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NON-ISOTROPIC HAUSDORFF MEASURE AND EXCEPTIONAL SETS FOR HOLOMORPHIC SOBOLEV FUNCTIONS

BY

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Let B^n denote the unit ball in C^n with boundary S, the unit sphere. If f is holomorphic in B^n with homogeneous polynomial expansion

$$f(z) = \sum_{k=0}^{\infty} f_k(z)$$

then f has radial derivative

$$Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z) = \sum_{k=0}^{\infty} k f_k(z)$$

as defined in [7]. For $\beta > 0$ one is therefore led to the definition

$$R^\beta f(z) = \sum_{k=0}^{\infty} (1+k)^\beta f_k(z)$$

of the so called "fractional derivatives" of f; see [4]. As in [4], for β, $p > 0$, define the "holomorphic Sobolev spaces"

$$H_\beta^p(B^n) = \{ f\colon R^\beta f \in H^p(B^n) \}$$

where $H^p(B^n)$ is the usual Hardy space [7].

For $\zeta \in S$ and $\delta > 0$ let

$$B(\zeta, \delta) = \{ \eta \in S \text{ and } |1 - \langle \eta, \zeta \rangle| < \delta \}$$

be the Koranyi ball and

$$D_\alpha(\zeta) = \left\{ z \in B^n \text{ and } |1 - \langle z, \zeta \rangle| < \frac{\alpha}{2}(1 - |z|^2) \right\}$$

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be the admissible approach regions for $\zeta \in S$ and $\alpha > 1$. For a complex valued function f defined on B^n we have the maximal functions

$$M_\alpha f(\zeta) = \sup_{z \in D_\alpha(\zeta)} |f(z)|$$

where $\zeta \in S$. If f: $B^n \to C$, $\zeta \in S$ and

$$\lim_{z \to \zeta,\, z \in D_\alpha(\zeta)} f(z)$$

exists for all $\alpha > 1$, then we say that f has an *admissable limit* at ζ. Let $E(f)$ denote the exceptional set

$$E(f) = \{\zeta \in S \text{ and } f \text{ does not have an admissable limit at } \zeta\}.$$

In [3], Ahern proves the following result and its corollary; see also [1].

THEOREM A. *Let $0 < p \leq 1$ and $d = n - \beta p > 0$. Suppose $f \in H^p_\beta(B^n)$ and ν is a positive measure on S satisfying*

$$(*) \qquad \nu(B(\zeta, \delta)) \leq C\delta^d \quad \textit{for } \zeta \in S \textit{ and } \delta > 0,$$

for an absolute constant C. Then for each $\alpha > 1$ there is a constant $C = C(\alpha)$ such that

$$\int (M_\alpha f(\zeta))^p \, d\nu(\zeta) \leq C \|R^\beta f\|_p^p.$$

(Here, $\|g\|_p$ denotes the $H^p(B^n)$ norm of g).

COROLLARY A. *If ν satisfies condition $(*)$ of Theorem A and $f \in H^p_\beta(B^n)$ then*

$$\nu(E(f)) = 0.$$

On the basis of Corollary A, Ahern suggests in [3] that the exceptional sets for functions in $H^p_\beta(B^n)$ are those of "non-isotropic" d-dimensional Hausdorff capacity 0. In this note, we verify his conjecture for the case of compact subsets of S.

For $d > 0$ and $E \subseteq S$ compact, let H_d be the capacity on S defined by

$$H_d(E) = \inf\left\{ \sum_{A \in \mathcal{O}} \delta_A^d \right\},$$

where the infimum is taken over all countable covers $\mathcal{O}$ of E by balls

$$A = B(\zeta_A, \delta_A).$$

If $n > 1$ notice that the H_d capacity of a set depends on "directional considerations" because of the nature of the Koranyi ball $B(\zeta, \delta)$; see [7]. For this reason we refer to H_d as *non-isotropic* d-dimensional Hausdorff capacity.

To verify Ahern's conjecture we need the following "Frostman theorem" for H_d.

THEOREM 1. *Let E be a compact subset of S. Then $H_d(E) > 0$ if and only if E contains the support of a positive measure $\nu \neq 0$ satisfying condition* $(*)$.

Combining Theorem 1 with Corollary A gives the next fact.

COROLLARY 1. *Let $d = n - \beta p > 0$, where $0 < p \leq 1$ and $\beta > 0$. Suppose E is a compact subset of S and $E = E(f)$ for $f \in H_\beta^p(B^n)$. Then $H_d(E) = 0$.*

Our second main result completes the characterization of the compact exceptional sets for $H_\beta^p(B^n)$.

THEOREM 2. *Let $d = n - \beta p > 0$, where $0 < p \leq 1$ and $\beta > 0$, and suppose E is a compact subset of S for which $H_d(E) = 0$. Then there exists a function $f \in H_\beta^p(B^n)$ such that $E = E(f)$.*

It follows that the compact subsets of S which arise as exceptional sets for $H_\beta^p(B^n)$ functions are precisely the ones whose non-isotropic d-dimensional Hausdorff capacity is 0.

Our proof of Theorem 1 requires some machinery and ideas which also allow a proof of strong type capacitary results for holomorphic Sobolev functions analogous to real variable results found in [1] and [2]. These are pursued after the proof of Theorem 1 and stated as Theorem 3.

In the sequel we adopt the following conventions and terminology. The letter σ will denote surface area on the sphere, while the letter C will stand for various absolute constants whose values differ in each occurrence while remaining independent of stated variables.

Finally, the symbol $\doteq$ is used to indicate that two quantities are "comparable". That is, $A \doteq B$ if and only if there is a positive constant C such that $C^{-1}A \leq B \leq CA$.

Proof of Theorem 1. We will need the following notation. If $\zeta, \eta \in S$, let

$$d(\zeta, \eta) = |1 - \langle \zeta, \eta \rangle|^{1/2}.$$

Then d is a metric on S; see [7]. Let

$$Q(\zeta, \delta) = \{\eta \in S \text{ and } d(\eta, \zeta) < \delta\}.$$

Then $Q(\zeta, \delta) = B(\zeta, \delta^2)$ and if $0 \leq \delta \leq 2$

$$\sigma(Q(\zeta, \delta)) \doteq \delta^{2n};$$

for this last fact we again refer to [7]. For $0 < m \leq n$ and $K \subseteq S$ compact, it follows that

$$H_m(K) = \inf\left\{\sum \delta_k^{2m}\colon K \subseteq \bigcup_k Q(\zeta_k, \delta_k)\right\}.$$

Motivated by Frostman's proof of Theorem 1 for the case $n = 1$ and by the generalizations of his proof to $\mathscr{R}^n$, $n > 1$, one would like to find successive "dyadic decompositions" of the sphere into disjoint unions of sets that are essentially Koranyi balls of radius 2^{-k}, $k = 1, 2, \ldots$. It seems, however, that the non-isotropic nature of the metric d on S for $n > 1$ makes the situation intrinsically more complicated than the situation in $\mathscr{R}^n$. Larman, [6], has given a, somewhat complicated, decomposition of a finite dimensional compact metric space into a "sequence of nets" which has the usual properties of the familiar dyadic decomposition of $\mathscr{R}^n$. We prefer, however, to proceed in a sightly different way, which will have the advantage of being simpler and keeping the paper self contained.

The first step in the proof requires the construction of a "lattice" contained in S. Let ζ^0 be an arbitrary point in S. Set $L_0 = \{\zeta^0\}$. Clearly, $S = Q(\zeta^0, 3)$. A standard argument proves the following lemma.

LEMMA 1. *There exist subsets $L_1, L_2, \ldots$ of S satisfying the following properties*:

(1.1) $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$;

(1.2) *If $L_k = \{\zeta_s^k\}_{s=1}^{m_k}$ is a listing of the distinct elements of L_k then*

(i) $d(\zeta_s^k, \zeta_t^k) > 1/2^k$ *if $s \neq t$*,

(ii) $\bigcup_{s=1}^{m_k} Q(\zeta_s^k, 3/2^k) = S$.

With $\{L_k\}_{k=0}^{\infty}$ constructed as in Lemma 1, define a relationship between elements of L_k and L_{k-1} by saying that $\zeta_s^k < \zeta_t^{k-1}$ if t is the smallest index such that

$$\zeta_s^k \in Q\left(\zeta_t^{k-1}, \frac{3}{2^{k-1}}\right).$$

Fix a positive integer N. For $l = 1, 2, \ldots, N-1$ say that

$$\zeta_s^N < \zeta_t^{N-l} \tag{1}$$

if there exists a sequence of "inequalities"

$$\zeta_s^N < \zeta_{s_1}^{N-1} < \zeta_{s_2}^{N-2} < \cdots < \zeta_{s_{l-1}}^{N-(l-1)} < \zeta_t^{N-l}. \tag{2}$$

Notice that for a given $\zeta_s^N \in L_N$ there is a unique $\zeta_t^{N-l} \in L_{N-l}$ such that (1) holds.

Next, for $\zeta_s^{N-l} \in L_{N-l}$, where $l = 1, 2, 3, \ldots, N$, set

$$S_s^{N,l} = \left\{\zeta_t^N \in L_N \text{ and } \zeta_t^N < \zeta_s^{N-l}\right\}.$$

It is possible that $S_s^{N,l} = \emptyset$. Now let $S_s^{N,0} = \{\zeta_s^N\}$, for $s = 1, 2, \ldots, m_N$. Then we have the following lemma.

LEMMA 2. *Let* $0 \leq l, j < N$. *Then*:
(2.1) $S_s^{N,l} \cap S_t^{N,l} = \emptyset$, *if* $s \neq t$.
(2.2) $\bigcup_{s=1}^{m_{N-l}} S_s^{N,l} = L_N$.
(2.3) *If* $S_s^{N,l} \cap S_t^{N,j} \neq \emptyset$ *and* $j < l$, *then* $S_t^{N,j} \subseteq S_s^{N,l}$.
(2.4) $S_s^{N,l} \subseteq Q(\zeta_s^{N-l}, 6/2^{N-l})$.

Proof. We only discuss (2.3) and (2.4).

For (2.3), if $\zeta_{s_0}^N \in S_s^{N,l} \cap S_t^{N,j}$, then $\zeta_{s_0}^N < \zeta_t^{N-j}$ and $\zeta_{s_0}^N < \zeta_s^{N-l}$ and therefore we have "inequalities"

$$\zeta_t^{N-j} < \zeta_{t_1}^{N-(j+1)} < \cdots < \zeta_s^{N-l},$$

which shows that each ζ_i^N in $S_t^{N,j}$ is also in $S_s^{N,l}$.

For (2.4), if $\zeta_t^N < \zeta_s^{N-l}$, then by (2) and the triangle inequality,

$$\begin{aligned} d\left(\zeta_t^N, \zeta_s^{N-l}\right) &\leq d\left(\zeta_t^N, \zeta_{s_1}^{N-1}\right) + d\left(\zeta_{s_1}^{N-1}, \zeta_{s_2}^{N-2}\right) + \cdots + d\left(\zeta_{S_{l-1}}^{N-(l-1)}, \zeta_s^{N-l}\right) \\ &< \frac{3}{2^{N-1}} + \frac{3}{2^{N-2}} + \cdots + \frac{3}{2^{N-l}} \\ &\leq \frac{6}{2^{N-l}}, \end{aligned}$$

as claimed.

LEMMA 3. *Let $0 \le l \le N-1$ and suppose $\zeta_{s_0}^{N-l} \in L_{N-l}$. Then there exists an absolute constant N_0, independent of l or N such that if*

$$G = \left\{ \zeta_s^{N-l} \colon S_s^{N,l} \cap Q\left(\zeta_{s_0}^{N-l}, \frac{3}{2^{N-l}}\right) \neq \emptyset \right\},$$

then the cardinality of G is less than N_0.

Proof. By Lemma 2, if $\zeta_s^{N-l} \in G$ then $d(\zeta_s^{N-l}, \zeta_{s_0}^{N-l}) < 10/2^{N-l}$. The collection

$$\left\{ Q\left(\zeta_s^{N-l}, \frac{1}{4 \cdot 2^{N-l}}\right) \right\}$$

where $\zeta_s^{N-l} \in G$ is therefore a pairwise disjoint collection of balls contained in $Q(\zeta_{s_0}^{N-l}, 20/2^{N-l})$. Taking surface area measure σ of the union gives the inequality

$$k\left[\frac{1}{4 \cdot 2^{N-l}}\right]^{2n} \le C\left[\frac{20}{2^{N-l}}\right]^{2n}$$

where k is the cardinality of G and C is an absolute constant. This gives the result.

The proof of Lemma 3 actually yields the following corollary.

COROLLARY 2. *Let $\zeta \in S$ and*

$$H = \left\{ \zeta_s^{N-l} \colon Q\left(\zeta_s^N, \frac{3}{2^{N-l}}\right) \cap Q\left(\zeta, \frac{1}{2^{N-l}}\right) \neq \emptyset \right\}.$$

Then the cardinality of H is less than N_1 where N_1 is an absolute constant independent of ζ, N, or l.

We are now ready to construct the measure μ that Theorem 1 asserts exists. Suppose that $E \subseteq S$ is compact and $H_d(E) > 0$. Let N be a fixed positive integer. Set

$$I_E = \left\{ s \colon Q\left(\zeta_s^N, \frac{3}{2^N}\right) \cap E \neq \emptyset \right\}.$$

Note that

$$E \subseteq \bigcup_{s \in I_E} Q\left(\zeta_s^N, \frac{3}{2^N}\right).$$

Let

$$\mu_0 = \sum_{s \in I_E} \left(\frac{1}{2^N}\right)^{2d} \delta_{\zeta_s^N},$$

where δ_ζ is point mass at ζ. Define μ_1 so it satisfies

$$\mu_1(S_s^{N,1}) = \begin{cases} \left(\frac{1}{2^{N-1}}\right)^{2d} & \text{if } \mu_0(S_s^{N,1}) > \left(\frac{1}{2^{N-1}}\right)^{2d} \\ \mu_0(S_s^{N,1}) & \text{otherwise.} \end{cases}$$

by, in the first case, redefining μ_0 on $S_s^{N,1}$ by multiplying the restriction of μ_0 to $S_s^{N,1}$ by the appropriate number λ, $0 < \lambda < 1$. Define μ_2 in a similar way so it satisfies

$$\mu_2(S_s^{N,2}) = \begin{cases} \left(\frac{1}{2^{N-2}}\right)^{2d} & \text{if } \mu_1(S_s^{N,2}) > \left(\frac{1}{2^{N-2}}\right)^{2d} \\ \mu_1(S_s^{N,2}) & \text{otherwise.} \end{cases}$$

Continue this process and construct μ_N.

By virtue of Lemma 2, for each $s \in I_E$ there is a largest number l such that $\zeta_s^N \in S_t^{N,l}$ and

$$\mu_N(S_t^{N,l}) = \left(\frac{1}{2^{N-l}}\right)^{2d}.$$

Call such a set $S_t^{N,l}$ "maximal". Let the maximal sets be denoted by $S_{t_i}^{N,l_i}$, $i = 1, \dots, k$. By Lemma 2, these sets are pairwise disjoint. It is also true that

$$\bigcup_{i=1}^{k} S_{t_i}^{N,l_i} \supseteq \{\zeta_s^N : s \in I_E\}.$$

By Lemma 2, (2.4),

$$\bigcup_{\zeta_s^N \in S_{t_i}^{N,l_i}} Q\left(\zeta_s^N, \frac{3}{2^N}\right) \subseteq Q\left(\zeta_{t_i}^{N-l_i}, \frac{24}{2^{N-l_i}}\right)$$

and therefore

$$E \subseteq \bigcup_i Q\left(\zeta_{t_i}^{N-l_i}, \frac{24}{2^{N-l_i}}\right),$$

implying that

$$H_d(E) \le \sum_i \left(\frac{24}{2^{N-l_i}}\right)^{2d} = (24)^{2d}\mu_N(S),$$

since the maximal sets are disjoint.

If we use Lemma 3 and its corollary and let $\zeta \in S$, then

$$Q\left(\zeta, \frac{1}{2^{N-l}}\right) \cap Q\left(\zeta_s^{N,l}, \frac{3}{2^{N-l}}\right) \ne \emptyset$$

for at most N_1 elements ζ_s^{N-l} in L_{N-l}. For each such ζ_s^{N-l}, let C_s be the set of all $S_t^{N,l}$ which intersect $Q(\zeta_s^{N-l}, 3/2^{N-l})$. Each C_s has at most N_0 elements, by Lemma 3. It follows now that

$$\begin{aligned}\mu_N\left(Q\left(\zeta_s^{N-l}, \frac{3}{2^{N-l}}\right)\right) &\le \sum_{C_s} \mu_N(S_t^{N,l}) \\ &\le N_0\left(\frac{1}{2^{N-l}}\right)^{2d}.\end{aligned}$$

Therefore

$$\mu_N\left(Q\left(\zeta, \frac{1}{2^{N-l}}\right)\right) \le N_1 N_0 \left(\frac{1}{2^{N-l}}\right)^{2d}$$

for all $\zeta \in S$, and $l = 0, 1, \dots, N$.

We now have a sequence of measures $\{\mu_N\}$ satisfying the inequalities

$$(24)^{-2d} H_d(E) \le \mu_N(S) \le 1.$$

We may find a weak $*$ convergent subsequence $\{\mu_{N_k}\}$ and a measure μ such that

$$(24)^{-2d} H_d(E) \le \mu(S) \le 1, \tag{3}$$

$$\mu \text{ is supported on } E, \tag{4}$$

$$\mu\left(Q\left(\zeta, \frac{1}{2^N}\right)\right) \le N_1 N_0 \left(\frac{1}{2^N}\right)^{2d}, \tag{5}$$

for all $\zeta \in S$ and $N = 0, 1, 2, \dots$.

This completes the proof of Theorem 1.

We turn now to the question of strong type capacitary inequalities for functions in $H_\beta^p(B^n)$. In the following discussion we will assume that $0 < p \le 1$ and $d = n - \beta p > 0$.

Let $f \in H^p_\beta(B^n)$. In [3], Ahern obtains the estimate

$$[MF(\zeta)]^p \leq \sum_{k=1}^{\infty} \lambda_k(\delta_k)^{-2d} u_k(\zeta) \tag{6}$$

for $\zeta \in S$, where $\Sigma\lambda_k \leq C\|R^\beta f\|_p^p$ and u_k is the characteristic function of a ball $Q(\zeta_k, \delta_k)$. For the purposes of this paper, however, it will be convenient to realize that we may assume that $0 \leq u_k \leq 1$ is *continuous* and supported in a ball $Q(\zeta_k, \delta_k)$. If X is a subset of S, extend the function H_d to X by letting

$$H_d(X) = \sup H_d(E)$$

where the supremum is taken over all compact sets $E \subseteq X$. We have the following lemma.

LEMMA 4. *Let $K_m \subseteq K_{m+1}$, $m = 1, 2, \ldots$ be compact subsets of S. Then*

$$H_d\left(\bigcup_{m=1}^{\infty} K_m\right) \doteq \lim_{m\to\infty} H_d(K_m).$$

Proof. It is obvious that $H_d(\bigcup_{n=1}^{\infty} K_m) \geq \lim_{m\to\infty} H_d(K_m)$. Now let K be a compact subset of $\bigcup_{m=1}^{\infty} K_m$. From the proof of Theorem 1 it follows that

$$H_d(K) \doteq \sup \mu(K)$$

where the supremum is taken over all measures μ supported on K satisfying condition $(*)$ with $C = 1$. Choose such a μ so

$$H_d(K) \leq C\mu(K).$$

Since μ is a measure and $K \subseteq \bigcup_{m=1}^{\infty} K_m$, $\mu(K) = \lim_{m\to\infty} \mu(K_m)$. For each m, find a cover $O_m = \{B(\zeta_i^m, \delta_i^m)\}$ such that

$$\sum_i (\delta_i^m)^d \leq H_d(K_m) + 2^{-m}.$$

Then

$$\mu(K_m) \leq \sum_i \mu(B(\zeta_i^m, \delta_i^m)) \leq \sum_i (\delta_i^m)^d$$

and therefore

$$\mu(K) \leq \lim_{m\to\infty} H_d(K_m),$$

which proves the lemma.

For $f \in H^p_\beta$ let $J((MF)^p)$ be the Choquet integral of $(Mf)^p$,

$$J\big((MF)^p\big) = \int_0^\infty H_d\{(MF)^p \geq t\}\,dt.$$

For $\varepsilon > 0$,

$$\begin{aligned} J\big((MF)^p\big) &= \sum_{m=0}^{\infty} \int_m^{m+1} H_d\{(MF)^p \geq t\}\,dt \\ &= \sum_{m=0}^{\infty} \int_m^{m+1} \varepsilon H_d\{(MF)^p \geq \varepsilon t\}\,dt \end{aligned}$$

and therefore

$$\left| J\big((MF)^p\big) - \sum_{m=0}^{\infty} \varepsilon H_d\{(MF)^p \geq \varepsilon m\} \right| \leq \varepsilon H_d(S).$$

To estimate $J((MF)^p)$ it is therefore enough to estimate the sum

$$\sum_{m=0}^{\infty} \varepsilon H_d\{(MF)^p \geq \varepsilon m\}$$

for ε small. Use Ahern's inequality (6) to see that

$$\begin{aligned} \text{(7)} \qquad \sum_{m=0}^{M} \varepsilon H_d\{(MF)^p \geq \varepsilon m\} &\leq \sum_{m=0}^{M} \varepsilon H_d\Big\{\sum_k \lambda_k \delta_k^{-2d} u_k \geq \varepsilon m\Big\} \\ &\leq C \sum_{m=0}^{M} \varepsilon H_d\Big\{\sum_{k=1}^{k_M} \lambda_k \delta_k^{-2d} u_k \geq \varepsilon m\Big\} \end{aligned}$$

for an integer k_M depending on M, where we have used Lemma 4 and the continuity of each u_k.

We need now to define approximating capacities H_d^N, $N = 1, 2, 3, \ldots$. Our notation will refer back to the proof of Theorem 1. For a positive integer N and $s = 1, 2, \ldots, M_N$ let

$$X(\zeta_s^N) = Q\Big(\zeta_s^N, \frac{3}{2^N}\Big) \Big\backslash \bigcup_{t<s} Q\Big(\zeta_t^N, \frac{3}{2^N}\Big).$$

The collection $\{X(\zeta_s^N)\}_{s=1}^{m_N}$ is therefore pairwise disjoint and the union of all such sets (for fixed N) is S. If $0 \leq l \leq N$, l an integer, and $s = 1, 2, \ldots, m_{N-l}$, let

$$Y_{N,l,s} = \bigcup X(\zeta_t^N)$$

where the union is over all indices for which $\zeta_t^N \in S_s^{N,l}$. If $E \subseteq S$ define

$$H_d^N(E) = \inf\left\{ \sum \left(\frac{1}{2^{N-l}} \right)^{2d} : E \subseteq \bigcup Y_{N,l,s} \right\}.$$

It is easy to see that if $E = \bigcup_{k=1}^{\infty} E_k$ and $E_k \subseteq E_{k+1}$, then

$$H_d^N(E) = H_d^N(E_j)$$

for all $j \geq l$, where l is sufficiently large.

It is also immediate from the construction that if any two sets $Y_{N,l,s}$ (with fixed N) intersect, then one is contained in the other. These last two properties and R. Fefferman's argument in [5] prove the following result.

LEMMA 5. *Let F and G be non-negative functions. Then*

$$J_N(F+G) \leq J_N(F) + J_N(G)$$

where

$$J_N(h) = \int_0^{\infty} H_d^N\{h \geq t\}\, dt$$

for $h \geq 0$.

We now estimate $J((MF)^p)$. With M fixed as in (7), choose N_2 so large that for $N > N_2$,

$$\begin{aligned}
\sum_{m=0}^{M} \varepsilon H_d\left\{ \sum_{k=1}^{k_m} \lambda_k \delta_k^{-2d} u_k \geq \varepsilon m \right\} &\leq C \sum_{m=0}^{M} \varepsilon H_d^N\left\{ \sum_{k=1}^{k_M} \lambda_k \delta_k^{-2d} u_k \geq \varepsilon m \right\} \\
&\leq C J_N\left(\sum_{k=1}^{k_M} \lambda_k \delta_k^{-2d} u_k \right) \\
&\leq C \sum_{k=1}^{k_M} \lambda_k J_N\left(\delta_k^{-2d} u_k \right) \\
&\leq C \sum_{k=1}^{k_M} \lambda_k,
\end{aligned}$$

provided N is sufficiently large so $2^{-N} \ll \delta_k$, $k = 1, \dots, k_M$. Since this holds all M we have proved the following strong type capacitary inequality.

THEOREM 3. *There is an absolute constant C such that*

$$J\big((MF)^p\big) \le C\|R^\beta\|_p^p,$$

for all functions f holomorphic on B^n.

The remainder of the paper concerns the proof of Theorem 2.

The proof of Theorem 2 requires some preliminary lemmas. The first two follow from standard estimates of the type found in [7], Chapter 5.

LEMMA 6. *Let $m > n$ and $0 < r < 1$. Then for t large and r close to 1 there is an absolute constant C such that with $\zeta \in S$,*

$$\int_X \frac{(1-r)^{m-n}}{|1-\langle z,\zeta\rangle|^m}\,d\sigma(z) \le Ct^{n-m},$$

where $X = \{z \in S \text{ and } |1-\langle z,\zeta\rangle| > t(1-r)\}$.

LEMMA 7. *Let $\zeta, \eta \in S$ and suppose $B(\zeta,\delta) \cap B(\eta,\rho) = \phi$. Assume that $0 < 1 - r_1 < \delta$, $0 < 1 - r_2 < \rho$ and let $z \in B(\eta, \rho/2)$. Then for δ and ρ sufficiently small,*

$$|1-\langle z, r_1\zeta\rangle| \doteq |1-\langle r_2\eta, r_1\zeta\rangle| \doteq |1-\langle \eta,\zeta\rangle|.$$

LEMMA 8. *Let $\{B(\zeta_i,\delta_i)\}$ be a collection of pairwise disjoint balls contained in δ. Set*

$$\tilde{\zeta}_i = \left(1 - \frac{\delta_i}{t}\right)\zeta_i \quad \textit{where } t > 1,$$

and let

$$r_i = 1 - \frac{\delta_i}{t}.$$

Then there is an absolute constant $C = C(m)$ such that if $m > n$ then

$$\sum_{j \ne i} \frac{(1-r_i)^{m-n}(1-r_j)^n}{|1-\langle\tilde{\zeta}_i,\tilde{\zeta}_j\rangle|^m} \le Ct^{-m}.$$

Proof. By Lemma 6, with $X = \{z \in S \text{ and } |1-\langle z,\zeta_i\rangle| > \delta_i\}$,

$$\int_X \frac{(1-r_i)^{m-n}}{|1-\langle z,\zeta_i\rangle|^m}\,d\sigma(z) \le C\left(\frac{\delta_i}{1-r_i}\right)^{m-n} = Ct^{n-m}.$$

Let $B_j = B(\zeta_j, \delta_j/2)$. Since the collection $\{B_j\}$ is pairwise disjoint,

$$\sum_{j\neq i}\int_{B_j}\frac{(1-r_i)^{m-n}}{|1-\langle z,\zeta_i\rangle|^m}\,d\sigma(z)\leq Ct^{n-m}.$$

Lemma 7 now allows the estimate

$$\sum_{j\neq i}\frac{(1-r_i)^{m-n}}{|1-\langle\tilde{\zeta}_j,\tilde{\zeta}_i\rangle|^m}\sigma(B_j)\leq Ct^{n-m}$$

and therefore

$$\sum_{j\neq i}\frac{(1-r_i)^{m-n}2^{-n}\delta_j^n}{|1-\langle\tilde{\zeta}_j,\tilde{\zeta}_i\rangle|^m}\leq Ct^{n-m}.$$

Since $\delta_j = (1-r_j)t$, the result follows.

LEMMA 9. *Let $\mathscr{C} = \{B(\zeta_i, \delta_i)\}$ be a finite collection of pairwise disjoint balls in S. Then there exists a function $F = F(z, \mathscr{C}, t)$ defined on B^n associated with $\mathscr{C}$ and the number t with the property that, for t sufficiently large,*

$$|F(\tilde{\zeta}_i)|\geq\frac{1}{3}$$

where

$$\tilde{\zeta}_i=\left(1-\frac{\delta_i}{t}\right)\zeta_i$$

as in Lemma 8.

Proof. Reorder the sequence $\{\zeta_i\}$ so $\delta_1\geq\delta_2\geq\cdots$. By Lemma 8 it follows that

$$\sum_{j>i}\frac{(1-r_j)^m}{|1-\langle\tilde{\zeta}_i,\tilde{\zeta}_j\rangle|^m}\leq C(m)t^{-m} \tag{7}$$

where $1-r_j=\delta_j/t$ as in Lemma 8, and $m>n$. Define

$$g_j(z)=\frac{2^m(1-r_j)^m}{\left(1-\langle z,\tilde{\zeta}_j\rangle\right)^m}$$

and notice that

$$|g_j(\tilde{\zeta}_j)| \geq 1.$$

Let $\omega_1 = 1$ and define $\omega_2, \omega_3, \ldots,$ inductively by the rule

$$\omega_k = \begin{cases} 0 & \text{if } \left| \sum_{j<k} \omega_j g_j(\tilde{\zeta}_k) \right| \geq \frac{1}{2} \\ 1 & \text{if } \left| \sum_{j<k} \omega_j g_j(\tilde{\zeta}_k) \right| < \frac{1}{2}. \end{cases}$$

Let $F(z) = \sum \omega_j g_j(z)$. Using (7) and the definition of ω_k, it follows that, for $k \geq 2$, in the case where $\omega_k = 0$,

$$|F(\tilde{\zeta}_k)| \geq \frac{1}{2} - C(m)t^{-m}$$

and in the case where $\omega_k = 1$,

$$|F(\tilde{\zeta}_k)| > 1 - \frac{1}{2} - C(m)t^{-m}.$$

For $k = 1$ it is also true that

$$|F(\tilde{\zeta}_1)| > 1 - C(m)t^{-m}.$$

For a fixed $m > n$, we may choose t sufficiently large and get the desired function.

Notice also that with F constructed as above, there is the easy pointwise estimate

$$|F(z)| \leq \sum_j \frac{2^m(\delta_j)^m}{(1 - |z|)^m} \quad \text{for } |z| < 1. \tag{8}$$

In what follows, the letters m and t and the notation

$$\tilde{\zeta} = \left(1 - \frac{\delta}{t}\right)\zeta$$

refer back to Lemma 9.

Now suppose that the hypotheses of Theorem 2 hold, i.e., $d = n - \beta p$, $0 < p \leq 1$, $\beta > 0$ and E is a compact subset of S for which $H_d(E) = 0$. We are ready to construct the function $f \in H^p_\beta(B^n)$ whose existence proves Theorem 2.

We first claim that it is possible to find a sequence of finite open covers of E, $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \ldots,$ and positive constants $K, m_1, m_2, m_3, \ldots,$ satisfying the following conditions:

(9) $\mathscr{C}_j = \{B(\zeta_{ij}, K\delta_{ij}\}_{i=1}^{N^j}$ where $\{B(\zeta_{ij}, \delta_{ij})\}_{i=1}^{N_j}$ is a pairwise disjoint collection of balls in S.

(10) If F_j is the function associated with the disjoint collection obtained from $\mathscr{C}_j$ as in Lemma 4, then

$$\frac{m_j}{4} \geq 8\left(\sum_{i<j} m_i \|F_i\|_\infty\right). \tag{11}$$

$$\frac{2^m}{\left(1 - |\tilde{\zeta}_{il}|\right)^m} \sum_{a=1}^{N_j} \left(\delta_{aj}\right)^m m_j < \frac{1}{2^j}\frac{1}{1{,}000} \tag{12}$$

for $1 \leq i \leq N_l$, $1 \leq l \leq j-1$, where

$$\tilde{\zeta}_{il} = \left(1 - \frac{\delta_{il}}{t}\right)\zeta_{il}.$$

$$m_j > m_{j-1} + 1. \tag{13}$$

$$m_j^p \sum_{i=1}^{N_j} \left(\delta_{ij}\right)^{n-\beta p} < 2^{-j}. \tag{14}$$

To prove the claim, first set $m_0 = 0$. That K, $\mathscr{C}_1$ and m_1 may be found is clear from standard covering arguments and the fact that $H_d(E) = 0$.

Assume inductively that $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_{k-1}$ and $m_1, m_2, \ldots, m_{k-1}$ have been chosen. Choose m_k so

$$m_k > m_{k-1} + 1 \tag{15}$$

and

$$\frac{m_k}{4} \geq 8\left(\sum_{i<k} m_i \|F_i\|_\infty\right), \tag{16}$$

which is possible since each $F_i \in H^\infty$. Now use the hypothesis that $H_d(E) = 0$ and the standard covering argument to obtain a finite cover of E by balls $\{B(\zeta_{ik}, K\delta_{ik})\}$ where $\{B(\zeta_{ik}, \delta_{ik})\}$ is a pairwise disjoint collection and the δ_{ik} are so small that (11), (12), and (14) hold with k replacing j. By induction, the claim is proved.

The desired function can now be constructed. Set

$$f(z) = \sum_{k=1}^{\infty} m_k F_k(z)$$

where m_k and F_k are as in (9)–(14). From the construction above and Lemma 9 as well as the pointwise estimate (8), it follows that

$$
\begin{aligned}
|f(\tilde{\zeta}_{ik})| &\geq m_k|F_k(\tilde{\zeta}_{ik})| - \Big|\sum_{j<k} m_j F_j(\tilde{\zeta}_{ik})\Big| - \Big|\sum_{j>k} m_j F_j(\tilde{\zeta}_{ik})\Big| \\
&\geq \frac{m_k}{3} - \sum_{j<k} m_j \|F_j\|_\infty - \frac{1}{1000}\sum_{j>k}\frac{1}{2^j} \\
&\geq Cm_k.
\end{aligned}
$$

Using this last inequality we show that $M_\alpha f \equiv \infty$ on E, provided α is sufficiently large. Let $\zeta \in E$. Fix k and find a ball $B(\zeta_{ik}, K\delta_{ik})$ in the cover of E, $\mathscr{C}_k$, which contains ζ. Since

$$|1 - \langle\zeta, \zeta_{ik}\rangle| < K\delta_{ik} \quad \text{and} \quad 1 - |\tilde{\zeta}_{ik}| = \frac{\delta_{ik}}{t}$$

it follows that

$$
\begin{aligned}
|1 - \langle\zeta, \tilde{\zeta}_{ik}\rangle| &= |1 - \langle\zeta, \zeta_{ik}\rangle + \langle\zeta, \zeta_{ik}\rangle - \langle\zeta, \tilde{\zeta}_{ik}\rangle| \\
&\leq |1 - \langle\zeta, \zeta_{ik}\rangle| + |\zeta_{ik} - \tilde{\zeta}_{ik}| \\
&\leq \left(K + \frac{1}{t}\right)\delta_{ik}
\end{aligned}
$$

and therefore

$$(1 - |\tilde{\zeta}_{ik}|) \geq \frac{1}{t\left(K + \frac{1}{t}\right)}|1 - \langle\zeta, \tilde{\zeta}_{ik}\rangle|,$$

i.e., $\tilde{\zeta}_{ik} \in D_\alpha(\zeta)$ for α sufficiently large independent of ζ or k. Thus

$$(M_\alpha f)(\zeta) \geq |f(\tilde{\zeta}_{ik})| \geq Cm_k$$

and $M_\alpha f \equiv \infty$ on E.

From the construction of f it is also apparent that f extends to be continuous at every point $\zeta \in S \setminus E$. Therefore $E = E(f)$.

Finally, we must show that $R^\beta f \in H^p(B^n)$. Since

$$f(z) = \sum_k m_k \sum_i \frac{\omega_{ik} 2^m t^{-m} (\delta_{ik})^m}{(1 - \langle z, \tilde{\zeta}_{ik}\rangle)^m}$$

where $\omega_{ik} = 1$ or 0, the fact that $0 < p \leq 1$ and the triangle inequality shows that

$$\|R^\beta f\|_p^p \le C \sum_k m_k^p \sum_i (\delta_{ik})^{mp} \|R^\beta C(\cdot, \tilde{\zeta}_{ik})\|_p^p$$

where

$$C(z, r\zeta) = \frac{1}{(1 - \langle z, r\zeta \rangle)^m}$$

for $z \in B^n$ and $\zeta \in S$. By (14), it is therefore sufficient to obtain the estimate

$$\int_S (1 - r)^{mp} |R^\beta C(\cdot, r\zeta)|^p \, d\sigma \le C(1 - r)^{n - \beta p}$$

for a constant C depending only on m, where $0 < r < 1$ and $\zeta \in S$. We will consider only the case where $0 < \beta < 1$ since only simple modifications of our argument are needed in general.

Let $g(z_1) = \sum_{n=1}^\infty a_n z_1^n$ be holomorphic for z_1 in the unit disk of the complex plane. The classical 'fractional derivative'

$$(D^\beta g)(z_1) = \sum_{n=0}^\infty (n + 1)^\beta a_n z_1^n, \ \beta > 0$$

has the well known representation

$$(D^\beta g)(z_1) = \frac{1}{\Gamma(1 - \beta)} \int_0^1 \left[\log \frac{1}{t}\right]^{-\beta} (D^1 f)(t z_1) \, dt$$

valid if $0 < \beta < 1$, which follows from the formulas

$$(n + 1)^\beta = \frac{(n + 1)}{\Gamma(1 - \beta)} \int_0^1 \left[\log \frac{1}{t}\right]^{-\beta} t^n \, dt \tag{17}$$

for $n = 0, 1, 2, \ldots$. Fix $z \in B^n$ and let $\lambda \in B^1$. If f is holomorphic B^n with homogeneous polynomial expansion $f(z) = \sum_{k=0}^\infty f_k(z)$, then

$$\begin{aligned}(R^\beta f)(\lambda z) &= \sum_{k=0}^\infty (1 + k)^\beta \lambda^k f_k(z) \\ &= \frac{1}{\Gamma(1 - \beta)} \int_0^1 \left[\log \frac{1}{t}\right]^{-\beta} D_\lambda^1 f(t z \lambda) \, dt,\end{aligned}$$

where D_λ^1 is the classical derivative operator defined above and is applied to the function (of $\lambda) f(t z \lambda)$; here we have used formula (17) above. Letting λ go to 1 yields an integral representation for $(R^{\beta}f)(z)$ analogous to the one for $(D^{\beta}g)(z_1)$ mentioned before. If we apply this representation to $f(z) = C(z, r\zeta)$ we obtain the estimate

$$|R^{\beta}C(z; r\zeta)| \leq C\int_0^1 \left[\log\frac{1}{t}\right]^{-\beta} \frac{1}{|1 - t\langle z, r\zeta\rangle|^{m+1}}\,dt$$
$$\leq C\frac{1}{|1 - \langle z, r\zeta\rangle|^{m+\beta}}.$$

Since

$$\int_S \frac{(1-r)^{mp}}{|1 - \langle z, r\zeta\rangle|^{(m+\beta)p}}\,d\sigma(z) \leq \frac{C(1-r)^{mp}}{(1-r)^{mp+\beta p-n}}$$

if $(m + \beta)p > n$, we have the desired inequality.

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