G}_2) = c_1\big(\mathscr{O}_{F(2)}(1)\big) + c_1(R_2).$$

On the other hand recalling the Euler sequence

$$0 \to R_2(1) \to f(2)^*\mathscr{G}_1 \to \mathscr{O}_{F(2)}(1) \to 0$$

we note that $c_1(R_2) = c_1(\mathscr{O}_{F(2)}(-2)) + x$, where x comes from $\operatorname{Pic}(F(1))$. Therefore

$$c_1(\mathscr{T}) - c_1(\mathscr{G}_2) = c_1\big(\mathscr{O}_{F(2)}(1)\big) + y,$$

where y comes from $\operatorname{Pic}(F(1))$. It is clear that $c_1(\mathscr{O}_{F(2)}(1)) + y$ is not divisible by 2, for instance because the restriction of this class to a fibre of $f(2): F(2) \to F(1)$ has degree 1.

Remark. The splitting of $\mathscr{S}$ in the case of $\mathbf{P}^2$ is the basic fact which allows the computation of the ring $A(S)$ in [11].

(1.7) *Secant bundles on F, S, T and their Chern classes.* In the following U denotes either F, S, T and $n = 2, 3, 4$ respectively and $g: U \to X$ is the structure map.

We consider the "universal" closed subscheme Y of $U \times X$, with projections $p_U: Y \to U$ and $q: Y \to X$, such that for any $u \in U$ the fibre $p_U^{-1}(u)$ is the subscheme of length n determined by u; equivalently Y is the scheme associated with the sheaf of algebras $\mathscr{Q}$ on F, $\mathscr{D}$ on S, $\mathscr{N}$ on T. Let $\mathscr{X}$ be a line bundle on X and let $E(\mathscr{X}, U) = p_{U*}q^*(\mathscr{X})$, a bundle of rank n on U; of course $E(\mathscr{O}, F) = \mathscr{Q}$, $E(\mathscr{O}, S) = \mathscr{D}$, $E(\mathscr{O}, T) = \mathscr{N}$. We call bundles of type $E(\ ,\ )$ secant bundles. There is some interest in computing the Chern classes of secant bundles. In our case the computation is reduced to the case of the Chern classes of $\mathscr{Q}$, $\mathscr{D}$, $\mathscr{N}$, because of the following:

PROPOSITION. $c.(E(\mathscr{X}, U)) = c.(E(\mathscr{O}, U) \otimes g^*(\mathscr{X}))$.

Proof. Let $Z = \operatorname{supp}(Y) \subset U \times X$; Z is a smooth variety naturally isomorphic with U, because it is the graph of $g: U \to X$. By devissage the inclusion $i: Z \to Y$ induces an isomorphism of the Grothendieck groups of coherent sheaves, $i_*: K.(Z) \approx K.(Y)$ (cf. [10]). Therefore $p * K.(Y) \to K.(U)$ is also an isomorphism, because $pi: Z \to U$ is the identity. There are two maps, $(gp)^*$ and q^*, from $K.(x)$ to $K.(Y)$; the statement of the proposition is

$$(+) \qquad p * q^*(\text{class } \mathscr{X}) = p * (gp)^*(\text{class } \mathscr{X}).$$

Let y = class $\mathcal{O}_Y$ in $K.(Z) = K.(U)$; since $p*$ is the inverse of $i*$ then the right hand side of $(+)$ is the product $yg^*(\text{class } \mathcal{X})$, the left hand side is the product $y(q|_Z)^*(\text{class } \mathcal{X})$, therefore they are equal because $q|_Z = g$.

(1.8) *Some computations.* We recall the following standard facts, see [7].

Let $\mathscr{E}$ be a vector bundle of rank e on a scheme Y. Define the Chern polynomial

$$c_t(\mathscr{E}) = 1 + c_1(\mathscr{E})t + c_2(\mathscr{E})t^2 + \cdots + c_e(\mathscr{E})t^e + 0$$

where $c_i(\mathscr{E})$ is the i-th Chern class of $\mathscr{E}$.

If Y is a nonsingular variety then $\mathbf{P}(\mathscr{E})$ is nonsingular and the Chow ring $A^{\cdot}(\mathbf{P}(\mathscr{E}))$ is an algebra over $A(Y)$ which can be described by $A^{\cdot}(\mathbf{P}(\mathscr{E})) = A^{\cdot}(Y)[z]/I$, where I is the principal ideal

$$I = \left(z^e + (-1)c_1(E)z^{e-1} + \cdots + (-1)^e c_e(E)\right)$$

and $z = c_1(\mathcal{O}_{\mathbf{P}(\mathscr{E})}(1))$.

(1.8.1) Let

$$k = c_1(\Omega_x^1), \eta = c_2(\Omega_x^1), \lambda = c_1(\mathscr{L}), \varphi = c_1(\mathcal{O}_F(1)),$$
$$\sigma = c_1(\mathcal{O}_s(1)), \tau = c_1(\mathcal{O}_T(1)).$$

Note that $\lambda + \varphi = k$, $\lambda\varphi = \eta$.

From (1.5) above we have

$$c_t(\mathscr{S}) = 1 - (\lambda + 2\varphi)t + 2\lambda\varphi t^2,$$
$$c_t(\mathscr{T}) = 1 - (3\varphi + \lambda)t - \sigma(3\varphi + \lambda + \sigma)t^2.$$

Then using the standard theory,

$$A^{\cdot}(F) = A^{\cdot}(X)[\varphi]/(\varphi^2 - k\varphi + \eta),$$
$$A^{\cdot}(S) = A^{\cdot}(F)[\sigma]/(\sigma^2 + (\varphi + k)\sigma + 2\eta),$$
$$A^{\cdot}(T) = A^{\cdot}(S)[\tau]/(\tau^2 + (2\varphi + k)\tau + (-\varphi\sigma + 2\eta).$$

(1.8.2) We specialize to the case $X = \mathbf{P}^2$. Then $A^{\cdot}(\mathbf{P}^2) = \mathbf{Z}[\alpha]/(\alpha^3)$, $k = -3\alpha$, $\eta = 3\alpha^2$, where α is the class of a line. In this case F is the incidence correspondence line-point. The inclusion $F \hookrightarrow \mathbf{P}^2 \times (\mathbf{P}^2)^\vee$ corresponds to the epimorphism of $\mathbf{P}^2$ sheaves $(\mathcal{O}_{\mathbf{P}^2})^{\oplus 3} \to T_{\mathbf{P}^2}(-1) \to 0$, so that F is also the projectivized bundle $\mathbf{P}(T_{\mathbf{P}^2}(-1))$; indeed

$$\Omega^1_{\mathbf{P}^2} \otimes \mathcal{O}_{\mathbf{P}^2}(3) \approx T_{\mathbf{P}^2}.$$

Let $\beta = c_1(\mathcal{O}_{(\mathbf{P}^2)^\wedge}(2))$. Then $\varphi = \beta - 2\alpha$ and $\lambda = -\alpha - \beta$. Note that

$$A^{\cdot}(F) = \mathbf{Z}[\alpha, \beta]/(\alpha^3, \beta^3, \alpha^2 - \alpha\beta + \beta^2),$$
$$A^{\cdot}(S) = \mathbf{Z}[\alpha, \beta, \sigma]/(\alpha^3, \beta^3, \alpha^2 - \alpha\beta + \beta^2, \sigma^2 + (\beta - 5\alpha)\sigma + 6\alpha^2).$$

In this case we know that there are two sections of $S \to F$, one which we called Σ, associated to the "big points", and the other which we called Λ or linear in (1.6.5) above. We have class(Σ) $= \sigma + \lambda$, from (1.5). Similarly, using say Example 3.2.16 in [7], we compute class(Λ) $= \sigma + 2\varphi$. Using the identification of S with the Semple bundle $F(2)$ above, we recover in this way some of the results in §1 and 2 of [11].

(1.8.3) We compute the Chern polynomials of $E(\mathcal{O}_{\mathbf{P}^2}(n), F)$ and $E(\mathcal{O}_{\mathbf{P}^2}(n), S)$.

$$\begin{aligned} c_t E(\mathcal{O}_{\mathbf{P}^2}(n), F) &= c_t(\mathcal{O}_{\mathbf{P}^2}(n) \otimes \mathcal{Q}) \\ &= c_t(\mathcal{P}^1(n))(1 + (\lambda + n\alpha)t)^{-1} \\ &= (1 + (3n-3)\alpha t + 3(n-1)^2\alpha^2 t^2) \\ &\quad \times (1 - (\lambda + n\alpha)t + (\lambda + n\alpha)^2 t^2 + (\lambda + n\alpha)^3 t^3) \end{aligned}$$

Note that $c_1 E(\mathcal{O}_{\mathbf{P}^2}(n), F) = 2n\alpha + \varphi$. Also

$$\begin{aligned} c_t E(\mathcal{O}_{\mathbf{P}^2}(n), S) &= c_t(\mathcal{O}_{\mathbf{P}^2}(n) \otimes \mathcal{D}) \\ &= c_t(\mathcal{O}_{\mathbf{P}^2}(n) \otimes \mathcal{O}_F(2) \oplus \mathcal{P}^1(n))(1 + (-\sigma + n\alpha)t)^{-1} \\ &= (1 + (3n-3)\alpha t + 3(n-1)^2\alpha^2 t^2)(1 + (2\varphi + n\alpha)t) \\ &\quad \times (1 + (-\sigma + n\alpha)t)^{-1}. \end{aligned}$$

Note

$$\begin{aligned} c_1 E(\mathcal{O}_{\mathbf{P}^2}(n), S) &= (3n-3)\alpha + 2\varphi + \sigma = (3n-7)\alpha + 2\beta + \sigma, \\ c_2 E_0(S) &= 3\alpha\beta - 9\beta^2 + (\beta - 2\alpha)\sigma, \\ c_2 E_1(S) &= -4\alpha\beta + 2\beta^2 + \beta\sigma, \\ c_2 E_2(S) &= -5\alpha\beta + 7\beta^2 + (\beta + 2\alpha)\sigma, \quad c_3 E_0 = 0, \\ c_3 E_1(S) &= -4\alpha\beta^2 + \beta^2\sigma, \quad c_3 E_2(S) = -4\alpha\beta^2 + 2\alpha\beta\sigma. \end{aligned}$$

Later we shall need the following degrees in S:

$$(c_2 E_0)^2 = 15, \quad c_2 E_0 c_2 E_1 = 0, \quad c_2 E_0 c_2 E_2 = -9, \quad (c_2 E_2)^2 = 15.$$

Part 2

We shall repeatedly use the following procedure in order to compute the degree of the monomials in the Chern classes against the proposed generators. Given a subvariety, call it W, in $\mathrm{Hilb}_d(\mathbf{P}^2)$ we build some manageable desingularization U of W. Next we consider the "universal" closed subscheme I of $U \times \mathbf{P}^2$, with projections p_U: $I \to U$ and q: $I \to \mathbf{P}^2$, such that for any $u \in U$ the fibre $p_U^{-1}(u)$ is the subscheme of length n determined by u. Let $\mathcal{O}(m)$ be the line bundle on $\mathbf{P}^2$, we let $E(\mathcal{O}(m), U) = p_U * q^*(\mathcal{O}(m))$. From the universal properties of the Hilbert scheme it follows that p_U: $I \to U$ is a flat map, the pull back of the universal family over $\mathrm{Hilb}(\mathbf{P}^2)$; moreover $E(\mathcal{O}(m), U)$ is the pull back to U of $E(\mathcal{O}(m), \mathrm{Hilb}(\mathbf{P}^2))$ by the property of flat base change (cf. lectures 14, 15 in *Lectures on curves on an algebraic surface* by D. Mumford, Annals of Mathematical Studies, vol. 59, 1966). Therefore the Chern classes of $E(\mathcal{O}(m), U)$ are the pull back to U of the Chern classes of $E(\mathcal{O}(m), \mathrm{Hilb}(\mathbf{P}^2))$. It follows that the degree over W of the monomials of the appropriate weight in the Chern classes of $E(\mathcal{O}(m), \mathrm{Hilb}(\mathbf{P}^2))$ is equal to the degree over U of the same monomials in the Chern classes of $E(\mathcal{O}(m), U)$. We compute this degree using the "easy" geometry of U.

(2.1) *List of generators.* We now describe some subvarieties of dim 1 and 2 on $\mathrm{Hilb}_d(\mathbf{P}^2)$ which turn out to be generators for the Chow groups $A_1(\mathrm{Hilb}_d(\mathbf{P}^2))$ and $A_2(\mathrm{Hilb}_d(\mathbf{P}^2))$. Although we shall not indicate how, these generators were motivated by our reading of [5].

In order to save time we adopt the following conventions. We fix a point P_0 which we will refer to as the origin; next we fix d points $Q_1, Q_2, \ldots, Q_d$, and two lines L, M. We assume that the points and the lines are in general position. We shall let $Q(m) := Q_1 \cup Q_2 \cup \cdots \cup Q_m$, the subscheme of $\mathbf{P}^2$ made of the first m points. In the following, P will denote either a point or the subscheme of length 1 supported at P. U_i denotes a curve in $\mathrm{Hilb}_d(\mathbf{P}^2)$, D_k denotes a surface in $\mathrm{Hilb}_d(\mathbf{P}^2)$. In defining a subvariety Z of $\mathrm{Hilb}_d(\mathbf{P}^2)$ we shall write $Z := [X \cup Y \cup \cdots \cup W]$ to mean Z is the subvariety in $\mathrm{Hilb}_d(\mathbf{P}^2)$ which is the closure of the set of points representing closed subschemes S of the plane of length d, where $S = X \cup Y \cup \cdots \cup W$, X a subscheme with the property x, Y a subscheme with the property $y, \ldots$, W a subscheme with the property w, and the reduced supports of $X, Y, \ldots, W$ are pairwise disjoint.

(2.1.1) $U_1 := [P \cup Q(d-1)]$ with $P \in L$; $U_2 := [Y \cup Q(d-2)]$ with length$(Y) = 2$ and support$(Y) = P_0$.

Note that $U_1 \approx \mathbf{P}^1$ and $E(\mathcal{O}_{\mathbf{P}^2}(n), U_1) = \mathcal{O}_{\mathbf{P}^1}(n) \oplus (\mathcal{O}_{\mathbf{P}^1})^{\oplus(d-1)}$. Also U_2 is $\mathbf{P}^1$, because it is the fibre A of the bundle F over P_0. We have computed $c_t E(\mathcal{O}_{\mathbf{P}^2}(n), F)$ in (1.8.3); hence $c_t(E(\mathcal{O}_{\mathbf{P}^2}(n), F) = (1 + \varphi t)$, where φ is the class of a point.

	U_1	U_2
c_1E_0	0	1
c_1E_1	1	1
c_1E_2	2	1

FIG. 2

Therefore we obtain table of degrees in Fig. 2.

(2.2) We define 5 surfaces, $D_1 \cdots D_5$, in $\mathrm{Hilb}_d(\mathbf{P}^2)$, $d \geq 3$, and need a sixth surface D_0 for $\mathrm{Hilb}_d(\mathbf{P}^2)$, $d \geq 4$. We let D_0 be the subvariety of $\mathrm{Hilb}_d(\mathbf{P}^2)$, $d \geq 4$, which parametrizes the folowing subschemes $Z \subset \mathbf{P}^2$.

We lexicographically order the monomials of the same degree in x and y; in particular $x < y$. We order monomials of different degree according to the degree. Let Set $= \{1, x, y, \dots\}$ be the set of the first $(d + 2)$ monomials and let J be the ideal of $\mathbf{C}\{x, y\}$ generated by the monomials which are not in Set. Let V be the vector space generated by the last 3 monomials in Set. Given any subvector space $W \subset V$ of codimension 1, the ideal $J + W$ is the ideal of a closed subscheme $Z(W)$ of length d with support the origin.

Clearly $D_0 \approx \mathbf{P}^2$, and $E(\mathcal{O}_{\mathbf{P}^2}(n), D_0) \approx \mathcal{O}^{\oplus(d-1)} \oplus \mathcal{O}(1)$.

In the definition of the other surfaces $D_i, 5 \geq i \geq 1$, we continue to use the notations introduced above.

$D_1 = [Y \cup Q(d - 3)]$ where length $(Y) = 3$, support$(Y) = P_0$. The desingularization of surface D_1 is isomorphic to the fibre B of S over P_0; see (1.4).

$D_2 = [Y \cup Q(d - 2)]$ where Y varies in the family of the closed subschemes of L which are of length 2. Note that $D_2 \approx \mathbf{P}^2$, the second symmetric product of $\mathbf{P}^1$.

$D_3 = [Y \cup Q(d - 2)]$ where length $(Y) = 2$, support (Y) is a varying point in L, so Y is not reduced. D_3 is isomorphic with the restriction of F to L; see (1.5).

$D_4 = [Y \cup X \cup Q(d - 3)]$ where length $(Y) = 2$, support $(Y) = P_0$, and X is a varying point in L.

$D_5 = [P \cup Q(d - 1)]$ where P varies in $\mathbf{P}^2$.

(2.2.1) We have the table of degrees in Fig. 3.

Note that the determinant of the 6×6 matrix is -1. When $d = 3$ we exclude the first column, because D_0 is not defined for $\mathrm{Hilb}_3(\mathbf{P}^2)$; in this case the last 5 columns and the rows $1, 2, 3, 4, 6$ give a matrix of determinant -1.

	D_0	D_1	D_2	D_3	D_4	D_5
$c_2 E_0$	0	1	0	0	0	0
$c_2 E_1$	0	1	0	1	1	0
$c_2 E_2$	0	1	1	2	2	0
$(c_1 E_1)^2$	1	3	0	1	2	$1-(d-1)$
$(c_1 E_1)(c_1 E_2)$	1	3	0	3	3	$2-(d-1)$
$(c_1 E_2)^2$	1	3	1	5	4	$4-(d-1)$

FIG. 3

We explain briefly how the intersection numbers are computed, the careful reader should be able to fill in the details.

(2.2.2) The intersection numbers with D_0 come from the explicit identifications $D_0 \approx \mathbf{P}^2$, $E(\mathcal{O}_{\mathbf{P}^2}(n), D_0) \approx \mathcal{O}^{\oplus(d-1)} \oplus \mathcal{O}(1)$.

(2.2.3) The desingularization of the surface D_1 is isomorphic to the fibre B of S over P_0. Hence

$$A^{\cdot}(B) = \mathbf{Z}[\alpha, \varphi, \sigma]/(\alpha, \varphi^2, \sigma^2 + \varphi\sigma)$$

because $A^{\cdot}(B) = A^{\cdot}(S)/(\alpha)$. Recalling (1.8.3), we have

$$c_t E(\mathcal{O}_{\mathbf{P}^2}(n), B) = (1 + (2\varphi)t)(1 + (-\sigma)t)^{-1}$$

and to finish we only need to remark that $\varphi\sigma$ has degree 1 on B.

(2.2.4) We recall that $D_2 \approx \mathbf{P}^2$, the second symmetric product of $\mathbf{P}^1$. The inclusion of the universal family $I \subset D_2 \times \mathbf{P}^2$ factors through $D_2 \times L$ and I is a divisor of bidegree $(1,2)$ in $D_2 \times L$. The computations of $c_t E(\mathcal{O}_{\mathbf{P}^2}(n), D_2)$ is then standard using relative duality:

$$c_t E(\mathcal{O}_{\mathbf{P}^2} D_2) = c_t(\mathcal{O}_{D_2}(-1)), \quad c_t E(\mathcal{O}_{\mathbf{P}^2}(1), D_2) = 1,$$
$$c_t E(\mathcal{O}_{\mathbf{P}^2}(2), D_2) = c_t(\mathcal{O}_{D_2}(-1))^{-1}.$$

(2.2.5) D_3 is simply the restriction of F to L, therefore

$$A^{\cdot}(D_3) = \mathbf{Z}[\alpha, \beta]/(\alpha^2, \beta^3, -\alpha\beta + \beta^2).$$

From (1.8),

$$\begin{aligned} c_t E(\mathcal{O}_{\mathbf{P}^2}(n), F) &= (1 + (3n-3)\alpha t)\left(1 - (\lambda + n\alpha)t + (\lambda + n\alpha)^2 t^2\right) \\ &= 1 + (2n\alpha + \varphi)t + (n\alpha\varphi)t^2 \end{aligned}$$

where $\varphi = \beta - 2\alpha$, $\lambda = -\alpha - \beta$, and degree $\alpha\varphi = 1$.

(2.2.6) D_4 is isomorphic to a smooth quadric surface, being the product $L \times A$, where $A \approx \mathbf{P}^1$ is the fibre of F over P_0. We have

$$c_t E(\mathcal{O}_{\mathbf{P}^2}(1), D_4) = c_t(\mathcal{O}_L(n)) \cdot c_t E(\mathcal{O}_{\mathbf{P}^2}(n), A) = (1 + n\alpha t)(1 + \varphi t)$$

where α and φ are the classes of the two lines in the quadric.

(2.2.7) D_5 is isomorphic to the surface, M say, which is the blow-up of $\mathbf{P}^2$ along the points $Q_1, Q_2, \ldots, Q_{d-1}$. The points of the exceptional lines L_i in M represent subschemes

$$Z = W \cup Q_1 \cup Q_2 \ldots \cup Q_{i-1} \cup Q_{i+1}, \ldots, \cup Q_{d-1},$$

where length$(W) = 2$, support$(W) = Q_i$. The universal family $I \subset M \times \mathbf{P}^2$ is the union

$$I = G \cup M_1 \cup M_2 \cup \cdots \cup M_{d-1},$$

where G is the graph of the map $M \to \mathbf{P}^2$, $G \approx M \approx M_i$, $G \cap M_i \approx L_i$, $M_i \cap M_k = \varnothing$, support$(q(M_i)) = Q_i$. The decomposition of I gives a Mayer-Vietoris sequence

$$0 \to \mathcal{O}_I \to \mathcal{O}_G \oplus \left(\oplus \mathcal{O}_{M_i}\right) \to \oplus \mathcal{O}_{L_i} \to 0;$$

hence also

$$0 \to \mathcal{O}_I \otimes q^*\mathcal{O}_{\mathbf{P}^2}(n) \to \mathcal{O}_G(n) \oplus \left(\oplus \mathcal{O}_{M_i}\right) \to \oplus \mathcal{O}_{L_i} \to 0,$$

where $\mathcal{O}_G(n)$ is the pull back to G of $\mathcal{O}_{\mathbf{P}^2}(n)$ via $G \approx M \to \mathbf{P}^2$. Since

$$E(\mathcal{O}_{\mathbf{P}^2}(n), D_5) = p_*(\mathcal{O}_I \otimes q^*\mathcal{O}_{\mathbf{P}^2}(n)),$$

using the property that p is finite we compute

$$c_t E(\mathcal{O}_{\mathbf{P}^2}(n), D_5) = (1 + n\mu t) \cdot (1 - \lambda_1 t) \cdot \ldots \cdot (1 - \lambda_{(d-1)} t)$$

where $\mu =$ class $\mathcal{O}_M(1)$ and $\lambda_i =$ class(L_i).

(2.3) From the table we see that in $\text{Hilb}_3(\mathbf{P}^2)$, $c_2E_1 = 3(c_1E_1)(c_1E_2)$. Using Porteous's formulas one could check that c_2E_1 is the class of the locus in $\text{Hilb}_3(\mathbf{P}^2)$ of the subschemes of length 3 which are subschemes of lines moving in a given pencil. Similarly c_2E_2 is the class of the locus in $\text{Hilb}_3(\mathbf{P}^2)$ of the subschemes of length 3 which are subschemes of conics moving in a given pencil. Also c_2E_0 has geometric meaning. Let $D(2)$ denote the fourfold in $\text{Hilb}_3(\mathbf{P}^2)$ which is the locus of subschemes supported on a single varying point. Clearly $D(2) \cap D_j = \varnothing$ if $j \neq 1$; hence class$(D(2)) = xc_2E_0$, for some integer x. We determine x by restriction to the subvariety $\mathbf{P}^3 \approx \text{Hilb}_3(L) \subset \text{Hilb}_3(\mathbf{P}^2)$. This variety is called T_4 later below and we compute there that degree c_2E_0 is 1 on T_4; on the other hand the restriction of $D(2)$ to $\mathbf{P}^3$ is transversal and gives a twisted cubic in $\mathbf{P}^3$. Hence class$(D(2)) = 3c_2E_0$.

Part III

(3.1) $A_3(\text{Hilb}_3(\mathbf{P}^2))$. We describe here some subvarieties of dim 3 on $\text{Hilb}_d(\mathbf{P}^2)$, $d \geq 3$, which turn out to form a basis for the Chow group $A_3(\text{Hilb}_3(\mathbf{P}^2))$. The same threefolds will also appear in the description of a set of generators for $A_3(\text{Hilb}_d(\mathbf{P}^2))_{\mathbf{Q}}$.

We keep the notations and the conventions used above in part 2. This is the list:

$T_1 := [W \cup Y]$ where $W \cup Y$ is a closed subscheme of length d inside a line moving in the pencil of centre P_0 having the property that length $W = d - 2$ and support$(W) = P_0$.

$T_2 := [Q \cup P \cup Q(d-2)]$ where Q is a point which varies in L, P is a point which varies in $\mathbf{P}^2$.

$T_3 := [W \cup P \cup Q(d-3)]$ where W is a varying closed subscheme of length 2 supported at P_0, P is a point which varies in $\mathbf{P}^2$.

$T_4 := [P_1 \cup P_2 \cup P_3 \cup Q(d-3)]$ where P_i are varying points of L.

$T_5 := [P \cup Y \cup Q(d-3)]$ where P varies in L, Y is a closed not reduced subscheme of length 2 for which the supporting point varies in a line M.

$T_6 := [Y \cup Q(d-3)]$ where Y is a varying closed subscheme of length 3 supported at P_0.

Using the computations outlined below we find the table for the degrees in the case of $\text{Hilb}_3(\mathbf{P}^2)$; see Fig. 4. Since E_0 is the direct image of the structure sheaf of the universal family there is a splitting map $\mathcal{O}_{\text{Hilb}} \to E_0$; hence $c_3E_0 = 0$ in our case and we have omitted the corresponding row. In the table we have computed instead the degrees on c_3E_3 which we shall need in some computations. The cycles K_i are defined below in (3.1.6).

The determinant of the submatrix obtained by deleting rows 2, 3, 4, 9, 10 and the last three columns is -3, while the determinant of the submatrix obtained

Degrees for Hilb_3

	T_1	T_2	T_3	T_4	T_5	T_6	K_2	K_3
c_3E_1	0	0	0	0	1	1	0	1
c_3E_2	0	0	0	0	4	2	4	20
c_3E_3	0	0	0	1	9	3	20	84
$(c_1E_1)^3$	6	-3	-5	-1	-5	-9	0	0
$(c_1E_2)^3$	0	9	4	0	22	18	8	54
$(c_2E_0)(c_1E_1)$	3	0	-2	-1	-2	-6	0	3
$(c_2E_0)(c_1E_2)$	3	0	-2	0	-2	-3	2	6
$(c_2E_1)(c_1E_1)$	1	-1	-1	0	0	0	0	0
$(c_2E_1)(c_1E_2)$	0	0	0	0	3	3	0	3
$(c_2E_2)(c_1E_1)$	0	0	0	0	6	6	0	6
$(c_2E_2)(c_1E_2)$	0	4	2	0	12	9	6	36

FIG. 4

by deleting rows 3, 6, 7, 9, 10 and the last three columns is 4. Since 3 and 4 are coprime the Chern monomials generate a lattice L of rank 6, with the property that the matrix of intersection of L with the lattice generated by the T_i is unimodular. Using Poincaré's duality we see that L is exactly $A_3(\mathrm{Hilb}_3(\mathbf{P}^2))$.

(3.1.1) *Geometry on* T_1. In the following we simply write T instead of T_1. Recall that T is the subvariety of Hilb_d which parametrizes closed subschemes Z with the following properties.

(1) Z is a closed subscheme of a line moving in the pencil of centre P_0.

(2) There is a closed subscheme $X \subset Z$ with length $X = d - 2$ supported at P_0.

We start from some useful considerations. Let L be the line at infinity in the plane, so that L is also the parameter space for the lines through the origin. Let $A \subset L \times \mathbf{P}^2$ be the incidence correspondence; if (α, β) are coordinates on L and (x, y, z) are coordinates in $\mathbf{P}^2$, then A is the divisor of the equation $\alpha x + \beta y = 0$. We write $f\colon A \to L$, the projection. The surjection $\mathcal{O}_L^{\oplus 3} \to \mathcal{O}_L \oplus \mathcal{O}_L(1)$, induced from $(\alpha, \beta)\colon \mathcal{O}_L^{\oplus 2} \to \mathcal{O}_L(1)$, corresponds to the closed immersion $A \subset L \times \mathbf{P}^2$. Therefore $A = \mathbf{P}(E)$, where $E := \mathcal{O}_L \oplus \mathcal{O}_L(1)$. We let λ

be the class of the pull back to A of $\mathcal{O}_{\mathbf{P}^2}(1)$ (so that $\lambda = \mathcal{O}_{\mathbf{P}(E)}(1)$) and $h = c_1\mathcal{O}_L(1)$, then $A^{\cdot}(A) = \mathbf{Z}[h, \lambda]/(h^2, \lambda^2 - h\lambda)$.

Since $f_*\mathcal{O}_A(1) = E$, the variety $V := \mathbf{P}(\mathrm{Sym}^2(E^{\vee}))$ parametrizes the subschemes of length 2 supported on the lines in the pencil through P_0, the origin. Let $v = c_1\mathcal{O}_V(1)$; then $A^{\cdot}(V) = \mathbf{Z}[h, v]/(h^2, v^3 + 3hv^2)$.

More generally we define $W = \mathbf{P}(\mathrm{Sym}^d(E^{\vee}))$; W parametrizes the subschemes of length d supported on the lines in the pencil through P_0.

Our variety T is a subvariety of W and it is isomorphic to V. More precisely

$$\mathrm{Sym}^2(E^{\vee}) \otimes \mathcal{O}_L(2 - d)$$

is a direct summand of $\mathrm{Sym}^d(E^{\vee})$ and this gives an isomorphic embedding of V in W, the image being T; equivalently we have

$$V \approx T := \mathbf{P}\big(\mathrm{Sym}^2(E^{\vee}) \otimes \mathcal{O}_L(2 - d)\big) \subset \mathbf{P}\big(\mathrm{Sym}^d(E^{\vee})\big).$$

Let $z = c_1\mathcal{O}_V(1)$; then

$$A^{\cdot}(W) = \mathbf{Z}[h, z]/\big(h^2, z^{d+1} + (d(d+1)/2)z^d h\big)$$

and the pull back to V of z is $z|_W = v + (2 - d)h$.

The incidence family $I \subset W \times \mathbf{P}^2$ is a divisor in $M := W \times_L \mathbf{P}(E)$, and M is a divisor in $W \times \mathbf{P}^2$. Since $M = \mathbf{P}(E_W)$ then $\mathrm{Pic}(M) = \mathbf{Z}h \oplus \mathbf{Z}\lambda \oplus \mathbf{Z}z$. The divisor I is the zero locus of the composite map

$$\mathcal{O}_w(-1) \to \mathrm{Sym}^d(E) \to \mathcal{O}_A(d).$$

Hence class $I = d\lambda + z$, equivalently the ideal sheaf of I is $\mathcal{O}_w(-1) \otimes \mathcal{O}_A(-d)$,

$$0 \to \mathcal{O}_w(-1) \otimes \mathcal{O}_A(-d) \to \mathcal{O}_M \to \mathcal{O}_I \to 0.$$

Taking tensor products with $\mathcal{O}_A(n)$ one has

$$0 \to \mathcal{O}_w(-1) \otimes \mathcal{O}_A(-d + n) \to \mathcal{O}_M(n) \to \mathcal{O}_I(n) \to 0.$$

Pushing down via $f_w\colon W \times_L \mathbf{P}(E) \to W$ the sequence gives

$$\begin{aligned} 0 &\to \mathcal{O}_w(-1) \otimes R^0 f_*\mathcal{O}_A(-d + n) \to \mathrm{Sym}^n(E) \\ &\to \mathcal{E}\big(W, \mathcal{O}_{\mathbf{P}^2}(n)\big) \to \mathcal{O}_w(-1) \\ &\qquad \otimes \mathcal{O}_L(-1) \otimes R^1 f_*(\mathcal{O}_A(-d + n) \otimes \mathcal{O}_L(1)) \to 0. \end{aligned}$$

Noting that the relative canonical divisor for $M \to W$ is $K_{M/W} = \mathcal{O}_A(-2) \otimes \det E$ (see [8, Example 8.4]), we compute using the relative duality isomorphism

$$R^1f_*\mathcal{M} = \left(R^0f_*\left(\mathcal{M}^\vee \otimes K_{M/W}\right)\right)^\vee$$

as follows.

$d = 2$. We have

$$c_t(\mathcal{E}(V, \mathcal{O}_{\mathbf{P}^2}(0)) = (1 - (v + h)t), \quad c_t(\mathcal{E}(V, \mathcal{O}_{\mathbf{P}^2}(1)) = (1 + ht),$$
$$c_t(\mathcal{E}(V, \mathcal{O}_{\mathbf{P}^2}(2))$$
$$= (1 + (3h)t)(1 + vt + v^2t^2 + v^3t^3) = \left(1 + (3h + v)t + (v^2 + 3hv)t^2\right)$$

$d = 3$. We have

$$c_t(\mathcal{E}(W, \mathcal{O}_{\mathbf{P}^2}(0)) = (1 - (z + h)t)(1 - (z + 2h)t),$$
$$c_t(\mathcal{E}(W, \mathcal{O}_{\mathbf{P}^2}(1)) = (1 + ht)(1 - (z + h)t),$$
$$c_t(\mathcal{E}(W, \mathcal{O}_{\mathbf{P}^2}(2)) = (1 + (3h)t).$$

$d = 4$ and $n \leq 2$. We have

$$0 \to \mathrm{Sym}^n(E) \to \mathcal{E}(W, \mathcal{O}_{\mathbf{P}^2}(n))$$
$$\to \mathcal{O}_W(-1) \otimes \mathcal{O}_L(-1) \otimes \mathrm{Sym}^{d-n-2}(E^\vee) \to 0.$$

$d = 3$. We obtain

$$c_t(\mathcal{E}(T, \mathcal{O}_{\mathbf{P}^2}(0)) = (1 - vt)(1 - (v + h)t),$$
$$c_t(\mathcal{E}(T, \mathcal{O}_{\mathbf{P}^2}(1)) = (1 - vt)(1 + ht),$$
$$c_t(\mathcal{E}(T, \mathcal{O}_{\mathbf{P}^2}(2)) = (1 + (3h)t)$$

and we compute the first column of degrees in the diagram of Hilb_3.

$d = 4$. We obtain

$$\begin{aligned} c_t(\mathcal{E}(T, \mathcal{O}_{\mathbf{P}^2}(0)) &= (1 + (h - vt))(1 - vt)(1 - (v + h)t) \\ &= 1 - 3vt + 3v^2t^2 + 3hv^2t^3, \\ c_t(\mathcal{E}(T, \mathcal{O}_{\mathbf{P}^2}(1)) &= (1 + ht)(1 + (-v + h)t)(1 - vt) \\ &= 1 + 2(h - v)t + (v^2 - 3hv)t^2 + hv^2t, \\ c_t(\mathcal{E}(T, \mathcal{O}_{\mathbf{P}^2}(2)) &= (1 + (3h)t)(1 + (h - v)t) = 1 + (4h - v)t - 3hvt^2. \end{aligned}$$

$d \geq 4$, $n \geq 0$. In this more general case we have

$$\begin{aligned} c_t(\mathcal{E}(T, \mathcal{O}_{\mathbf{P}^2}(n)) = &\left(1 + \left(\tfrac{1}{2}n(n + 1)h\right)t\right)(1 - (v + (3 - d)h)t) \\ &\times (1 - (v + (4 - d)h)t) \ldots (1 - (v - (n - 1)h)t). \end{aligned}$$

For any integer $s > 0$ let

$$g(s) = (1 - (v - sh)t)(1 - (v + (1 - s)h)t)\dots(1 - (v - 1h)t),$$

and let $g(0) = 1$. Then in $A(V)$ we compute

$$\begin{aligned} g(s) &= (1 - vt)^s + (1/2)s(s + 1)ht(1 - vt)^{(s-1)} \\ &= 1 - svt + \tfrac{1}{2}s(s + 1)ht + \tfrac{1}{2}s(s - 1)v^2t^2 - \tfrac{1}{2}s(s - 1)(s + 1)hvt^2 \\ &\quad + \tfrac{1}{4}s(s - 1)(s - 2)(s + 3)hv^2t^3. \end{aligned}$$

Therefore we can express the total Chern class as a polynomial in d:

$$\begin{aligned} c_t(\mathscr{E}(T, \mathcal{O}_{\mathbf{P}^2}(0)) &= (1 - (v + h)t)(1 - vt)g(d - 3), \\ c_t(\mathscr{E}(T, \mathcal{O}_{\mathbf{P}^2}(1)) &= (1 + ht)(1 - vt)g(d - 3), \\ c_t(\mathscr{E}(T, \mathcal{O}_{\mathbf{P}^2}(2)) &= (1 + 3ht)g(d - 3). \end{aligned}$$

The process to determine the degrees is now elementary; according to the computer we obtain the following expressions for the degrees on T_1:

$$\begin{aligned}
z_1 &:= \text{degree of } c_3E_0 &&= \tfrac{1}{4}(d^4 - 8d^3 + 23d^2 - 28d + 12), \\
z_2 &:= \text{degree of } c_3E_1 &&= \tfrac{1}{4}(d^4 - 10d^3 + 37d^2 - 60d + 36), \\
z_3 &:= \text{degree of } c_3E_2 &&= \tfrac{1}{4}(d^4 - 12d^3 + 53d^2 - 102w + 72), \\
z_4 &:= \text{degree of } (c_1E_1)^3 &&= \tfrac{1}{2}(3d^4 - 21d^3 + 60d^2 - 84d + 48), \\
z_5 &:= \text{degree of } (c_1E_2)^3 &&= \tfrac{1}{2}(3d^4 - 27d^3 + 99d^2 - 189d + 162), \\
z_6 &:= \text{degree of } (c_2E_0)(c_1E_1) &&= \tfrac{1}{4}(3d^4 - 20d^3 + 51d^2 - 58d + 24), \\
z_7 &:= \text{degree of } (c_2E_0)(c_1E_2) &&= \tfrac{1}{4}(3d^4 - 22d^3 + 63d^2 - 80d + 36), \\
z_8 &:= \text{degree of } (c_2E_1)(c_1E_1) &&= \tfrac{1}{4}(3d^4 - 24d^3 + 75d^2 - 110d + 64), \\
z_9 &:= \text{degree of } (c_2E_1)(c_1E_2) &&= \tfrac{1}{4}(3d^4 - 26d^3 + 89d^2 - 146d + 96), \\
z_{10} &:= \text{degree of } (c_2E_2)(c_1E_1) &&= \tfrac{1}{4}(3d^4 - 28d^3 + 101d^2 - 172d + 120), \\
z_{11} &:= \text{degree of } (c_2E_2)(c_1E_2) &&= \tfrac{1}{4}(3d^4 - 30d^3 + 117d^2 - 222d + 180), \\
z_{12} &:= \text{degree of } (c_1E_1)(c_1E_2)^2 &&= \tfrac{1}{4}(6d^4 - 50d^3 + 170d^2 - 294d + 216), \\
z_{13} &:= \text{degree of } (c_1E_1)^2(c_1E_2) &&= \tfrac{1}{4}(6d^4 - 46d^3 + 144d^2 - 224d + 144).
\end{aligned}$$

For the sake of completeness we also record:

$$
\begin{aligned}
c_1E_0 &= \left(2 + \tfrac{1}{2}(-5d + d^2)\right)h + (1-d)v,\\
c_1E_1 &= \left(4 + \tfrac{1}{2}(-5d + d^2)\right)h + (2-d)v,\\
c_1E_2 &= \left(6 + \tfrac{1}{2}(-5d + d^2)\right)h + (3-d)v,\\
c_2E_0 &= \tfrac{1}{2}(8 - 14d + 7d^2 - d^3)hv + \tfrac{1}{2}(2 - 3d + d^2)v^2,\\
c_2E_1 &= \tfrac{1}{2}(22 - 23d + 8d^2 - d^3)hv + \tfrac{1}{2}(6 - 5d + d^2)v^2,\\
c_2E_2 &= \tfrac{1}{2}(42 - 32d + 9d^2 - d^3)hv + \tfrac{1}{2}(12 - 7d + d^2)v^2.
\end{aligned}
$$

(3.1.2) *Geometry on T_2.* T_2 is the closure of the subset in Hilb_d which parametrizes the subschemes of the type $Z = Q \cup P \cup Q(d-2)$, where Q moves in L and P varies in $\mathbf{P}^2 - (L \cup Q(d-2))$. T_2 has a natural desingularization W which we describe as follows.

Let $\mathbf{P}^+$ be the blow up of $\mathbf{P}^2$ at $Q(d-2)$, let E_i be the exceptional line mapping to Q_i. Let M be the diagonal in $L \times L \subset L \times \mathbf{P}^+$; the desingularization W is the blow-up of $L \times \mathbf{P}^+$ along M. We let $g: W \to \mathbf{P}^+$ and $f: W \to L$ be the natural projections, D the exceptional divisor in W, Q_i^+ the divisors $L \times E_i$. Note that for $x \in L$, $\mathbf{P}_x^+ := f^{-1}(x)$ is the blow-up of $\mathbf{P}^2$ at $Q(d-2) \cup \{x\}$. We let E_x be the exceptional divisor for $\mathbf{P}_x^+ \to \mathbf{P}^+$.

A general point z of W represents a subscheme Z as above, $g(z) = P$, $f(z) = Q$. When P comes to coincide with a Q_i then $g(Z) \in E_i$ represents the tangent direction determined by the scheme of length 2 supported at Q_i. When P becomes a point P_1 of L then $g(Z) = P_1$ and $f(Z) = Q$; if Q and P_1 coincide in x then they determine tangent directions parametrized by $E_x \subset \mathbf{P}_x^+ \subset W$.

The Chow ring $A^\cdot(W)$ is computed easily. Let

$$h = c_1\mathcal{O}_{\mathbf{P}^2}(1), \quad z = c_1\mathcal{O}_L(1), \quad \delta = \mathrm{class}(D), \quad q_i = \mathrm{class}(Q_i^+).$$

From the standard theory of the Chow ring of a blowing-up it follows that $A^\cdot(W)$ is a quotient of $\mathbf{Z}[h, z, q_i, \delta]$. There are the obvious relations $hq_i = 0$; $q_jq_i = 0$, $i \neq j$; $z^2 = 0$; $h^3 = 0$; $h^2\delta = 0$; $\delta q_i = 0$; $zh\delta = 0$; $q_i^3 = 0$; $q_i^2 = -\mathrm{class}(L \times \{\text{point}\})$; $\mathrm{degree}(h^2z) = 1$; $\mathrm{degree}(q_i^2z) = -1$. In order to compute powers of δ we need to determine the Chern classes of the normal bundle, say $\mathcal{N}$, of the diagonal $L \approx M$ in $L \times \mathbf{P}^+$. Looking at the inclusions $M \subset L \times L \subset L \times \mathbf{P}^+$, we see that $c_1(\mathcal{N}) = 3z$. It follows that degree $\delta^3 = -3$ and further that $\mathrm{degree}(\delta^2z) = -1$ and $\mathrm{degree}(\delta^2h) = -1$.

Let I be the universal family in $W \times \mathbf{P}^2$, $p: I \to W$ the projection, $q: I \to \mathbf{P}^2$.

I decomposes in components as $I = C \cup B \cup A_1 \cup \cdots \cup A_{d-2}$, where $q(A_i) = Q_i$, $q(B) = L$, $q(C) = \mathbf{P}^2$. Note that $A_i \approx B \approx C$; $A \cap B = \varnothing$; $A_i \cap C = L \times E_i = Q_i^+$, by definition, $B \cap C \approx D$.

The composition of I corresponds to a Majer-Vietoris sequence

$$0 \to \mathcal{O}_I \to \mathcal{O}_C \oplus \mathcal{O}_B \oplus \left(\oplus \mathcal{O}_{A_i}\right) \to \oplus \mathcal{O}_D \oplus \left(\oplus \mathcal{O}_{Q_i}\right) \to 0.$$

Tensoring with $q^*(c_1\mathcal{O}_{\mathbf{P}^2}(n))$ gives

(a)
$$0 \to \mathcal{O}_I(n) \to \mathcal{O}_C(nh) \oplus \mathcal{O}_B(nz) \oplus \left(\oplus \mathcal{O}_{A_i}\right) \to \oplus \mathcal{O}_D(nz) \oplus \left(\oplus \mathcal{O}_{Q_i}\right) \to 0.$$

We have abused notations here by writing $\mathcal{O}_C(nh)$ to mean the pull-back of $\mathcal{O}_{\mathbf{P}^2}(n)$, and $\mathcal{O}_B(nz)$, $\mathcal{O}_D(nz)$ to mean the pull-back of $\mathcal{O}_L(n)$. Pushing down via p_*, (a) yields

$$\begin{aligned} 0 &\to \mathcal{E}(W, \mathcal{O}_{\mathbf{P}^2}(n)) \to \mathcal{O}_w(nh) \oplus \mathcal{O}_w(nz) \oplus \left(\oplus_i \mathcal{O}_w\right) \\ &\to \oplus \mathcal{O}_D(nz) \oplus \left(\oplus \mathcal{O}_{Q_i}\right) \to 0. \end{aligned}$$

Noting that $\mathcal{O}_D(nz) \approx \mathcal{O}_D(nh)$ we obtain

$$\begin{aligned} &c_t(\mathcal{E}(W, \mathcal{O}_{\mathbf{P}^2}(n)) \\ &\quad = (1 + nzt)(1 - q_1 t) \cdots \left(1 - q_{(d-2b)} t\right) \cdot \left(1 + (nh - \delta)t\right) \\ &\quad = 1 + \left(n(z + h) - \delta - \left(q_1 + \cdots + q_{(d-2b)}\right)\right)t \\ &\qquad + \left(nz\left(nh - \delta - \left(q_1 + \cdots + q_{(d-2b)}\right)\right)\right)t^2. \end{aligned}$$

Note that $c_3(\mathcal{E}(W, \mathcal{O}_{\mathbf{P}^2}(n)) = 0$, $c_2(\mathcal{E}(W, \mathcal{O}_{\mathbf{P}^2}(0)) = 0$.
Elementary computations give

$$\begin{aligned} &\operatorname{degree}\left(c_1\left(\mathcal{E}(W, \mathcal{O}_{\mathbf{P}^2}(n))\right) \cdot c_1\left(\mathcal{E}(W, \mathcal{O}_{\mathbf{P}^2}(m))\right) \cdot c_1\left(\mathcal{E}(W, \mathcal{O}_{\mathbf{P}^2}(s))\right)\right) \\ &\qquad = 3nms + (n + m + s)(-2 - (d - 2)) + 3, \\ &\operatorname{degree}\left(c_1\left(\mathcal{E}(W, \mathcal{O}_{\mathbf{P}^2}(n))\right) \cdot c_2\left(\mathcal{E}(W, \mathcal{O}_{\mathbf{P}^2}(m))\right)\right) = m^2 n - m - (d - 2)m. \end{aligned}$$

(3.1.3) *Geometry on the variety* T_3. T_3 is the subvariety of $\mathrm{Hilb}_d(\mathbf{P}^2)$, $d \geq 3$, which is the closure of the set of points parametrizing subschemes of $\mathbf{P}^2$ of the type $Z = W \cup P \cup Q(d - 3)$, where W is a varying scheme of length 2 supported at P_0 and P varies in $\mathbf{P}^2 - (\{P_0\} \cup Q(d - 3))$.

Let $\mathbf{P}$ be the blown-up of $\mathbf{P}^2$ at $Q(d - 3)$, and let $R_i, 1 \leq i \leq (d - 3)$, be the exceptional line in $\mathbf{P}$ over Q_i. R_i represents naturally the family of subschemes of length 2 supported at Q_i. Let A be the fibre of the variety F of first order data over P_0 (see (1.4)); so A represents naturally the family of subschemes of length 2 supported at P_0. We write W_x to denote the subscheme of length 2 parametrized by $x \in A$. There is a natural birational correspondence between T_3 and $A \times \mathbf{P}$. In $A \times \mathbf{P}$ the correspondence is not defined along $A \times \{P_0\}$. In T_3 the correspondence is not defined along D_1, the surface of "triple points" at P_0; see (2.2).

We produce below a desingularization $E \to T_3$, and E is in fact the closure of the graph of the correspondence in $T_3 \times (A \times \mathbf{P})$. Let $a: E \to A$ be the induced map, $x \in A$; then $E_x := a^{-1}(x)$ is a surface which is a desingularization of the surface $D_x := \{Z \in T_3: W_x \subset Z\}$. More precisely E_x is obtained by blowing up $\mathbf{P}$ twice, first along P_0, and next along the point on the exceptional fibre over P_0 which represents the tangent direction determined by x. D_x is then obtained by blowing down the proper transform of the first exceptional line, which is contracted to the point which represents the "big point", see (1.4).

We construct E by the following process. Let $\mathbf{P}^+$ be the blown-up of $\mathbf{P}$ at the origin P_0, let R_0 be the exceptional line. There is a natural identification $R_0 \cong A$. Let N be the diagonal in $A \times R_0 \subset A \times \mathbf{P}^+$. E is obtained by blowing up $A \times \mathbf{P}^+$ along N. We shall let T be the exceptional divisor in E, and M the proper transform of $M^+ := A \times R_0$; for brevity we write $M_i := A \times R_i, 1 \leq i \leq (d-3)$.

Via $E \to T_3$ the divisor T maps onto the surface D_1 and M is contracted to a point.

The Chow ring of E is computed using the standard theory of blow-ups. The following divisorial classes are generators for the ring:

$$z = c_1\mathcal{O}_A(1), \quad h = c_1\mathcal{O}_{\mathbf{P}^2}(1),$$
$$\tau = \text{class}(T), \quad \mu = \text{class}(M), \quad \mu_i = \text{class}(M_i), 1 \leq i \leq (d-3).$$

It is easy to establish the following set of relations:

$$0 = h^3 = z^2 = \mu_i h = \mu h = \tau h = \mu_i \tau = \mu\mu_i = \mu_i\mu_j, \quad \text{for } i \neq j;$$
$$1 = \text{degree}(h^2 z) = \text{degree}(\tau\mu z) = -\text{degree}\big((\mu_i)^2 z\big).$$

This set of relations is not complete yet. To complete it we need to compute the Chern class of the conormal sheaf $\mathcal{N}$ of N in $A \times \mathbf{P}^+$. This is done using the inclusion of divisors $N \subset M^+ \subset A \times \mathbf{P}^+$; it gives $c_1(\mathcal{N}) = -\rho$, where ρ is the class of a point in N. Therefore $A^{\cdot}(T) = \mathbf{Z}[\rho, \sigma]/(\rho^2, \sigma^2 + \rho\sigma)$, where σ is the tautological class in $T = \mathbf{P}(\mathcal{N})$. Using the well known isomorphism $\mathcal{O}_E(T) \otimes \mathcal{O}_T \approx \mathcal{O}_T(-1)$, we obtain the relations: degree(τ^3) = degree$((-\sigma)^2)$ $= -1$. We have also $(\tau + \mu)^3 = 0$, because $(\tau + \mu)$ is the pull-back to E of the class of M^+ in $A \times \mathbf{P}^+$ and M^+ is the pull-back to $A \times \mathbf{P}^+$ of the divisor R_0 in the surface $\mathbf{P}^+$. The same argument gives $(\tau + \mu)^2\tau = 0$. Since class(M^+) restricts to $-\rho$ in $A^{\cdot}(N)$, we have

$$\text{degree}\big(\tau^2(\tau + \mu)\big) = \text{degree}((-\rho)(-\sigma)) = 1.$$

Therefore

$$\text{degree}(\tau^3) = -1, \quad \text{degree}(\mu\tau^2) = 2, \quad \text{degree}(\tau\mu^2) = -3, \quad \text{degree}(\mu^3) = 4,$$

and also

$$\text{degree}(\tau\mu z) = 1, \quad \text{degree}(\tau^2 z) = -1, \quad \text{degree}(z\mu^2) = -2.$$

In order to compute the Chern classes of the secant bundles we need to control the incidence family $I \subset E \times \mathbf{P}^2$ with the projections $p: I \to E, q: I \to \mathbf{P}^2$. I decomposes as $I = X \cup Y \cup Z_1 \cup \cdots \cup Z_{(d-3)}$, where $q(X) = \mathbf{P}^2, q(Y) = R_0, q(Z_i) = Q_i$.

If z is a general point in E then $p^{-1}(z) \cap X = P$, the varying point in $\mathbf{P}^2$, $p^{-1}(z) \cap Y = W$ the scheme of length 2 supported on P_0. This means that $p: Y \to E$ is just the pull-back to E of the universal family on A, and therefore $p_*\mathcal{O}_Y = \mathcal{O}_E \oplus \mathcal{O}_E(z)$, where $\mathcal{O}_E(z)$ denotes the pull-back to E of $\mathcal{O}_A(1)$.

Clearly $E \approx X$. Using this identification the components of E intersect in this way: $X \cap Y = S$, $X \cap Z_i = M_i$, $Y \cap Z_i = \varnothing$ where S is a subscheme of cod 1 in E. Via $E \approx X \to T_3$, S maps to D_1, the surface described in (2.2). The decomposition of I corresponds to the Mayer-Vietoris sequence

$$0 \to \mathcal{O}_I \to \mathcal{O}_X \oplus \mathcal{O}_Y \oplus \left(\oplus \mathcal{O}_{Z_i}\right) \to \mathcal{O}_S \oplus \left(\oplus \mathcal{O}_{M_i}\right) \to 0$$

which yields (tensoring first with $q^*\mathcal{O}_{\mathbf{P}^2}(n)$ and then projecting via p_*)

$$\begin{aligned} 0 &\to \mathcal{E}(n, E) \to \mathcal{O}_E(nh) \oplus \mathcal{O}_E \oplus \mathcal{O}_E(z) \oplus \left(\oplus(\mathcal{O}_E)_i\right) \\ &\to \mathcal{O}_S \otimes \mathcal{O}_E(nh) \oplus \left(\oplus \mathcal{O}_{M_i}\right) \to 0, \end{aligned}$$

where $\mathcal{O}_E(nh)$ denotes the pull back to E of $\mathcal{O}_{\mathbf{P}^2}(n)$.

To compute $c_t(\mathcal{E}(n, E))$ we need only determine the class of the divisor S in E; in fact $\mathcal{O}_S \otimes \mathcal{O}_E(nh) \approx \mathcal{O}_S$ because $q(S)$ is supported in P_0. It is enough to compute on Hilb_3. In particular we have

(a) $\quad c_1(\mathcal{E}(0, E)) = z - \text{class}(S), c_1(\mathcal{E}(1, E)) = z + h - \text{class}(S).$

To compute class(S) we remark first that S is supported on $M \cup T$, so that class$(S) = \alpha\tau + \beta\mu$, where α and β are non negative integers, to be determined. To do this we restrict everything to the fibre of $E \to A$, which we called E_x. This surface represents the family of subschemes of the type

$$Z = P \cup W_x \cup Q(d-3).$$

E_x is obtained by blowing up $\mathbf{P}^+$ along the point x of R_0 which corresponds to the chosen tangent direction. Let L_1 be the proper transform of R_0 in E_x, let L_2 be the exceptional divisor over x. We have $M \cdot E_x = L_1, T \cdot E_x = L_2$; (a) gives

$$c_1(\mathscr{E}(0, E_x)) = -(\alpha(\text{class}(L_2)) + \beta(\text{class}(L_1))).$$

Now the curve L_1 is contracted under the map $E_x \to \text{Hilb}_d(\mathbf{P}^2)$, so that

$$c_1(\mathscr{E}(0, E_x)) \cdot L_1 = 0;$$

hence $2\beta = \alpha$. On the other hand the same map sends L_2 to the family of "triple points" supported at P_0 with fixed tangent direction x. Using our (incompatible) notations of (1.4) this is the fibre of $S \to F$. From our computations in part 1 it follows that degree$(c_1(\mathscr{E}(0, L_2)) = 1$, so $\beta - \alpha = -1$. Therefore class$(S) = 2\tau + \mu$.

Standard computations give the following table for the degree of the Chern monomials taken in the order used for the matrix of degrees in Hilb_d. Let $a = 3b$, $b = -(d-3)$:

$$0\ 0\ 0\ (-5+a)\ (4+a)\ (-2+b)\ (-2+b)\ (-1+b)\ (0+b)$$
$$(0+b)\ (2-b)\ (a)\ (-3+a).$$

(3.1.4) *Geometry on the variety* T_4. T_4 is just the variety of schemes of length 3 on $\mathbf{P}^1$; i.e. T_4 is $\text{Hilb}_3(\mathbf{P}^1)$, the third symmetric product of $\mathbf{P}^1$. In other words T_4 is isomorphic with $\mathbf{P}^3$. Let α, β be coordinates for $\mathbf{P}^1$ and let x, y, z, w be coordinates for $\mathbf{P}^3$. Then

$$x\alpha^3 + y\alpha^2\beta + z\alpha\beta^2 + w\beta^3 = 0$$

is the equation for the incidence family $I \subset \mathbf{P}^1 \times \mathbf{P}^3$, so I is a divisor of type (3,1). Using the exact sequence

$$0 \to \mathscr{O}_{\mathbf{P}^3}(-1) \otimes \mathscr{O}_{\mathbf{P}^1}(n-3) \to \mathscr{O}_{\mathbf{P}^1 \times \mathbf{P}^3} \otimes \mathscr{O}_{\mathbf{P}^1}(n) \to \mathscr{O}_I \otimes \mathscr{O}_{\mathbf{P}^1}(n) \to 0$$

we have

$$c_t(\mathscr{E}(T_4, \mathscr{O}_{\mathbf{P}^2}(0)) = c_t(\mathscr{O}_{\mathbf{P}^3}(-1))^2, \quad c_t(\mathscr{E}(T_4, \mathscr{O}_{\mathbf{P}^2}(1)) = c_t(\mathscr{O}_{\mathbf{P}^3}(-1))$$

and

$$c_t(\mathscr{E}(T_4, \mathscr{O}_{\mathbf{P}^2}(2)) = 1.$$

Note that if $n \geq 3$,

$$c_3(\mathscr{E}(T_4, \mathscr{O}_{\mathbf{P}^2}(n)) = \binom{n}{3}.$$

the coefficient of t^3 in $c_t(\mathscr{O}_{\mathbf{P}^3}(-1))^{(-n+2)}$.

(3.1.5) *Geometry on the variety* T_6. T_6 is isomorphic with the variety of "triple points" supported on L; hence a natural desingularization of T_6 is the restriction, say R, to L of the variety of second order data S (see part 1). The Chow ring of R is obtained by adding to $A^{\cdot}(S)$ the further relation $\alpha^2 = 0$. Elementary computations based on (1.8) yield the following degrees on T_6:

$$\text{degree } c_3E_1 = 1, \quad \text{degree } c_3E_2 = 2, \quad \text{degree } c_3E_3 = 3,$$

$$\text{degree}(c_1E_1)^3 = -9, \quad \text{degree}(c_1E_2)^3 = 18, \quad \text{degree}(c_2E_0)(c_1E_1) = -6,$$

$$\text{degree}(c_2E_0)(c_1E_2) = -3, \quad \text{degree}(c_2E_1)(c_1E_1) = 0,$$
$$\text{degree}(c_2E_1)(c_1E_2) = 3,$$

$$\text{degree}(c_2E_2)(c_1E_1) = 6, \quad \text{degree}(c_2E_2)(c_1E_2) = 9.$$

(3.1.6) *Geometry on the variety* T_5. It is enough to compute the degree of the monomials for the case of T_5 in $\mathrm{Hilb}_3(\mathbf{P}^2)$, since all T_5 are isomorphic and the associated secant bundles differ only for trivial factors when d varies.

Instead of working directly on T_5 we find more convenient to express class(T_5) as a linear combination with rational coefficient of T_4, T_6 and K_2, this last denoting the subvariety of $\mathrm{Hilb}_3(\mathbf{P}^2)$ which parametrizes the subschemes of length 3 supported on a general conic C.

More generally we let K_n be the subvariety of $\mathrm{Hilb}_3(\mathbf{P}^2)$ which parametrizes the subschemes of length 3 supported on a general curve C_n of degree n. Note that $K_1 = T_4$. The computation of the degrees of the monomials of the secant bundles on the K_n is a standard exercise in the theory of symmetric product of curves; indeed K_n is the third symmetric product of C_n and one can copy Lemma (2.5) in VIII of [1]. In particular, identifying K_2 with $\mathbf{P}^3$, and letting $\alpha = \text{class}(\mathcal{O}_{\mathbf{P}^3}(1))$, we have

$$c_t(\mathscr{E}(K_2, \mathcal{O}_{\mathbf{P}^2}(n))) = (1 - \alpha t)^{2n-3}.$$

We note:

LEMMA. Class$(K_n) = c_3(\mathscr{E}(\mathrm{Hilb}_3(\mathbf{P}^2), \mathcal{O}_{\mathbf{P}^2}(n))$ *in* $A^{\cdot}(\mathrm{Hilb}_3(\mathbf{P}^2))$.

Proof. This is an application of Porteous formulas [7]. On $\mathrm{Hilb}_3(\mathbf{P}^2)$ there is a natural free bundle F of rank $\binom{n+2}{n}$, with basis the set of monomials of degree n in the plane, and there is a natural surjection

$$F \to \mathscr{E}(\mathrm{Hilb}_3(\mathbf{P}^2), \mathcal{O}_{\mathbf{P}^2}(n));$$

at a point z parametrizing a subscheme Z this is just evaluation of the monomials. To give a curve C_n amounts to fixing a section φ of F, hence a section of

$$\mathscr{E}(\mathrm{Hilb}_3(\mathbf{P}^2), \mathcal{O}_{\mathbf{P}^2}(n));$$

K_n is exactly the locus where the section vanishes. It is also easy to verify that the section vanishes with multiplicity 1 on K_n, for instance by restricting everything to T_4 if $n \geq 3$ and to K_3 for $n = 1, 2$.

Therefore we can compute

$$\operatorname{degree}(K_n \cdot K_m) = \binom{nm}{3},$$

which we need only for $1 \leq n \leq m \leq 3$. In particular K_1, K_2, K_3 are linearly independent in $A_3(\mathrm{Hilb}_3(\mathbf{P}^2))$. Now one can easily see that:

(1) K_1, K_2, K_3 are orthogonal to T_1, T_2, T_3;

(2) $\operatorname{degree}(T_n \cdot T_m) = 0$ for $n = 3$ and $m = 2$, for $n = 3$ and $m = 3$;

(3) $\operatorname{degree}(T_n \cdot T_m) > 0$ for $n = 3$ and $m = 1$, for $n = 2$ and $m = 2$.

This means that T_1, T_2, T_3 are linearly independent and that the lattice generated by T_1, T_2, T_3 is orthogonal to the lattice generated by K_1, K_2, K_3. Since $A_3(\mathrm{Hilb}_3(\mathbf{P}^2))$ is a free group of rank 6 then $K_1, K_2, K_3, T_1, T_2, T_3$ generate

$$A_3(\mathrm{Hilb}_3(\mathbf{P}^2)) \otimes \mathbf{Q}.$$

Since T_5 is orthogonal to T_1, T_2, T_3 we have

$$T_5 = aK_1 + bK_2 + cK_3,$$

and similarly

$$T_6 = eK_1 + fK_2 + gK_3,$$

as elements of $A_3(\mathrm{Hilb}_3(\mathbf{P}^2)) \otimes \mathbf{Q}$. The coefficients $a, \ldots, g$ are found by computing the numbers $\operatorname{degree}(T_n \cdot K_m), 1 \leq m \leq 3, 5 \leq n \leq 6$. The case $n = 6$ is done in (3.1.5); $\operatorname{degree}(T_6 \cdot K_m) = m$.

To compute $\operatorname{degree}(T_5 \cdot K_m)$ we remark first that T_5 is birational to the product $D \times L$, where D denotes the surface which is the fibre of the variety F over the line M; and in fact T_5 and $D \times L$ are locally isomorphic on the open subset of the points which parametrize the subschemes of $\mathbf{P}^2$ of the type $Z = P \cup Y$ with $P \neq L \cap M \neq \operatorname{supp}(Y), P \in L, Y \in D$. The intersections $T_5 \cap K_m$ are contained in this open set if the curve C_m is general enough with respect to L and M; in this case we see by local considerations that the intersections are in fact transversal. The computation of the number of points in $T_5 \cap K_m$ is elementary, and gives

$$\operatorname{degree}(T_5 \cdot K_1) = 1, \quad \operatorname{degree}(T_5 \cdot K_2) = 4, \quad \operatorname{degree}(T_5 \cdot K_3) = 9.$$

We obtain

$$2T_6 = 18K_1 - 9K_2 + 2K_3, T_5 = 5K_1 - 4K_2 + K_3,$$

hence

$$2T_5 - 2T_6 = -8K_1 - K_2.$$

We find the degrees of the monomials on T_5 using the computations done for T_6, $K_1 = T_4$, K_2.

(3.2) *List of generators for* $A_3(\mathrm{Hilb}_4(\mathbf{P}^2))$. We propose the following set of 10 generators for the Chow group $A_3(\mathrm{Hilb}_4(\mathbf{P}^2))$: $T_1, T_2, \ldots, T_6, T_7, T_8, T_9, T_{10}$. The generators $T_1, T_2, \ldots, T_6$, have been described in (3.1); there it is also indicated how to compute their intersection degree with the monomials in the Chern classes of the secant bundles. We describe the new generators $T_7, \ldots, T_{10}$, as threefolds in $\mathrm{Hilb}_d(\mathbf{P}^2)$.

$T_7 := [Y \cup W]$, where W is a varying closed subscheme of length 2 supported at P_0, and Y is parametrized by $D_2 \subset \mathrm{Hilb}_{d-2}(\mathbf{P}^2)$.

$T_8 := [Y \cup W]$, where W is a varying closed subscheme of length 2 supported at P_0, and Y is parametrized by $D_3 \subset \mathrm{Hilb}_{d-2}(\mathbf{P}^2)$.

$T_9 := [P \cup W]$, where W is parametrized by the surface $D_1 \subset \mathrm{Hilb}_{d-1}(\mathbf{P}^2)$, and P varies in L.

$T_{10} := [W \cup Q(d-4)]$ where W is a varying closed subscheme of length 4 supported at P_0. Note that $C \to T_{10}$ is a desingularization, where C is the fibre over P_0 of the variety T of third order data introduced in (1.5).

The computation of the degrees of intersection is simple; note that the degrees do not vary with d.

The Chow ring of the desingularization C of T_{10} is obtained from the ring $A^{\cdot}(T)$ by adding the further relation $\alpha = 0$ hence

$$A^{\cdot}(C) = \mathbf{Z}[\varphi, \sigma, \tau]/(\varphi^2, \sigma^2 + \varphi\sigma, \tau^2 + 2\varphi\tau - \varphi\sigma).$$

The class of a point is $\varphi\sigma\tau$; hence

$$\mathrm{degree}(\tau^3) = 1, \quad \mathrm{degree}(\sigma\tau^2) = -2,$$
$$\mathrm{degree}(\sigma^2\tau) = -1, \quad \mathrm{degree}(\varphi\tau^2) = 0, \quad \mathrm{degree}(\varphi\sigma^2) = 0.$$

The secant bundles $\mathscr{E}(T_{10}, \mathscr{O}_{\mathbf{P}^2}(n))$ are all isomorphic to the restriction, say N, of $\mathscr{E}(T, \mathscr{O}_{\mathbf{P}^2}(0)) = \mathscr{N}$ to C. We compute $c_t(N)$ by chasing through the exact sequences which produce $\mathscr{N}$ in (1.5). From (1.5.14),

$$c_t(\mathscr{N}) = c_t(\mathscr{M})(1 - \tau t)^{-1}, \quad c_t(\mathscr{M}) = c_t(\mathscr{C})(1 + (3\varphi + \lambda + \sigma)t);$$

from (1.5.6) it follows that

$$\begin{aligned} c_t(N) &= (1 - \tau t)^{-1}(1 + (2\varphi + \sigma)t)(1 + 2\varphi t) \\ &= (1 - \tau t)^{-1}(1 + (4\varphi + \sigma)t + 2\varphi\sigma t^2) \\ &= \left(1 + \tau t + (\varphi\sigma - 2\varphi\tau)t^2 + \varphi\sigma\tau t^3\right)\left(1 + (4\varphi + \sigma)t + 2\varphi\sigma t^2\right) \\ &= \left(1 + (4\varphi + \sigma + \tau)t + (3\varphi\sigma + 2\varphi\tau + \sigma\tau)t^2 + (\varphi\sigma\tau)t^3\right). \end{aligned}$$

Therefore

$$\text{degree } c_3\big(\mathscr{E}(T_{10}, \mathcal{O}_{\mathbf{P}^2}(n)\big) = 1.$$

Also

$$\begin{aligned}\big(c_1(\mathscr{E}(T_{10}, \mathcal{O}_{\mathbf{P}^2}(n)))\big)^3 &= (4\varphi + \sigma + \tau)^3 = 12\varphi(\sigma + \tau)^2 + (\sigma + \tau)^3 \\ &= 24\varphi\sigma\tau + 3\sigma^2\tau + 3\sigma\tau^2 + \tau^3 = 16\varphi\sigma\tau.\end{aligned}$$

Therefore

$$\text{degree}\Big(\big(c_1(\mathscr{E}(T_{10}, \mathcal{O}_{\mathbf{P}^2}(n)))\big)^3\Big) = 16.$$

Similarly

$$c_1c_2 = (4\varphi + \sigma + \tau)(3\varphi\sigma + 2\varphi\tau + \sigma\tau) = 6\varphi\sigma\tau,$$

so

$$\text{degree}\Big(\big(c_1(\mathscr{E}(T_{10}, \mathcal{O}_{\mathbf{P}^2}(n)))\big)\big(c_2(\mathscr{E}(T_{10}, \mathcal{O}_{\mathbf{P}^2}(m)))\big)\Big) = 6.$$

The varieties T_7, T_8, T_9, are products; in fact $T_7 = U_2 \times D_2, T_8 = U_2 \times D_3, T_9 = D_1 \times U_1$. We use the computations done for the factors in order to compute the new degrees, which we write in Fig. 5.

On T_7,

$$c_t\big(\mathscr{E}(T_7, \mathcal{O}_{\mathbf{P}^2}(n))\big) = c_t\big(\mathscr{E}(U_2, \mathcal{O}_{\mathbf{P}^2}(n))\big)c_t\big(\mathscr{E}(D_2, \mathcal{O}_{\mathbf{P}^2}(n))\big).$$

Therefore

$$c_t\big(\mathscr{E}(T_7, \mathcal{O}_{\mathbf{P}^2}(0))\big) = (1 + zt)(1 - ht), \quad c_t\big(\mathscr{E}(T_7, \mathcal{O}_{\mathbf{P}^2}(1))\big) = (1 + zt),$$
$$c_t\big(\mathscr{E}(T_7, \mathcal{O}_{\mathbf{P}^2}(2))\big) = (1 + zt)(1 + ht + h^2t^2),$$

where z is the class of a point in $U_2 = \mathbf{P}^1$ and h is the class of a line in $D_2 = \mathbf{P}^2$.

Similarly on T_8,

$$c_t\big(\mathscr{E}(T_8, \mathcal{O}_{\mathbf{P}^2}(n)\big) = c_t\big(\mathscr{E}(U_2, \mathcal{O}_{\mathbf{P}^2}(n))\big)c_t\big(\mathscr{E}(D_3, \mathcal{O}_{\mathbf{P}^2}(n))\big)\big).$$

Proceeding as before we note that on $T_9 = D_1 \times U_1$,

$$c_t\big(\mathscr{E}(T_9, \mathcal{O}_{\mathbf{P}^2}(n))\big) = c_t\mathscr{E}(D_1, \mathcal{O}_{\mathbf{P}^2}(n))(1 + nzt),$$

and we use the desingularization $B \times U_1$ of T_9 to compute the degrees.

Degrees for Hilb$_4$

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
$c_3 E_0$	3	0	0	0	0	0	0	0	0	1
$c_3 E_1$	1	0	0	0	1	1	0	1	1	1
$c_3 E_2$	0	0	0	0	4	2	1	2	2	1
$(c_1 E_1)^3$	48	-6	-8	-1	-5	-9	0	3	9	16
$(c_1 E_2)^3$	15	3	1	0	22	18	3	15	18	16
$(c_2 E_0)(c_1 E_1)$	24	0	-3	-1	-2	-6	0	-1	1	6
$(c_2 E_0)(c_1 E_2)$	21	0	-3	0	-2	-3	-1	1	2	6
$(c_2 E_1)(c_1 E_1)$	14	-2	-2	0	0	0	0	2	4	6
$(c_2 E_1)(c_1 E_2)$	10	-1	-1	0	3	3	0	4	5	6
$(c_2 E_2)(c_1 E_1)$	6	-2	-1	0	6	6	1	5	7	6
$(c_2 E_2)(c_1 E_2)$	3	2	1	0	12	9	2	7	8	6
$(c_1 E_1)(c_1 E_2)^2$	24	-5	-3	0	9	9	1	11	15	16
$(c_1 E_1)^2(c_1 E_2)$	36	-7	-6	0	0	0	0	7	12	16

FIG. 5

(3.3) *List of generators for* $A_3(\mathrm{Hilb}_5(\mathbf{P}^2))$ *and* $A_3(\mathrm{Hilb}_d(\mathbf{P}^2))$. We propose the following set of 12 generators for the Chow group $A_3(\mathrm{Hilb}_5(\mathbf{P}^2))$: $T_1, T_2, \dots, T_6, T_7, \dots, T_{10}, T_{11}, T_{12}$. The generators $T_1, T_2, \dots, T_{10}$, have been described above and it was shown how to compute their intersection degree with the monomials in the Chern classes of the secant bundles. The new generators T_{11}, T_{12} are:

$T_{11} := [P \cup Y]$, where P is a varying point of L and Y is parametrized by $\mathbf{P}^2 = D_0 \subset \mathrm{Hilb}_{d-1}(\mathbf{P}^2)$; i.e. Y is a subscheme of length $d-1$ supported at the origin of the type described in (2.2).

$T_{12} := [W \cup Y]$, where W is a varying closed subscheme of length 2 supported at P_0, and Y is parametrized by $D_1 \subset \mathrm{Hilb}_{d-2}(\mathbf{P}^2)$.

Using the previous computations we have

$$c_t(\mathscr{E}(T_{11}, \mathscr{O}_{\mathbf{P}^2}(n))) = c_t\mathscr{E}(D_0, \mathscr{O}_{\mathbf{P}^2}(n))(1 + nzt) = (1 + ht)(1 + nzt),$$

Degrees for Hilb_5

	T_1	T_2^0	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}
c_3E_0	18	0	0	0	0	0	0	0	0	1	0	1
c_3E_1	9	0	0	0	1	1	0	1	1	1	0	1
c_3E_2	3	0	0	0	4	2	1	2	2	1	0	1
$(c_1E_1)^3$	189	6	-11	-1	-5	-9	0	3	9	16	3	9
$(c_1E_2)^3$	96	27	-2	0	22	18	3	15	18	16	6	9
$(c_2E_0)(c_1E_1)$	96	0	-4	-1	-2	-6	0	-1	1	6	0	4
$(c_2E_0)(c_1E_2)$	84	0	-4	0	-2	-3	-1	1	2	6	0	4
$(c_2E_1)(c_1E_1)$	66	2	-3	0	0	0	0	2	4	6	1	4
$(c_2E_1)(c_1E_2)$	54	3	-2	0	3	3	0	4	5	6	1	4
$(c_2E_2)(c_1E_1)$	40	6	-2	0	6	6	1	5	7	6	2	4
$(c_2E_2)(c_1E_2)$	30	10	0	0	12	9	2	7	8	6	2	4
$(c_1E_1)(c_1E_2)^2$	124	15	-6	0	9	9	1	11	15	16	5	9
$(c_1E_1)^2(c_1E_2)$	156	9	-9	0	0	0	0	7	12	16	4	9

FIG. 6

where h is the class of a line in $D_0 = \mathbf{P}^2$ and z is the class of a point in L. We get the degrees on T_{11} by easy computations.

Similarly

$$c_t(\mathscr{E}(T_{12}, \mathscr{O}_{\mathbf{P}^2}(n))) = c_t\mathscr{E}(D_1, \mathscr{O}_{\mathbf{P}^2}(n))(1 + zt),$$

where z is the class of a point in $U_2 = \mathbf{P}^1$.

We add to the previous list the following threefold T_{13} in $\mathrm{Hilb}_d(\mathbf{P}^2)$, $d \geq 6$,

$T_{13} := [W \cup Y]$, where W is as in T_{12}, and Y is as in T_{11}. Therefore

$$c_t(\mathscr{E}(T_{13}, \mathscr{O}_{\mathbf{P}^2}(n))) = (1 + ht)(1 + zt),$$

where z is as in T_{12} and h is as in T_{11}.

In the table for $\mathrm{Hilb}_5(\mathbf{P}^2)$ we have replaced the second column with the column of the degrees on the cycle $T_2^0 = T_2 + 5T_{11}$. Similarly in the table for

Degrees for Hilb_d

	T_1	T_2^0	T_3^0	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}
c_3E_0	z_1	0	0	0	0	0	0	0	0	1	0	1	0
c_3E_1	z_2	0	0	0	1	1	0	1	1	1	0	1	0
c_3E_2	z_3	0	0	0	4	2	1	2	2	1	0	1	0
$(c_1E_1)^3$	z_4	6	-5	-1	-5	-9	0	3	9	16	3	9	3
$(c_1E_2)^3$	z_5	27	4	0	22	18	3	15	18	16	6	9	3
$(c_2E_0)(c_1E_1)$	z_6	0	-2	-1	-2	-6	0	-1	1	6	0	4	1
$(c_2E_0)(c_1E_2)$	z_7	0	-2	0	-2	-3	-1	1	2	6	0	4	1
$(c_2E_1)(c_1E_1)$	z_8	2	-1	0	0	0	0	2	4	6	1	4	1
$(c_2E_1)(c_1E_2)$	z_9	3	0	0	3	3	0	4	5	6	1	4	1
$(c_2E_2)(c_1E_1)$	z_{10}	6	0	0	6	6	1	5	7	6	2	4	1
$(c_2E_2)(c_1E_2)$	z_{11}	10	2	0	12	9	2	7	8	6	2	4	1
$(c_1E_1)(c_1E_2)^2$	z_{12}	15	0	0	9	9	1	11	15	16	5	9	3
$(c_1E_1)^2(c_1E_2)$	z_{13}	9	-3	0	0	0	0	7	12	16	4	9	3

FIG. 7

$\mathrm{Hilb}_d(\mathbf{P}^2)$ we have replaced the second column with the column of the degrees on the cycle $T_2^0 = T_2 + dT_{11}$, and the third column with the column of the degrees on $T_3^0 = T_3 + (d-3)T_{13}$.

The z_i in the first column have been computed above in (3.1.1) and we recall that they are polynomials of degree 4 in d.

Now according to the computer the determinant of the intersection matrix (see Fig. 7) of $\mathrm{Hilb}_d(\mathbf{P}^2)$, $d \geq 6$, is -1.

By Poincaré duality it follows that $T_1, \dots, T_{13}$ are a basis of $A_3(\mathrm{Hilb}_d)$ and that the 13 Chern monomials of weight 3 which appear in the table are in fact a basis of $A^3(\mathrm{Hilb}_d)$, $d \geq 6$.

The cases $d = 4$ and $d = 5$ reduce to the computation for $d \geq 6$. Indeed we can see the tables of intersection as giving vectors $T_1, \dots, T_{13}$ in the lattice $\mathbf{Z}^{13}$. From T_4 to T_{13} the vectors do not depend on d, and the vectors $T_2^0 = T_2 + dT_{11}$ and $T_3^0 = T_3 + (d-3)T_{13}$ are also constant (this is also true for $d = 4, 5$

where, if undefined, we define T_{11}, T_{12}, T_{13} to be the vectors in the table of Hilb_6). Therefore for all $d \geq 4$ we have a set of 13 vectors $T_1, T_2^0, T_3^0, T_4, \ldots, T_{13}$ in the lattice $\mathbf{Z}^{13}$, and only T_1 varies, its coordinates being polynomials of degree 4 in d. Since the determinant is the constant -1, then also for $d = 4$, $T_1, T_2^0, T_3^0, T_4, \ldots, T_{13}$ form a basis of $\mathbf{Z}^{13}$, so the 13 Chern monomials form a basis of the dual lattice. Now for $d = 4$ (and for all d) the lattice M, say, spanned by $T_1, T_2, T_3, T_4, \ldots, T_{10}$, is a direct summand of $\mathbf{Z}^{13}$; therefore the 13 Chern monomials also generate the dual lattice of M. By duality it follows that $T_1, \ldots, T_{10}$, are a basis of $A_3(\mathrm{Hilb}_4)$ and that the 13 Chern monomials of weight 3 generate $A^3(\mathrm{Hilb}_4)$. The proof for Hilb_5 is based on the same argument.

(3.4) *The degrees of monomials of weight* 6 *on* $\mathrm{Hilb}_3(\mathbf{P}^2)$. I have been informed by Prof. Kleiman that several people have computed the Chow ring of $\mathrm{Hilb}_3(\mathbf{P}^2)$ (see [6], [4]). Our computations and Poincaré duality would allow us to do this once more; we only indicate how to compute the degrees of the monomials of weight 6 on $\mathrm{Hilb}_3(\mathbf{P}^2)$. In (3.1.6) we have noted that T_1, T_2, T_3 generate a lattice orthogonal to the lattice generated by T_4, T_5, T_6. The intersection matrix on $A_3(\mathrm{Hilb}_3(\mathbf{P}^2))$ is

	T_1	T_2	T_3	T_4	T_5	T_6
T_1	-1	1	1	0	0	0
T_2	1	1	0	0	0	0
T_3	1	0	0	0	0	0
T_4	0	0	0	0	1	1
T_5	0	0	0	1	-2	0
T_6	0	0	0	1	0	3

The table for T_4, T_5, T_6, has been obtained by using the corresponding table for the K_i. The degrees of intersection of T_1, T_2, T_3 can be computed by remarking:

(1) the determinant of the first 3×3 block is unimodular, by Poincaré duality;

(2) the intersection of the T_i is clearly empty where we have put a 0;

(3) $T_1 \cap T_2$ is a point and one can check by local considerations that the intersection is transversal;

(4) in $\mathrm{Hilb}_3(\mathbf{P}^2)$ the divisor Λ of the subschemes of $\mathbf{P}^2$ which are subschemes of some line has class $c_1 E_1$ (use Porteous formulas).

$T_1 \subset \Lambda$ and two copies of T_1 intersect transversally in Λ along a line $l \approx \mathbf{P}^1$ which parametrizes triples $\{P, Q_1, Q_2\}$, where P varies in the line $[Q_1, Q_2]$;

hence by the excess intersection formula [7],

$$\text{degree}(T_1 \cdot T_1)_{\text{Hilb}_3} = \text{degree}_l(c_1E_1) = -1.$$

Using the table of the degrees over the T_i's we find the components of the Chern monomials in the basis of the T_i's. We use row notation and the ordered basis $T_1, \dots, T_6$.

$$c_3E_1 = (0,0,0,1,0,0), \quad c_3E_2 = (0,0,0,8,2,-2),$$
$$(c_1E_1)^3 = (-5,2,-1,-3,1,-2),$$
$$(c_1E_2)^3 = (4,5,-1,30,4,-4), \quad (c_2E_0)(c_1E_1) = (-2,2,-1,0,1,-2),$$
$$(c_2E_0)(c_1E_2) = (-2,2,-1,0,1,-1), \quad (c_2E_1)(c_1E_1) = (-1,0,0,0,0,0),$$
$$(c_2E_1)(c_1E_2) = (0,0,0,3,0,0), \quad (c_2E_2)(c_1E_1) = (0,0,0,6,0,0),$$
$$(c_2E_2)(c_1E_2) = (2,2,0,18,3,-3).$$

Note that

$$(c_2E_1)(c_1E_2) = 3c_3E_1, \quad (c_2E_2)(c_1E_1) = 6c_3E_1.$$

We also know that

$$c_3E_0 = 0, \quad (c_1E_1)(c_1E_2) = 3c_2E_1.$$

The intersection table for the T_i's allows us to compute the degree of all monomials of weight 6, but for possibly the degree of $(c_2E_0)^a(c_2E_2)^b$, $a + b = 3$. In Part 2 we saw that the class of the image of the variety of third order data S in Hilb_3 is $3c_2E_0$, and we computed the degrees on S of $(c_2E_0)^a(c_2E_2)^b$, $a + b = 2$ in (1.8.3) above; it follows that in Hilb_3,

$$\text{degree}(c_2E_0)^3 = 5, \quad \text{degree}(c_2E_0)^2(c_2E_2) = -3, \quad \text{degree}(c_2E_0)(c_2E_2)^2 = 5.$$

We can reduce the computation of the degree of $(c_2E_2)^3$ to the other numbers. The bundle E_2 in $\text{Hilb}_3(\mathbf{P}^2)$ is a quotient of the free bundle F of rank 6, which has basis the monomials of degree 2. The 4-th Chern class of the kernel of $F \to E_2$ is 0; hence

$$2(c_1E_2)(c_3E_2) + (c_2E_2)^2 - 3(c_1E_2)^2(c_2E_2) + (c_1E_2)^4 = 0.$$

Therefore

$$\begin{aligned}\text{degree}(c_2E_2)^3 &= -\text{degree}(c_2E_2)\big(2(c_1E_2)(c_3E_2) \\ &\qquad -3(c_1E_2)^2(c_2E_2) + (c_1E_2)^4\big) \\ &= 9.\end{aligned}$$

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