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THE EQUIVALENCE PROBLEM FOR COMPLEX FOLIATIONS OF COMPLEX SURFACES

BY

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1. Introduction

In [4] we began a systematic study of the geometry of complex foliations (a *complex foliation* is a foliation of a complex manifold by complex submanifolds). In the case where the complex dimension of the underlying manifold M is 2 and the foliation $\mathscr{F}$ is not holomorphic we showed that its leaves come equipped with a metric of constant curvature. In this paper we continue this study by examining in detail the local geometry of complex foliations of complex 2-dimensional manifolds. More precisely, we will solve the Cartan equivalence problem for complex foliations of complex surfaces (see [3], [5] and [7] for discussions of the equivalence problem).

Note that when the foliation $\mathscr{F}$ is holomorphic the equivalence problem is trivial in the sense that for any point $p \in M$ there is a biholomorphism ϕ: $U \to \Delta^2$ between a neighborhood of p and the polydisk $\Delta^2 \subset \mathbf{C}^2$ sending the leaves of the restriction of $\mathscr{F}$ to U onto sets of the form $\{(z, w) \in \Delta^2 | w = \text{const}\}$; i.e., all holomorphic foliations are locally equivalent to the foliation of $\mathbf{C}^2$ by parallel lines.

When the foliation $\mathscr{F}$ is non-holomorphic the geometry of $\mathscr{F}$ is determined by the *anti-holomorphic torsion tensor* introduced by Bedford and Burns [1], [4]. To define it let L denote the complex tangent bundle of $\mathscr{F}$ and let pr: $TM \to Q$ be the projection map onto the complex normal bundle of $\mathscr{F}$. The *anti-holomorphic torsion* is the the section of the vector bundle

$$L^* \otimes \overline{Q}^* \otimes Q$$

defined by the map

$$\tau: \begin{cases} L \otimes \overline{Q} \to Q \\ Y \otimes \overline{X} \mapsto \operatorname{pr}\left(\bar{\partial}\tilde{Y}(\overline{X'})\right) \end{cases} \tag{1.1}$$

where $\tilde{Y}$ is any vector field extension of the vector Y and X' is any vector such that $\operatorname{pr}(X') = X$. One easily checks that τ is well-defined. It is easily shown [1]

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that $\mathscr{F}$ is holomorphic at all points if and only if the tensor τ vanishes everywhere. Consequently $\mathscr{F}$ is said to be *non-holomorphic at p* if $\tau_p \neq 0$ and $\mathscr{F}$ is said to be *non-holomorphic on M* if it is non-holomorphic at all points of M. Unless expressly stated all foliations will be assumed to be non-holomorphic.

The importance of the anti-holomorphic torsion is that it gives a way of choosing a distinguished class of framings of the holomorphic tangent bundle of M.

DEFINITION 1.2. A pair of independent vectors $X, Y \in T_{1,0}M$ based at a point $p \in M$ is called an *adapted frame* if (i) $Y \in L$ and (ii) $\tau(Y \otimes \mathrm{pr}(X)) = \mathrm{pr}(X)$. A coframe dual to an adapted frame is called an *adapted coframe*. The *bundle of adapted frames*, denoted by $P(M, \mathscr{F}) \to M$ (or simply by P when no confusion is likely to arise), is the bundle of all adapted frames. It is a right principal G-bundle where $G \subset GL(2, \mathbf{C})$ is the group of matrices of the form

$$\begin{pmatrix} a & 0 \\ b & a/\bar{a} \end{pmatrix}.$$

To see that P is a right G-principal bundle just observe that if (X_k, Y_k), $k = 1, 2$ are two adapted frames based at p then condition (i) above shows that there are uniquely defined complex numbers $a \neq 0$, b and $c \neq 0$ with the property that

$$X_2 = aX_1 + bY_1, \quad Y_2 = cY_1$$

but condition (ii) implies the further restriction $c\bar{a}/a = 1$.

Note that if f: $(M, \mathscr{F}) \to (M', \mathscr{F}')$ is an isomorphism of two complex foliations (i.e., a biholomorphism respecting foliations) then the derivative of f defines a diffeomorphism $\tilde{f}$: $P \to P'$. Our main result is the following theorem.

THEOREM 1.3. *Let $(M, \mathscr{F})$ be a pair consisting of a complex two dimensional manifold and a foliation by complex curves which in non-holomorphic on M. Then the manifold $P(M, \mathscr{F})$ has a complex structure with respect to which*

$$\pi\colon P(M, \mathscr{F}) \to M$$

is a holomorphic fibration and there is a global framing $(\theta, \eta, \phi, \psi)$ of the cotangent bundle of $P = P(M, \mathscr{F})$ by forms of type $(1,0)$.

Let $(M', \mathscr{F}')$ be another such foliation. Then the mapping $f \mapsto \tilde{f}$ is a bijection between the set of all isomorphisms between $(M, \mathscr{F})$ and $(M', \mathscr{F}')$ and the set of all diffeomorphisms between P and P' satisfying the condition

$$\tilde{f}^*((\theta', \eta', \phi', \psi')) = (\theta, \eta, \phi, \psi).$$

The importance of Theorem 1.3 is that it enables us to identify elements of $\mathrm{Aut}(M, \mathcal{F})$ with the set of automorphisms of $P(M, \mathcal{F})$ which preserve the framing. We can now employ the fundamental theorem on the automorphism group of manifolds with global framings, Theorem 3.2 of [6], to obtain the next corollary.

COROLLARY 1.4. *The group* $\mathrm{Aut}(M, \mathcal{F})$ *has the structure of a Lie group. In fact let* $p \in P(M, \mathcal{F})$ *be any point, then the mapping*

$$\mathrm{Aut}(M, \mathcal{F}) \to P(M, \mathcal{F}),$$
$$f \mapsto \tilde{f}(p)$$

is an embedding of $\mathrm{Aut}(M, \mathcal{F})$ *as a closed submanifold of* $P(M, \mathcal{F})$. *In particular, the inequality*

$$\dim \mathrm{Aut}(M, \mathcal{F}) \leq 8$$

holds.

More is true. The group G embeds as a subgroup of $SL(3, \mathbf{R})$ and the framing $(\theta, \eta, \phi, \psi)$ can be used to define an $\mathfrak{sl}(3, \mathbf{R})$-valued 1-form ω on P. Let H denote the subgroup of G consisting of matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & a/\bar{a} \end{pmatrix}$$

and set $E \equiv P/H$. The fibration $P \to E$ is an H-principal bundle.

THEOREM 1.5. *The* $\mathfrak{sl}(3, \mathbf{R})$*-valued* 1*-form* ω: $TP \to \mathfrak{sl}(3, \mathbf{R})$ *is a Cartan connection on the right principal H-bundle* $P \to E$. *Moreover, when the fundamental invariant A in the structure equations* (1.6) *vanishes* ω *is in fact a Cartan connection on the right principal G-bundle* $P \to M$.

The outline of the paper is as follows:

In Section 2 we construct the framing $(\theta, \eta, \phi, \psi)$ and derive the following *structure equations*:

$$\begin{aligned} d\theta &= -\phi \wedge \theta + \eta \wedge \bar{\theta} \\ d\eta &= -\psi \wedge \theta - (\phi - \bar{\phi}) \wedge \eta \\ d\phi &= \psi \wedge \theta - 2\theta \wedge \bar{\psi} - \eta \wedge \bar{\eta} - 3A\theta \wedge \bar{\eta} \\ d\psi &= -\eta \wedge \psi - \psi \wedge \phi - 3A\eta \wedge \bar{\eta} \\ &\quad + (B\bar{\theta} + 2\bar{C}\eta + c\bar{\eta} + 2\bar{A}\psi + A\bar{\psi}) \wedge \theta. \end{aligned} \tag{1.6}$$

The complex-valued functions A, B and C are the fundamental invariants of a complex foliation.

A foliation for which A, B and C all vanish is called *flat* and in Section 3 we construct one. It is easily described: Let M be the complement of real projective space $\mathbf{RP}^2$ in complex projective space $\mathbf{CP}^2$. Every real line in $\mathbf{RP}^2$ is the restriction of exactly one complex line in $\mathbf{CP}^2$ and there is exactly one such line passing through each point of M; hence this construction defines a complex foliation $\mathscr{F}$. Note that the group $SL(3, \mathbf{R})$ has dimension 8 and acts transitively and effectively on $(M, \mathscr{F})$. It follows that $\mathscr{F}$ is a flat foliation (this can also be checked directly) and that the total space $P(M, \mathscr{F})$ is diffeomorphic to $SL(3, \mathbf{R})$. By applying the results of Section 2 we are able to relate the framing $(\theta, \eta, \phi, \psi)$ to the Maurer-Cartan form of the group $SL(3, \mathbf{R})$ and to show that in the flat case the structure equations 1.6 reduce to the structure equations for the Lie algebra $\mathfrak{sl}(3, \mathbf{R})$.

2. Solution of the equivalence problem

In this section we prove Theorem 1.3, derive the structure equations (1.6) and determine the transformation properties of the fundamental invariants, A, B and C.

Local coordinates

Many of our computations will be done in local coordinates which we choose as follows. Coordinates for $\mathbf{C}^2$ are (w, z) and holomorphic coordinates on an open set $U \subset M$ are chosen so that the z-axis is transverse to $\mathscr{F}$. Consequently, there is a smooth function $\lambda = \lambda(z, w)$ defined in the neighborhood of the origin such that the form

$$\theta_0 =_{\text{def}} dz - \lambda\, dw \tag{2.1}$$

is a normal form defining $\mathscr{F}$ and the vector field

$$Y_0 =_{\text{def}} \frac{\partial}{\partial w} + \lambda \frac{\partial}{\partial z}$$

is tangent to $\mathscr{F}$ is holomorphic if and only if Y is a holomorphic vector field.

It is straightforward to check that the local framing

$$\left(\frac{\partial}{\partial w}, \frac{1}{\lambda_{,\bar z}} \left(\frac{\partial}{\partial w} + \lambda \frac{\partial}{\partial z} \right) \right)$$

is an adapted framing whose dual coframe is

$$(2.2) \qquad \begin{pmatrix} \theta_0 \\ \eta_0 \end{pmatrix} =_{\text{def}} \begin{pmatrix} dz - \lambda\, dw \\ \lambda_{,\bar z}\, dw \end{pmatrix}.$$

This frame gives a local trivialization of the bundle of adapted frames

$$P_{|U} \cong U \times G.$$

The *tautological* 1 *forms* on P are the forms θ and η characterized by the property that for every local adapted frame, say $s \in \Gamma(U, P)$, the coframe $(s^*\theta, s^*\eta)$ is dual to s. With respect to the above trivialization the identity

$$(2.3) \qquad \begin{pmatrix} \theta \\ \eta \end{pmatrix} = g^{-1} \begin{pmatrix} \theta_0 \\ \eta_0 \end{pmatrix}, \quad g = \begin{pmatrix} a & 0 \\ b & a/\bar a \end{pmatrix}.$$

holds.

First reduction

We start by considering local frames of the cotangent bundle of M (coframes) consisting of pairs of forms of type $(1,0)$ whose first entries lie in the conormal bundle of $\mathscr{F}$. In local coordinates we may write such a coframe in the form

$$(2.4) \qquad \begin{pmatrix} \theta \\ \eta \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}^{-1} \begin{pmatrix} dz - \lambda\, dw \\ dw \end{pmatrix}$$

where

$$(2.5) \qquad g = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

is a matrix valued function. The set of all such matrices forms a subgroup of $GL(2, \mathbf{C})$.

LEMMA 2.6. *There is a natural reduction of the bundle of frames to a principal subbundle* $P = P(M, \mathscr{F})$ *with structure group* $G \subset GL(2, \mathbf{C})$ (see Definition 1.2).

Proof. Actually, we have already proved this lemma; but we will have need of expressions for the exterior derivative of the dual coframes so we will adopt a slightly different point of view which exhibits Cartan's method of reduction of structure group and prove it again. Let θ_0 be as in (2.1) and note that the integrability condition $[Y_0, \overline{Y}_0] = 0$ written in local coordinates assumes the form

$$\lambda_{,\bar{w}} + \bar{\lambda}\lambda_{,\bar{z}} = 0 \tag{2.7}$$

from which the identity

$$d\theta_0 = \lambda_{,z}\, dw \wedge \theta_0 + \lambda_{,\bar{z}}\, dw \wedge \bar{\theta}_0, \tag{2.8}$$

follows easily. Choose a coframe as in (2.4) and compute as follows:

$$\begin{aligned}
d\begin{pmatrix}\theta\\ \eta\end{pmatrix} &= -g^{-1}\,dg \wedge \begin{pmatrix}\theta\\ \eta\end{pmatrix} + g^{-1}\begin{pmatrix}d\theta_0\\ 0\end{pmatrix}\\
&= -g^{-1}\,dg \wedge \begin{pmatrix}\theta\\ \eta\end{pmatrix}\\
&\quad + g^{-1}\begin{pmatrix}(ac\lambda_{,z})\eta \wedge \theta + (c\bar{a}\lambda_{,\bar{z}})\eta \wedge \bar{\theta} + (b\bar{a}\lambda_{,\bar{z}})\theta \wedge \bar{\theta}\\ 0\end{pmatrix}\\
&= -g^{-1}\,dg \wedge \begin{pmatrix}\theta\\ \eta\end{pmatrix}\\
&\quad + \begin{pmatrix}(c\lambda_{,z})\eta - (\bar{a}b\lambda_{,\bar{z}}/a)\bar{\theta} & 0\\ (\bar{a}b^2\lambda_{,\bar{z}}/ac)\bar{\theta} & (\bar{a}b\lambda_{,\bar{z}}/a)\bar{\theta}\end{pmatrix} \wedge \begin{pmatrix}\theta\\ \eta\end{pmatrix}\\
&\quad + \begin{pmatrix}(\bar{a}c\lambda_{,\bar{z}}/a)\eta \wedge \theta\\ 0\end{pmatrix}.
\end{aligned}$$

This can be written in the form

$$d\begin{pmatrix}\theta\\ \eta\end{pmatrix} = -\begin{pmatrix}\phi & 0\\ \psi & \beta\end{pmatrix} \wedge \begin{pmatrix}\theta\\ \bar{\eta}\end{pmatrix} + \begin{pmatrix}(\bar{a}c\lambda_{,\bar{z}}/a)\eta \wedge \bar{\theta}\\ 0\end{pmatrix}. \tag{2.9}$$

The last term of this equation is an invariantly defined torsion term and a reduction in the structure group can be made by choosing only those frames whose dual coframes satisfy the normalization condition,

$$(\bar{a}/a)c\lambda_{,\bar{z}} = 1. \tag{2.10}$$

(Note that the requirement that $\mathscr{F}$ be non-holomorphic is essential here since in local coordinates it is equivalent to the condition $\lambda_{,\bar{z}} \neq 0$). One checks that the condition is satisfied if and only if the function g takes its values in G. ■

Note that the Lie algebra of G consists of all complex matrices of the form

$$\begin{pmatrix} a & 0 \\ b & a - \bar{a} \end{pmatrix}.$$

We want to write structure equations for the exterior derivatives of the forms θ and η which incorporate the structure of the Lie algebra. The next lemma shows how to do this.

LEMMA 2.11. *Given a local adapted coframe $(\theta, \eta)^t$ there are one forms ϕ and ψ such that the identity*

$$d\begin{pmatrix} \theta \\ \eta \end{pmatrix} = -\begin{pmatrix} \phi & 0 \\ \psi & \phi - \bar{\phi} \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \eta \end{pmatrix} + \begin{pmatrix} \eta \wedge \bar{\theta} \\ 0 \end{pmatrix} \tag{2.12}$$

holds. Moreover, the forms ϕ and ψ are determined up to the transformation

$$\psi \mapsto \psi - f\theta \tag{2.13}$$

where f is an arbitrary complex valued function.

Proof. A straightforward computation using the integrability condition (2.7) gives the equation

$$d\begin{pmatrix} \theta_0 \\ \eta_0 \end{pmatrix} = -\begin{pmatrix} \phi_0 & 0 \\ \psi_0 & \phi_0 - \bar{\phi}_0 \end{pmatrix} \wedge \begin{pmatrix} \theta_0 \\ \eta_0 \end{pmatrix} + \begin{pmatrix} \eta_0 \wedge \bar{\theta}_0 \\ 0 \end{pmatrix} \tag{2.14}$$

where

$$\phi_0 = -(\lambda_{,z}/\lambda_{,\bar{z}})\eta_0 + \overline{(\lambda_{,\bar{z}\bar{z}}/\lambda_{,\bar{z}})}\theta_0$$
$$\psi_0 = \left(\lambda_{,\bar{z}\bar{z}}/\lambda_{,\bar{z}} + \overline{\lambda_{,\bar{z}\bar{z}}/\lambda_{,\bar{z}}}\right)\eta_0.$$

Equation (2.14) can be used to derive a similar structure equation for the exterior derivative of the adapted coframe $g^{-1}(\theta_0, \eta_0)^t$, can now be computed:

$$
\begin{aligned}
d\begin{pmatrix}\theta\\ \eta\end{pmatrix} &= -g^{-1}\,dg \wedge \begin{pmatrix}\theta\\ \eta\end{pmatrix} + g^{-1} d\begin{pmatrix}\theta_0\\ \eta_0\end{pmatrix}\\
&= \left(-g^{-1}\,dg - g^{-1}\begin{pmatrix}\phi_0 & 0\\ \psi_0 & \phi_0 - \bar\phi_0\end{pmatrix} g\right) \wedge \begin{pmatrix}\theta\\ \eta\end{pmatrix}\\
&\quad + g^{-1}\begin{pmatrix}\eta_0 \wedge \bar\theta_0\\ 0\end{pmatrix}\\
&= \left(-g^{-1}\,dg - g^{-1}\begin{pmatrix}\phi_0 & 0\\ \psi_0 & \phi_0 - \bar\phi_0\end{pmatrix} g\right) \wedge \begin{pmatrix}\theta\\ \eta\end{pmatrix}\\
&\quad + g^{-1}\begin{pmatrix}((a/\bar a)\eta + b\theta) \wedge \bar a\bar\theta\\ 0\end{pmatrix}\\
&= \left(-g^{-1}\,dg - g^{-1}\begin{pmatrix}\phi_0 & 0\\ \omega_0 & \phi_0 - \bar\phi_0\end{pmatrix} g\right) \wedge (\theta\eta)\\
&\quad + \begin{pmatrix}\eta \wedge \bar\theta + (b\bar a/a)\theta \wedge \bar\theta\\ -(b\bar a/a)\eta \wedge \bar\theta - (b\bar a/a)^2\theta \wedge \bar\theta\end{pmatrix}\\
&= \left(-g^{-1}\,dg - g^{-1}\begin{pmatrix}\alpha_0 & 0\\ \psi_0 & \phi_0 - \bar\phi_0\end{pmatrix} g\right) \wedge \begin{pmatrix}\theta\\ \eta\end{pmatrix}\\
&\quad - \begin{pmatrix}2(\bar b a/\bar a)\theta + (b\bar a/a)\bar\theta & 0\\ -(b\bar a/a)^2\bar\theta + (\bar b a/\bar a)\eta & (\bar b a/\bar a)\theta - (b\bar a/a)\bar\theta\end{pmatrix} \wedge \begin{pmatrix}\theta\\ \eta\end{pmatrix}\\
&\quad + \begin{pmatrix}\eta \wedge \bar\theta\\ 0\end{pmatrix}.
\end{aligned}
$$

Note that this is of the form (2.12) where,

$$
\begin{aligned}
(2.15)\quad \begin{pmatrix}\phi & 0\\ \psi & \phi - \bar\phi\end{pmatrix}
&= g^{-1}\,dg + g^{-1}\begin{pmatrix}\phi_0 & 0\\ \psi_0 & \phi_0 - \bar\phi_0\end{pmatrix} g\\
&\quad + \begin{pmatrix}2(\bar b a/\bar a)\theta + (b\bar a/a)\bar\theta & 0\\ -(b\bar a/a)^2\bar\theta + (\bar b a/\bar a)\eta & (\bar b a/\bar a)\theta - (b\bar a/a)\bar\theta\end{pmatrix}.
\end{aligned}
$$

To see that the forms ϕ and ψ for which the relation (2.12) holds are determined up to a transformation of the form (2.13) assume that ϕ is replaced by the form $\phi' = \phi - g\theta$, g a complex valued function and that $\psi' = \psi +$ (a 1 form). (This is the most general transformation that preserves the form of the structure equation, $d\theta = -\phi \wedge \theta + \eta \wedge \bar{\theta}$.) Now expand the second row of equation (2.12):

$$\begin{aligned} d\eta &= -\psi \wedge \theta - (\phi - \bar{\phi}) \wedge \eta \\ &= \psi \wedge \theta - (\phi' + g\theta + \bar{\phi}' + \bar{g}\bar{\theta}) \wedge \eta \\ &= -(\psi + g\eta) \wedge \theta - (\phi' - \phi') \wedge \eta - g\theta \wedge \eta. \end{aligned}$$

The choice $\psi' = \psi + g\eta$ yields the equation

$$d\eta = -\psi' \wedge \theta - (\phi' - \bar{\phi}') \wedge \eta + \bar{g}\bar{\theta} \wedge \eta,$$

from which we see that the term $\bar{g}\bar{\theta} \wedge \eta$ cannot be removed. Hence we must choose $g = 0$. Having shown that the form ϕ is uniquely determined, it is clear that ψ is determined up to a change of the form $\psi - f\theta$, where f is an arbitrary complex valued function. ■

Prolongation

We have shown that given an adapted coframe it is possible to find forms ϕ and ψ so that equation (2.12) holds. In order to remove the ambiguity in the choice of the forms ϕ and ψ we have to move from the base space M to the total space of the bundle P (doing so is called *prolongation*).

We seek a canonical trivialization of the cotangent bundle of P. Since P is 8 dimensional we must find four complex 1-forms whose real and imaginary parts are independent. Two forms already exist, the tautological 1-forms θ and η. Using the local trivialization $P_{|U} \cong U \times G$ induced by local coordinates, we may reinterpret formula (2.15) as a formula on the total space of P defining two more forms on P, ϕ and ψ. The collection θ, η, ϕ and ψ is a complex framing of the complexified cotangent bundle of P (i.e., the real and imaginary parts of these forms yield a framing for the real cotangent bundle of P). There is a problem: this framing is *not* canonical—it depends on the initial choice of coordinates. The dependence is given by Lemma 2.11 which (reinterpreted as a statement about forms on P) shows that the set of forms satisfying (2.12) are only defined up to transformations of the form

$$\text{(2.16)} \qquad \begin{pmatrix} \theta \\ \eta \\ \phi \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} \theta \\ \eta \\ \phi \\ \psi - f\theta \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -f & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta \\ \eta \\ \phi \\ \psi \end{pmatrix},$$

where $f = f(w, z, g)$ is a complex-valued function.

The next step is to show how to make a canonical choice of the form ψ. The following lemma shows that this is possible and concludes the proof of Theorem 1.3.

LEMMA 2.17. *There is a unique complex framing* $(\theta, \eta, \phi, \psi)$ *of the cotangent bundle of P characterized by the equations*

$$d\theta = -\phi \wedge \theta + \eta \wedge \bar{\theta} \tag{2.18}$$

$$d\eta = -\psi \wedge \theta - (\phi - \bar{\phi}) \wedge \eta \tag{2.19}$$

$$d\phi = \psi \wedge \bar{\theta} - 2\theta \wedge \psi - \eta \wedge \bar{\eta} - 3A\theta \wedge \bar{\eta}, \tag{2.20}$$

where θ *and* η *are the tautological* 1*-forms on P and A is a complex valued function.*

Proof. We already know that ϕ is determined by the first two equations of the lemma; we need only show that ψ is determined by the third equation.

One way to do this is to differentiate (2.12) and then to use (2.12) to simplify the resulting expression. The computation goes like this:

$$\begin{aligned}
0 &= d^2\begin{pmatrix}\theta\\ \eta\end{pmatrix}\\
&= -d\begin{pmatrix}\phi & 0\\ \psi & \phi-\bar{\phi}\end{pmatrix}\wedge\begin{pmatrix}\theta\\ \eta\end{pmatrix}+\begin{pmatrix}\phi & 0\\ \psi & \phi-\bar{\phi}\end{pmatrix}\wedge d\begin{pmatrix}\theta\\ \eta\end{pmatrix}+d\begin{pmatrix}\eta\wedge\bar{\theta}\\ 0\end{pmatrix}\\
&= -\left\{d\begin{pmatrix}\phi & 0\\ \psi & \phi-\bar{\phi}\end{pmatrix}+\begin{pmatrix}\phi & 0\\ \psi & \phi-\bar{\phi}\end{pmatrix}\wedge\begin{pmatrix}\phi & 0\\ \psi & \phi-\bar{\phi}\end{pmatrix}\right\}\wedge\begin{pmatrix}\theta\\ \eta\end{pmatrix}\\
&\quad+\begin{pmatrix}\phi & 0\\ \psi & \phi-\bar{\phi}\end{pmatrix}\wedge\begin{pmatrix}\eta\wedge\bar{\theta}\\ 0\end{pmatrix}+\begin{pmatrix}d\eta\wedge\bar{\theta}-\eta\wedge d\bar{\theta}\\ 0\end{pmatrix}\\
&= \begin{pmatrix}-d\phi\wedge\theta\wedge\eta\wedge\bar{\theta}\\ (-d\psi+\psi\wedge\bar{\phi})\wedge\theta-d(\phi-\bar{\phi})+\psi\wedge\eta\wedge\bar{\theta}\end{pmatrix}\\
&\quad+\begin{pmatrix}(\psi\wedge\bar{\theta}-\eta\wedge\bar{\eta})\wedge\theta+\phi\wedge\bar{\theta}\wedge\eta\\ 0\end{pmatrix}.
\end{aligned}$$

Collecting terms results in the two identities

$$\{-d\phi + \psi \wedge \bar{\theta} - \eta \wedge \bar{\eta}\} \wedge \theta = 0 \tag{2.21}$$

$$-\{d\psi + \psi \wedge \bar{\phi}\} \wedge \theta + \{-(d\phi - d\bar{\phi}) - \psi \wedge \bar{\theta}\} \wedge \eta = 0. \tag{2.22}$$

It follows that $d\phi$ can be written in the form

$$\begin{aligned}d\phi &= \psi \wedge \bar{\theta} - \eta \wedge \bar{\eta}\\ &\quad+ (3A\bar{\eta} + B\bar{\theta} + C\eta + D\phi + E\bar{\phi} + F\psi + G\bar{\psi}) \wedge \theta\end{aligned} \tag{2.23}$$

where $A, B, \ldots, G$ are complex valued functions to be determined (the coefficient of A is chosen to simplify later calculations). Wedging equation (2.22) with θ, substituting into it the above formula of $d\phi$ and the conjugate formula for $d\bar{\phi}$ and simplifying we arrive at the identity

$$(2.24) \qquad \left(\bar{C}\bar{\eta} + \bar{D}\bar{\phi} + \bar{E}\phi + \bar{F}\bar{\psi} + (\bar{G} - 2)\psi\right) \wedge \bar{\theta} \wedge \eta \wedge \theta = 0.$$

The independence of the forms $\theta, \bar{\theta}, \eta, \bar{\eta}, \phi, \bar{\phi}, \psi$ and $\bar{\psi}$ yields the equalities

$$C = D = E = F = 0 \quad \text{and} \quad G = 2;$$

and equation (2.23) assumes the form

$$(2.25) \qquad d\phi = \psi \wedge \bar{\theta} - 2\theta \wedge \bar{\psi} - \eta \wedge \bar{\eta} - 3A\theta \wedge \bar{\eta} - B\theta \wedge \bar{\theta}.$$

This equation can be used to make a canonical choice for the form ψ—for replacing ψ by $\psi - f\theta$ yields the equation

$$d\phi = \psi \wedge \bar{\theta} - 2\theta \wedge \bar{\psi} - \eta \wedge \bar{\eta} - 3A\theta \wedge \bar{\eta} + \left(-f + 2\bar{f} - B\right)\theta \wedge \bar{\theta}$$

and the choice $f - 2\bar{f} = -B$ fixes the form ψ uniquely. With this choice we arrive at the identity (2.20). This concludes the proof of the lemma. ■

The function A is an invariantly defined function on P and as such is an invariant of the pair $(M, \mathscr{F})$. To determine the complete set of invariants the formula for the exterior derivative of ψ must be computed.

PROPOSITION 2.26. *The canonical framing of the cotangent bundle of $P(M, \mathscr{F})$ satisfies the structure equations*

$$(2.27) \qquad \begin{aligned} d\theta &= -\phi \wedge \theta + \eta \wedge \bar{\theta} \\ d\eta &= -\psi \wedge \theta - \left(\phi - \bar{\phi}\right) \wedge \eta \\ d\phi &= \phi \wedge \bar{\theta} - 2\theta \wedge \bar{\psi} - \eta \wedge \bar{\eta} - 3A\theta \wedge \bar{\eta} \\ d\psi &= -\eta \wedge \bar{\psi} - \psi \wedge \bar{\phi} - 3A\eta \wedge \bar{\eta} + \Phi \wedge \theta \end{aligned}$$

where

$$\Phi = \left(B\bar{\theta} + 2\bar{C}\eta + C\bar{\eta} + 2\bar{A}\psi + A\bar{\psi}\right) \wedge \theta$$

and A, B and C are complex valued functions satisfying the identity $C = \partial A/\partial\bar{\theta}$.

Proof. It remains to consider the last structure equation. Substitution of (2.20) into equation (2.22) yields, after simplification, the formula

$$\left(d\psi + \psi \wedge \bar{\phi} + \eta \wedge \bar{\psi} + 3A\eta \wedge \bar{\eta}\right) \wedge \theta = 0$$

from which it follows that $d\psi$ can be written uniquely in the form

$$d\psi = -\eta \wedge \bar{\psi} - \psi \wedge \bar{\phi} - 3A\eta \wedge \bar{\eta} + \Phi \wedge \theta \tag{2.28}$$

where the one form Φ is to be determined.

By computing the exterior derivative of equation (2.20) it is possible to determine the form of Φ. Specifically, expand as follows

$$\begin{aligned} 0 = d(d\phi) &= d(\psi \wedge \bar{\theta} - 2\theta \wedge \bar{\psi} - \eta \wedge \bar{\eta} - 3A\theta \wedge \bar{\eta}) \\ &= d\psi \wedge \bar{\theta} - \psi \wedge d\bar{\theta} - 2d\theta \wedge \bar{\psi} + 2\theta \wedge d\bar{\psi} - d\eta \wedge \bar{\eta} + \eta \wedge d\bar{\eta} \\ &\quad - 3(dA \wedge \theta \wedge \bar{\eta} - A\, d\theta \wedge \bar{\eta} + A\theta \wedge d\bar{\eta}), \end{aligned}$$

employ formulas (2.12), (2.20) and (2.28) and simplify to obtain the identity

$$\begin{aligned} &\{\Phi - 2\bar{\Phi} + 3A\bar{\psi}\} \wedge \theta \wedge \bar{\theta} \\ &\qquad + 3\{-dA + 2\bar{A}\eta + A\bar{\phi}\} \wedge \theta \wedge \bar{\eta} = 0. \end{aligned} \tag{2.29}$$

Denoting the framing of the manifold P dual to the coframe θ, η, ϕ, ψ by

$$\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\phi}, \frac{\partial}{\partial\psi},$$

we can write dA in the form

$$dA = \frac{\partial A}{\partial\theta}\theta + \frac{\partial A}{\partial\eta}\eta + \frac{\partial A}{\partial\phi}\phi + \frac{\partial A}{\partial\psi}\psi + \frac{\partial A}{\partial\bar{\theta}}\bar{\theta} + \frac{\partial A}{\partial\bar{\eta}}\bar{\eta} + \frac{\partial A}{\partial\bar{\phi}}\bar{\phi} + \frac{\partial A}{\partial\bar{\psi}}\bar{\psi}.$$

which can be substituted into (2.29) to obtain the equation

$$\begin{aligned} &3\left\{\left(2\bar{A} - \frac{\partial A}{\partial\eta}\right)\eta - \left(\frac{\partial A}{\partial\phi}\right)\phi + \left(A - \frac{\partial A}{\partial\bar{\phi}}\right)\bar{\phi} - \left(\frac{\partial A}{\partial\psi}\right)\psi - \left(\frac{\partial A}{\partial\bar{\psi}}\right)\bar{\psi}\right\} \wedge \theta \wedge \bar{\eta} \\ &\qquad + \left\{\Phi - 2\bar{\Phi} + 3A\bar{\psi} + 3\frac{\partial A}{\partial\bar{\theta}}\bar{\eta}\right\} \wedge \theta \wedge \bar{\theta} = 0. \end{aligned}$$

This equation can be partially solved for Φ and dA to yield the identity

$$\Phi = B\bar{\theta} + 2\bar{C}\eta + C\bar{\eta} + \frac{\partial A}{\partial\theta}\eta 2ay + 2\bar{A}\psi + A\bar{\psi} \tag{2.30}$$

where B is a new complex valued function. (Note that this equation can also be used to derive the formula for the exterior derivative, dA, given below (2.32).) ∎

Using the structure equations (2.27) it is possible to derive explicit formulas for dA, dB and dC.

LEMMA 2.31. *The following identities hold*:

$$dA = \frac{\partial A}{\partial \theta}\theta + \frac{\partial A}{\partial \bar\theta}\bar\theta + 2\bar A\eta + \frac{\partial A}{\partial \bar\eta}\bar\eta + A\bar\phi \tag{2.32}$$

$$dB = \frac{\partial B}{\partial \theta}\theta + \frac{\partial B}{\partial \bar\theta}\bar\theta + \left(2\frac{\partial \bar C}{\partial \bar\theta} + \bar B - A\bar C\right)\eta + \left(\frac{\partial C}{\partial \bar\theta} - 2AC\right)\bar\eta \tag{2.33}$$
$$+ B\phi + 2B\bar\phi + 2\frac{\overline{\partial A}}{\partial \theta}\psi + 2(C - A^2)\bar\psi$$

$$dC = \frac{\partial C}{\partial \theta}\theta + \frac{\partial C}{\partial \bar\theta}\bar\theta + \frac{\partial C}{\partial \bar\eta}\bar\eta + \left(|A|^2 - \frac{\partial A}{\partial \bar\theta}\right)\eta - 2C\bar\phi + \frac{\partial \bar A}{\partial \eta}\bar\psi. \tag{2.34}$$

Proof. The formula for dA was derived in the proof of Proposition 2.26. To determine dB and dC expand the exterior derivative of the fourth structure equation in (2.27) and use equations (2.30) to obtain the equation

$$0 = d(d\psi) = \left\{d\Phi + \Phi \wedge \phi + \Phi \wedge \bar\phi - 3\frac{\partial A}{\partial \theta}\eta \wedge \bar\eta - 3A\psi \wedge \bar\eta\right\} \wedge \theta.$$

Expand Φ and $d\Phi$ in this equation and rearrange terms to arrive at the identity

$$\left\{dB - 2\frac{\overline{\partial A}}{\partial \theta}\psi + 2(A^2 - C)\bar\psi - B\phi - 2B\bar\phi + (A\bar C - \bar B)\eta + 2AC\bar\eta\right\}$$
$$\wedge\, \bar\theta \wedge \theta$$
$$+\left\{2d\bar C + 3\left(\frac{\partial A}{\partial \theta} - |A|^2\right)\bar\eta - 4\bar C\phi - 2\frac{\partial A}{\partial \bar\eta}\psi\right\} \wedge \eta \wedge \theta$$
$$+\left\{dC - \frac{\partial C}{\partial \bar\theta}\bar\theta - 2C\bar\phi - \frac{\partial A}{\partial \bar\eta}\bar\psi\right\} \wedge \bar\eta \wedge \theta = 0.$$

Expanding dB and dC and collecting linearly independent terms results in the formulas for dB and dC in the lemma. ■

Remark 2.35. The special case in which the invariants A, B and C are constant is of particular interest because in this case the group Aut $(M, \mathscr{F})$ has maximum dimension. In fact the only case to consider is the *flat* case where A, B and C vanish for from Lemma 2.31 we can extract the formulas

$$\frac{\partial A}{\partial \bar\phi} = A, \quad \frac{\partial B}{\partial \phi} = B \text{ and } \frac{\partial C}{\partial \bar\phi} = 2C$$

which clearly force vanishing of A, B and C.

3. The Cartan connection

In Riemannian geometry the curvature tensor of the Levi-Civita connection can be interpreted as the deviation of the metric from the flat metric on Euclidean space. Similarly, the analysis of this section will show that the structure equations 1.6 are those of a Cartan connection with values in the Lie algebra $\mathfrak{sl}(3,\mathbf{R})$. The fundamental invariants A, B and C can then be interpreted as curvature components and measure the deviation of a complex foliation from the flat foliation.

The flat model

We begin with an investigation of the flat foliation. Let M_0 denote the complement of real projective space in complex projective space and let $\mathscr{F}_0$ be the foliation of M_0 whose leaves are the restriction to M_0 of the complex projective lines in $\mathbf{CP}^2$ which intersect $\mathbf{RP}^2$ in real projective lines.

Let the group $SL(3,\mathbf{R}) \subset SL(3,\mathbf{C})$ act from the left on $\mathbf{CP}^2$ in the standard way by complex projective transformations. Because $SL(3,\mathbf{R})$ maps $\mathbf{RP}^2$ to itself and maps real projective lines to real projective lines it embeds in the group $\mathrm{Aut}(M_1, \mathscr{F}_0)$. In fact because $SL(3,\mathbf{R})$ is 8 dimensional it follows from Corollary 1.4 that the manifolds $SL(3,\mathbf{R})$ and $P(M_0, \mathscr{F}_0)$ are diffeomorphic, that $SL(3,\mathbf{R}) \cong \mathrm{Aut}(M_0, \mathscr{F}_0)$, and that the invariants, A, B and C are all constant. Finally, by Remark 2.35 the constants A, B and C vanish and $\mathscr{F}_0$ is a flat foliation with structure equations

$$
\begin{aligned}
d\theta &= -\phi \wedge \theta + \eta \wedge \bar{\theta} \\
d\eta &= -\psi \wedge \theta - (\phi - \bar{\phi}) \wedge \eta \\
d\phi &= \psi \wedge \bar{\theta} - 2\theta \wedge \bar{\psi} - \eta \wedge \bar{\eta} \\
d\psi &= -\eta \wedge \bar{\psi} - \phi \wedge \bar{\phi}.
\end{aligned} \tag{3.1}
$$

By virtue of the isomorphism between $SL(3,\mathbf{R})$ and $P(M_0, \mathscr{F}_0) \cong \mathrm{Aut}(M_0, \mathscr{F}_0)$ *these equations are equivalent to the structure equations for the Lie algebra* $\mathfrak{sl}(3,\mathbf{R})$. In particular, the framing $(\theta, \eta, \phi, \psi)$ can be expressed in terms of the components of the Maurer-Cartan form of $SL(3,\mathbf{R})$, ω_{SL}. (Recall that if g denotes a variable 3×3 matrix in $SL(3,\mathbf{R})$ then $\omega_{SL} \equiv g^{-1}\,dg$.)

To derive the form of this expression we must introduce some notation. Let $[\zeta^1, \zeta^2, \zeta^3]$ be homogeneous coordinates on $\mathbf{CP}^2$, let (w, z) be the affine coordinates, $w = \zeta^2/\zeta^1$, $z = \zeta^3/\zeta^1$ and fix once and for all the point $x_0 = [1, i, 0] \in M_0$. Let $g = (a^i_j)$ denote an arbitrary element of $SL(3,\mathbf{R})$ and let ω^i_j denote the ijth entry of ω_{SL}. Finally, let π be the surjective map

$$
\pi\colon \begin{cases} SL(3,\mathbf{R}) \to M_0 \\ g \mapsto g \cdot x_0 = \left[a^1_1 + ia^1_2,\, a^2_1 + ia^2_2,\, a^3_1 + ia^3_2\right] \end{cases}
$$

and observe that $\pi\colon SL(3,\mathbf{R}) \to M_0$ is a right G'-principal fiber bundle where

G' is the subgroup of $SL(3, \mathbf{R})$ defined by the formula

$$G' = \left\{ \begin{pmatrix} a & b & c \\ -b & a & d \\ 0 & 0 & 1/(a^2+b^2) \end{pmatrix} \middle| \, a, b, e, f \in \mathbf{R} \right\}. \tag{3.2}$$

Consequently, M_0 can be identified with the homogeneous space $SL(3, \mathbf{R})/G'$.

To construct a framing of the complexified tangent bundle of $SL(3, \mathbf{R})$ for which the structure equations (3.1) hold start with the local adapted coframing of TM_0^* given by the 1-forms

$$\theta_0 = dz - \frac{z - \bar{z}}{w - \bar{w}}\, dw \quad \text{and} \quad \eta_0 = \frac{-1}{w - \bar{w}}\, dw.$$

At the point x_0, the equations $\theta_0 = dz$ and $\eta_0 = \frac{1}{2} i\, dw$ hold. These covectors pull-back to a pair of covectors at the identity element of $SL(3, \mathbf{R})$ and extend by left translation to left invariant forms, denoted by θ and η. We leave it to the reader to check the identities

$$\theta = \omega_1^3 + i\omega_2^3 \tag{3.3}$$

$$\eta = \tfrac{1}{2}\left\{\left(\omega_1^1 - \omega_2^2\right) + i\left(\omega_2^1 + \omega_1^2\right)\right\}. \tag{3.4}$$

The forms ϕ and ψ are uniquely determined by (3.1) and can be found by computing $d\theta$ and $d\eta$ using the structure equations for $SL(3, \mathbf{R})$:

$$d\omega_j^i = -\omega_k^i \wedge \omega_j^k \tag{3.5}$$

$$\omega_1^1 + \omega_2^2 + \omega_3^3 = 0. \tag{3.6}$$

The result of the computation is

$$\phi = -\frac{3}{2}\left(\omega_1^1 + \omega_2^2\right) - \frac{i}{2}\left(\omega_2^1 - \omega_1^2\right) \tag{3.7}$$

$$\psi = +\frac{1}{2}\left(\omega_3^1 + i\omega_3^2\right). \tag{3.8}$$

The skeptical reader can check directly that the forms do indeed satisfy the identities (3.1).

Remark 3.9. There is an inclusion of Lie groups $G \hookrightarrow SL(3, \mathbf{R})$. Start with the Lie algebra isomorphism $\mathfrak{g}' \cong \mathfrak{g}$ between the Lie algebras of the groups G' and G given by the formula

$$\begin{pmatrix} a & b & c \\ -b & a & d \\ 0 & 0 & -2a \end{pmatrix} \mapsto \begin{pmatrix} -3a - ib & 0 \\ \dfrac{c + id}{2} & -2ib \end{pmatrix}$$

and exponentiate to give the Lie group isomorphism

$$\begin{pmatrix} A & B & C \\ -B & A & D \\ 0 & 0 & (A^2+B^2)^{-1} \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{1}{(A^2+B^2)(A+iB)} & 0 \\ \dfrac{C+iD}{2(A+iB)} & \dfrac{A-iB}{A+iB} \end{pmatrix}.$$

The inverse map gives the sought after inclusion $G \hookrightarrow SL(3, \mathbf{R})$:

$$(3.10)\quad \begin{pmatrix} a & 0 \\ b & a/\bar{a} \end{pmatrix}$$
$$\mapsto \begin{pmatrix} \mathrm{Re}(a)/|a|^{4/3} & -\mathrm{Im}(a)/|a|^{4/3} & 2\,\mathrm{Re}(a\bar{b})/|a|^{4/3} \\ \mathrm{Im}(a)/|a|^{4/3} & \mathrm{Re}(a)/|a|^{4/3} & 2\,\mathrm{Im}(a\bar{b})/|a|^{4/3} \\ 0 & 0 & |a|^{2/3} \end{pmatrix}$$

and allows us to identify the group G with the subgroup $G' \subset SL(3, \mathbf{R})$.

The connection

Now let $(M, \mathscr{F})$ be an arbitrary complex foliation of a two dimensional complex manifold. The formulas (3.3), (3.4), (3.7) and (3.8), applied now to the canonical framing of $P(M, \mathscr{F})$ by complex 1-forms can be inverted to furnish a canonical $\mathfrak{sl}(3, \mathbf{R})$-valued 1-form, ω.

DEFINITION 3.11. The *Cartan connection* of the foliated manifold $(M, \mathscr{F})$ is the $\mathfrak{sl}(3, \mathbf{R})$-valued 1-form, ω, on the total space $P(M, \mathscr{F})$, given as follows:

$$\omega = \begin{pmatrix} \mathrm{Re}(\eta - \phi/3) & \mathrm{Im}(\eta - \phi) & 2\,\mathrm{Re}(\psi) \\ \mathrm{Im}(\eta + \phi) & \mathrm{Re}(-\eta - \phi/3) & 2\,\mathrm{Im}(\psi) \\ \mathrm{Re}(\theta) & \mathrm{Im}(\theta) & 2\,\mathrm{Re}(\phi/3) \end{pmatrix}.$$

It remains to make precise the sense in which ω is a Cartan connection. Form the quotient bundle $E = P/H$ where $H \subset G$ is the group of matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & a/\bar{a} \end{pmatrix}.$$

A section of $E \to M$ is a splitting of the exact sequence of complex vector bundles

$$0 \to L \to TM \to Q \mapsto 0.$$

The bundle π': $P \to E$ is a right H-principal bundle and the forms θ, η and ψ are horizontal with respect to the map π'.

THEOREM 3.12. *The $\mathfrak{sl}(3,\mathbf{R})$-valued 1-form ω: $TP \to \mathfrak{sl}(3,\mathbf{R})$ is a Cartan connection on the right principal H-bundle $P \to E$. Moreover, when the fundamental invariant A in the structure equations* (1.6) *vanishes ω is a Cartan connection on the right principal g-bundle, $P \to M$.*

The curvature matrix, $\Omega = d\omega + \omega \wedge \omega$, is of the form

$$\Omega = \begin{pmatrix} \Omega_1^1 & -\Omega_1^2 & \Omega_3^1 \\ \Omega_1^2 & \Omega_1^1 & \Omega_3^2 \\ 0 & 0 & -2\Omega_1^1 \end{pmatrix} \tag{3.13}$$

with

$$3\Omega_1^1 + i\Omega_1^2 = 3A\theta \wedge \bar{\eta} \tag{3.14}$$

$$\Omega_3^1 + i\Omega_3^2 = -6A\eta \wedge \bar{\eta} + 2\left(B\bar{\theta} + 2\bar{C}\eta + C\bar{\eta} + 2\bar{A}\psi + A\bar{\psi}\right) \wedge \theta. \tag{3.15}$$

Proof. Begin with the definition of a Cartan connection (see [6a, 127–128]). Let R_g denote right multiplication by an element $g \in G$ and recall that the right action of G on P can be used to associate to each element $X \in \mathfrak{g}$ a vertical vector field $X^* \in \Gamma(P, TP)$; more precisely, for $p \in P$, X_p^* is the vector

$$X_p^* \equiv \left(\frac{dR_{\exp(tX)}p}{dt}\right)_{|t=0}.$$

The form ω is a Cartan connection on $P \to E$ (resp. $P \to M$) if the following three conditions hold:

(a) The components of ω form a framing of the cotangent bundle of P.
(b) For all $X \in \mathfrak{h}$ (resp. $X \in \mathfrak{g}$), $\omega(X^*) = X$.
(c) For all $g \in H$ (resp. $g \in G$), $R_g^*\omega = \mathrm{Ad}(g^{-1})\omega$.

Condition (a) is almost immediate. By construction, the real and imaginary parts of the forms θ, η, ϕ and ψ form a framing of T^*P and the components of ω are obtained from them by a non-singular linear transformation.

To prove condition (b) work over a trivializing neighborhood, $U \subset M$, so that $P_{|U} \cong U \times G$ and recall that with respect to such a trivialization, $X^* = 0 \times X$ where X is now thought of as a left invariant vector field on G. Examining formula (2.15) and recalling that subsequent modifications to the forms ϕ and ψ only involved the addition to ψ of a multiple of the form θ reveals that the restriction to a fiber (i.e., to $x \times G$, $x \in U$) of the matrix

$$\begin{pmatrix} \phi & 0 \\ \psi & \phi - \bar{\phi} \end{pmatrix}$$

is precisely the Maurer-Cartan form $g^{-1}\,dg$ and condition (b) becomes obvious, because $g^{-1}\,dg(X) = X$ for all $X \in \mathfrak{g}$.

It remains to check condition (c). Because G and H are connected we need only check the infinitesimal version

$$\mathscr{L}_{X^*}\omega = -\operatorname{ad}(X)\omega,$$

for all $X \in \mathfrak{h}$ (resp. $X \in \mathfrak{g}$), where $\mathscr{L}$ denotes Lie differentiation. But by virtue of property (b) the equation $\operatorname{ad}(X)\omega = i(X^*)\omega \wedge \omega$ holds and $\omega(X^*) = X \in \mathfrak{g}$ is constant. Therefore, from the standard formula for the Lie derivative

$$\mathscr{L}_X^*\omega = (d \circ i(X^*) + i(X^*) \circ d)\omega,$$

condition (c) reduces to the identity

$$i(X^*)(d\omega + \omega \wedge \omega) \equiv i(X^*)\Omega = 0,$$

for all $X \in \mathfrak{h}$ (resp. $X \in \mathfrak{g}$). In other words, condition (c) holds if and only if the curvature 2-form Ω is horizontal. Inspection of the formulas for the components of Ω given previously reveals that all components are linear combinations of the forms θ, η, ϕ and their conjugates. Hence Ω is horizontal with respect to the fibration $P \to E$. When $A = 0$ all curvature components are linear combinations of θ and η and their conjugates and Ω is horizontal with respect to the fibration $P \to M$.

The curvature identities are simply a translation of the structure equations (2.27). ∎

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