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AMENABILITY AND BOUNDED APPROXIMATE IDENTITIES IN IDEALS OF $A(G)$

BY

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1. Introduction

Let G be a locally compact group. In [6], P. Eymand defined the Fourier algebra $A(G)$ of G to be the linear subspace of $C_0(G)$ (the continuous complex-valued functions on G vanishing at infinity) consisting of all functions $(f*\tilde{g})^{\vee}$, where $f, g \in L^2(G)$, $f^{\vee}(x) = f(x^{-1})$ and $\tilde{f}(x) = \overline{f(x^{-1})}$. If $f \in L^2(G)$ and $x \in G$ define $L_x f(y) = f(x^{-1}y)$ for every $y \in G$. Let $VN(G)$ denote the closure in the weak operator topology of the linear span of $\{L_x;\ x \in G\}$ in $B(L^2(G))$, the algebra of bounded linear operators on $L^2(G)$. $A(G)$ is the unique predual of the von Neumann algebra $VN(G)$, [6, pp. 210 and 218]. Furthermore $A(G)$ with pointwise multiplication and

$$\|u\|_{A(G)} = \sup\{|\langle T, u\rangle|;\ T \in VN(G), \|T\| \le 1\}$$

is a commutative Banach algebra with spectrum $\Delta(A(G)) = G$ [6, p. 222].

In case G is abelian, $A(G)$ is isometrically isomorphic, by means of the Fourier transform $\hat{}$, to $L^1(\hat{G})$, the group algebra of $\hat{G}$, the dual group of G. Liu, van Rooij and Wang proved in [22, p. 479] that if G is a locally compact abelian group and I is a closed deal in $L^1(\hat{G}) \simeq A(G)$, then I has a bounded approximate identity if and only if

$$I = I(A) = \{f \in L^1(\hat{G});\ \hat{f}(x) = 0 \quad \text{for every } x \in A\},$$

where A is closed in G and A is an element of the ring of subsets of G generated by the left cosets of subgroups of G.

In this paper, we will attempt to determine which closed ideals in $A(G)$ have bounded approximate identities for an arbitrary locally compact group G. We shall show that the answer to this question is intimately related to the amenability of G. We will also establish algebraic and topological criteria for a closed ideal to be a candidate to possess a bounded approximate identity.

We characterize the weak-$*$ closed ideals in $A(G)$ with bounded approximate identities in §4. In §5 we undertake an investigation of the cofinite ideals in $A(G)$. A number of characterizations of amenable locally compact groups

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are given, including Theorem 5.9, which answers the analogue of Willis' and Dales' "weak automatic continuity" question [4, p. 397].

In §6, we will examine Banach modules over a Banach algebra with a bounded approximate identity. These results will be applied in §7 to study the nature of $VN(G)$ as a Banach $A(G)$-bimodule. The paper culminates in §8 with some applications to discrete groups.

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2. Definitions and notations

Throughout this paper, G denotes a locally compact group with a fixed left Haar measure μ_G. If A is a measurable subset of G, then $|A|$ is the measure of A. For any subset A of G, 1_A denotes the characteristic function of A.

G is amenable if there exists $m \in L^\infty(G)^*$ such that $m \geq 0$, $m(1_G) = 1$ and $m(xf) = m(f)$ for every $x \in G, f \in L^\infty(G)$, where $xf(y) = f(xy)$, $y \in G$. All abelian groups and all compact groups are amenable. The free group on two generators is not amenable.

Let $B(G)$ be the linear span of $P(G)$, the continuous positive definite functions on G. $B(G)$ can be identified with the dual of $C^*(G)$, the group C^*-algebra of G [cf. 6, p. 192]. With pointwise multiplication and the dual norm, $B(G)$ becomes a commutative Banach algebra, called the Fourier-Stieltjes algebra of G. The Fourier algebra $A(G)$, as defined in §1, is a closed ideal of $B(G)$ [6, p. 208]. For further properties of $A(G)$, $VN(G)$ and $B(G)$ see [6].

Let $\mathscr{A}$ be a Banach algebra. A net $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ in $\mathscr{A}$ is called a *bounded left* (resp. *right*) *approximate identity* if $\lim_\alpha \|u_\alpha u - u\| = 0$ (resp. $\lim_\alpha \|u u_\alpha - u\| = 0$) for every $u \in \mathscr{A}$, and if there exists an M such that $\|u_\alpha\| < M$ for every $\alpha \in \mathfrak{A}$. $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ is a *bounded approximate identity* if it is both a left bounded approximate identity and a right bounded approximate identity.

Let $\mathscr{A}$ be a commutative Banach algebra. Let $\Delta(\mathscr{A})$ denote the maximal ideal space of $\mathscr{A}$. By means of the Gelfand transform, $\mathscr{A}$ can be realized as a subalgebra of $C_0(\Delta(\mathscr{A}))$.

Let I be an ideal in $\mathscr{A}$. Define

$$Z(I) = \{x \in \Delta(\mathscr{A}), u(x) = 0 \quad \text{for every } u \in I\}.$$

Then $Z(I)$ is a closed subset of $\Delta(\mathscr{A})$. If E is a closed subset of $\Delta(\mathscr{A})$, define

$$I(E) = \{u \in \mathscr{A}, u(x) = 0 \quad \text{for every } x \in E\},$$
$$I_0(E) = \{u \in \mathscr{A}; \operatorname{supp} u \in \mathscr{F}(\mathscr{A})\},$$

where $\mathscr{F}(E) = \{K \subset \Delta(\mathscr{A});\ K$ is compact and $K \cap E = \varnothing\}$. $I_0(E)$ and $I(E)$ are ideals in $\mathscr{A}$. $I(E)$ is closed. Furthermore, if I is any ideal in $\mathscr{A}$ with $Z(I) = E$, then $I_0(E) \subseteq I \subseteq I(E)$.

A closed subset E of $\Delta(\mathscr{A})$ is said to be a *set of spectral synthesis*, or simply an *S-set*, if $I(E)$ is the only closed ideal I for which $Z(I) = E$. This is equivalent to the density of $I_0(E)$ in $I(E)$ [cf. 14, Theorem 39.18].

$\mathscr{A}$ is said to satisfy *Ditkin's Condition* if:

(i) For every $u \in \mathscr{A}$ and $x \in \Delta(\mathscr{A})$ such that $u(x) = 0$, there exists a sequence $\{v_n\}$ in $\mathscr{A}$ such that each v_n vanishes in some neighborhood of x and

$$\lim_n \|uv_n - u\| = 0.$$

(ii) If $\Delta(\mathscr{A})$ is not compact, then, in addition to (i), for every $u \in \mathscr{A}$ there exists a sequence $\{v_n\}$ in $\mathscr{A}$ such that each v_n has compact support and

$$\lim_n \|uv_n - u\| = 0.$$

3. Amenability and bounded approximate identities

In this section, we establish the connection between amenability and the existence of bounded approximate identities in ideals of $A(G)$. For a closed ideal I in $A(G)$, we will see that the existence of a bounded approximate identity in I is dependent on the algebraic and topological properties of the set $Z(I)$ of common zeros of I.

DEFINITION 3.1. Let A, B be closed subsets of G. Let

$$\mathscr{S}(A, B) = \{u \in B(G);\ u(A) \equiv 1,\ u(B) \equiv 0\},$$

$$s(A, B) = \begin{cases} \inf\{\|u\|_{B(G)};\ u \in \mathscr{S}(A, B)\} & \text{if } \mathscr{S}(A, B) \neq \emptyset \\ \infty & \text{if } \mathscr{S}(A, B) = \emptyset, \end{cases}$$

$$\mathscr{F}(A) = \{K \subset G;\ K \text{ is compact, } K \cap A = \emptyset\},$$

$$s(A) = \sup\{s(A, K);\ K \in \mathscr{F}(A)\}.$$

Since $1_G \in B(G)$ and $\|1_G\|_{B(G)} = 1$ it is clear that if $s(A, B) < \infty$, then $s(B, A) < \infty$ and

$$|s(A, B) - s(B, A)| \leq 1.$$

If K is compact and A is a closed subset of G disjoint from K, then $s(A, K) < \infty$ by the regularity of $A(G)$.

Our principal tools in the identification of closed ideals with bounded approximate identities are the next two propositions.

PROPOSITION 3.2. *Let G be amenable. Let A be a closed set of spectral synthesis. If $s(A) < \infty$, then $I(A)$ has an approximate identity $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ which satisfies*:

(i) $\|u_\alpha\|_{A(G)} \leq 2 + s(A)$ *for every* $\alpha \in \mathfrak{A}$,
(ii) $u_\alpha \in A(G) \cap C_{00}(G)$ *for every* $\alpha \in \mathfrak{A}$,
(iii) *if* $K \in \mathscr{F}(A)$, *there exists a sequence* $\{u_{K_n}\} \subseteq \{u_\alpha\}_{\alpha \in \mathfrak{A}}$ *such that*

$$\|vu_{K_n} - v\|_{A(G)} \leq \frac{1}{n}$$

for every $v \in A(G)$ *with* $\operatorname{supp} v \subseteq K$.

Proof. Let $K \in \mathscr{F}(A)$ and $\varepsilon > 0$. Since G is amenable, by a result of H. Leptin [20], there exists a compact set $U = U_{K,\varepsilon} \subseteq G$ such that

$$|U| > 0 \quad \text{and} \quad |KU| \leq (1+\varepsilon)^2 |U|.$$

Define

$$u_{K,\varepsilon}(x) = \frac{1}{(1+\varepsilon)|U|} 1_{KU} * 1_U^{\vee}(x).$$

Then $u_{K,\varepsilon} \in A(G)$ and $\|u_{K,\varepsilon}\|_{A(G)} \leq 1$. Also $\operatorname{supp} u_{k,\varepsilon} \subseteq KUU^{-1}$ is compact. If $v \in A(G)$ with $\operatorname{supp} v \subseteq K$, then

$$u_{K,\varepsilon} v = \frac{v}{1+\varepsilon}.$$

Suppose that $s(A) < \infty$. Then there exists $w_K \in \mathscr{S}(A, K)$ with

$$\|w_K\|_{B(G)} \leq s(A) + 1.$$

Define

$$v_{K,\varepsilon} = u_{K,\varepsilon} - u_{K,\varepsilon} w_K.$$

Then $v_{K,\varepsilon} \in I(A)$ and

$$\|v_{K,\varepsilon}\|_{A(G)} \leq \|u_{K,\varepsilon}\|_{A(G)} + \|u_{K,\varepsilon}\|_{A(G)} \|w_K\|_{B(G)} \leq 2 + s(A).$$

If $x \in K$, $v_{K,\varepsilon}(x) = u_{K,\varepsilon}(x)$. Therefore, if $\operatorname{supp} v \subseteq K$,

$$v_{K,\varepsilon} v = \frac{v}{1+\varepsilon}.$$

Define a partial order on $\mathscr{F}(A) \times \mathbf{R}^+$ by $(K, \varepsilon) \geq (K_1, \varepsilon_1)$ if and only if $K \subseteq K_1$ and $\varepsilon \geq \varepsilon_1$.

Let $u \in I(A)$ and $\varepsilon > 0$. Since A is a set of spectral synthesis, $I_0(A)$ is dense in $I(A)$ [14, Theorem 39.18]. Therefore, there exists $v \in A(G)$ with

$$\operatorname{supp} v = K \in \mathscr{F}(A), \ \|v\|_{A(G)} \leq 2\|u\|_{A(G)}, \ \|u - v\|_{A(G)} < \varepsilon.$$

Let $(K_1, \varepsilon_1) \geq (K, \varepsilon)$. Then

$$\begin{aligned} \|u - v_{K_1, \varepsilon_1} u\|_{A(G)} &\leq \|u - v\|_{A(G)} + \|u - v_{K_1, \varepsilon_1} v\|_{A(G)} + \|v_{K_1, \varepsilon_1} v - v_{K_1, \varepsilon_1} u\|_{A(G)} \\ &\leq \left(s(A) + 2 + 2\|u\|_{A(G)}\right)\varepsilon. \end{aligned}$$

Hence $\{v_{K, \varepsilon}\}_{\mathscr{F}(A) \times \mathbf{R}^+}$ is the desired approximate identity. □

LEMMA 3.3. *Let A be a closed subset of G. Suppose that $I(A)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ with $\|u_\alpha\|_{A(G)} \leq M$ for every $\alpha \in \mathfrak{A}$. Then $1_{G \setminus A}$ and 1_A belong to $B(G_d)$, where G_d denotes the algebraic group G with the discrete topology. Furthermore, $\|1_{G \setminus A}\|_{B(G_d)} \leq M$ and $\|1_A\|_{B(G_d)} \leq M + 1$.*

Proof. Let $u \in I(A)$. Since $\lim_\alpha \|u_\alpha u - u\|_{A(G)} = 0$,

$$\lim_\alpha \|u_\alpha u - u\|_\infty = 0.$$

Therefore, $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ converges to $1_{G \setminus A}$ in the pointwise topology. For each $\alpha \in \mathfrak{A}$,

$$u_\alpha \in B(G_d) \quad \text{and} \quad \|u_\alpha\|_{B(G_d)} = \|u_\alpha\|_{A(G)} \leq M \quad [6, \text{p. } 199].$$

Since the set $\{v \in B(G_d);\ \|v\|_{B(G_d)} \leq M\}$ is closed in the pointwise topology [6, p. 202],

$$1_{G \setminus A} \in B(G_d) \quad \text{and} \quad \|1_{G \setminus A}\|_{B(G_d)} \leq M.$$

Consequently, $1_A = 1_G - 1_{G \setminus A} \in B(G_d)$ and $\|1_A\|_{B(G_d)} \leq M + 1$. □

DEFINITION 3.4. For any locally compact group G, let $\mathscr{R}(G)$ denote the ring of subsets of G generated by the left cosets of open subgroups G. $\mathscr{R}(G)$ is called the *coset ring* of G. Define

$$\mathscr{R}_c(G) = \{A \subset G,\ A \in \mathscr{R}(G_d) \text{ and } A \text{ is closed in } G\}.$$

PROPOSITION 3.5. *Let A be a closed subset of G. If $I(A)$ has a bounded approximate identity, then $A \in \mathscr{R}_c(G)$.*

Proof. Apply Lemma 3.3, followed by Host's non-commutative generalization of Cohen's Idempotent Theorem [15]. □

LEMMA 3.6. *Let G be a locally compact group. Let H be a closed subgroup of G which is either* (i) *open*, (ii) *compact or* (iii) *normal. Then* $s(H) = 1$.

Proof. (i) If H is an open subgroup of G, then $1_H \in B(G)$ and $\|1_H\|_{B(G)} = 12$ [6, p. 205].

(ii) Assume that H is compact. Let $K \in \mathscr{F}(H)$. Since H is a subgroup, we can find an open symmetric neighborhood V of the identity such that V^- is compact and

$$K^{-1}H \cap HV^2 = \emptyset.$$

Let

$$u_V(x) = \frac{1_{HV} * 1_{HV}^{\vee}(x)}{|HV|} = \frac{1}{|HV|}\int_G 1_{HV}(xy)1_{HV}(y)\,dy$$
$$= \frac{|x^{-1}HV \cap HV|}{|HV|}.$$

Since u_V is positive definite (cf. [6, p. 189]) and $u_V(e) = 1$, $\|u_V\|_{B(G)} = 1$. If $x \in H$, $u_V(x) = 1$, whereas if $x \in K$, $u_V(x) = 0$. Hence $u_V \in \mathscr{S}(H, K)$.

(iii) Assume that H is normal. Let $\pi\colon G \to G/H$ be the canonical homomorphism. Assume that $K \in \mathscr{F}(H)$. Then

$$\pi(K) \in \mathscr{F}_{G/H}(\{eH\}).$$

By (ii), there exists $u_0 \in B(G/H)$ with $\|u_0\|_{B(G/H)} = 1$ and $u_0(eH) = 1$, while $u_0(xH) = 0$ for every $x \in K$. Let

$$u(x) = u_0(\pi(x)).$$

Then $u \in B(G)$, $\|u\|_{B(G)} = 1$ [6, p. 199], $u(x) = 1$ if $x \in H$ and $u(x) = 0$ if $x \in K$. □

PROPOSITION 3.7. *Let G be an amenable locally compact group. Let H be a closed subgroup of G which is either* (i) *open*, (ii) *compact or* (iii) *normal. Then $I(H)$ has a bounded approximate identity.*

Proof. Closed subgroups are S-sets [33, Theorem 3]. If H satisfies (i), (ii) or (iii), then $s(H) = 1$. By Proposition 3.2, $I(H)$ has a bounded approximate identity. □

For non-amenable groups, Proposition 3.7 is no longer valid. In fact, Proposition 3.7 serves to characterize the amenable locally compact groups (see Theorem 3.9).

LEMMA 3.8. *Let H be a closed subgroup of G. Then the quotient Banach algebra $A(G)/I(H)$ is isometrically isomorphic to $A(H)$.*

Proof. Let $u \in B(G)$. Let $u|_H$ denote the restriction of u to H. Then $u|_H \in B(H)$ [6, p. 199]. If $u \in A(G)$, then $u|_H \in A(H)$ [6, p. 199]. Furthermore, if $v \in A(H)$, then there exists $u \in A(G)$ such that $v = u|_H$ and

$$\|v\|_{A(H)} = \inf\{\|u\|_{A(G)};\ u|_H = v\}$$

with the infimum actually attained [12, Theorem 16].

Define ψ: $A(G)/I(H) \to A(H)$ by

$$\psi(u + I(H)) = u|_H \quad \text{for } u \in A(G).$$

If $u + I(H) = v + I(H)$, then $u - v \in I(H)$. Therefore $u|_H - v|_H = 0$ and ψ is well defined. Clearly, ψ is an algebra homomorphism. If $u|_H = 0$, $u \in I(H)$ so ψ is 1-1. Since $A(G)|_H = A(H)$, ψ is an algebra isomorphism.

Let $u \in A(G)$. Let $\|\cdot\|_Q$ be the quotient norm on $A(G)/I(H)$. Then

$$\|u + I(H)\|_Q = \inf\{\|v\|_{A(G)};\ v|_H = u|_H\} = \|u|_H\|_{A(H)}. \qquad \square$$

THEOREM 3.9. *Let G be a locally compact group. Then the following are equivalent*:

(i) *G is amenable.*
(ii) *$I(H)$ has a bounded approximate identity for some amenable closed subgroup H of G.*

Proof. Assume that G is amenable. Let $H = \{e\}$. Then $I(H)$ has a bounded approximate identity by Proposition 3.7.

Conversely, assume that H is an amenable closed subgroup of G such that $I(H)$ has a bounded approximate identity. Leptin's theorem [21] implies that $A(H)$ has a bounded approximate identity and hence that $A(G)/I(H)$ has a bounded approximate identity. If $I(H)$ and $A(G)/I(H)$ both have bounded approximate identities, so does $A(G)$ (cf. [5, p. 173]). Therefore G is amenable. $\square$

COROLLARY 3.10. *Let G be a locally compact group. Then G is amenable if and only if $I(\{e\})$ has a bounded approximate identity.*

Corollary 3.10 is due to A.T. Lau [18, Corollary 4.11]. His techniques are entirely different from ours, in that they rely heavily on the fact that $A(G)$ is a Banach algebra which is also the predual of a von Neumann algebra.

PROPOSITION 3.11. *Let G be a locally compact group. Let A be a compact subset of G. If $I_0(A)$ has a bounded approximate identity, then G is amenable.*

Proof. If G is compact, G is amenable. Therefore we may assume that G is non-compact.

Let $K \subset G$ be compact. Let V be a compact neighborhood of e. Let

$$v(x) = \frac{1}{|V|} 1_V * 1_{V^{-1}K}(x)$$
$$= \frac{1}{|V|} \int_G 1_V(xy) 1_{V^{-1}K}(y^{-1})\, dy.$$

If $x \in K$, $v(x) = 1$. Also $\operatorname{supp} v$ is compact. Let $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ be a bounded approximate identity for $I_0(A)$ with $\|u_\alpha\|_{A(G)} \le M$. By translating A if necessary, we may assume that $A \cap \operatorname{supp} v = \emptyset$. Therefore $v \in I_0(A)$ and $\lim_\alpha \|u_\alpha v - v\|_{A(G)} = 0$. Let $\varepsilon > 0$. There exists $\alpha_0 \in \mathfrak{A}$ such that

$$\inf\left\{\operatorname{Re} u_{\alpha_0}(x);\ x \in K\right\} \ge 1 - \varepsilon.$$

Let $\varphi \in C_{00}(G)$, $\varphi \ge 0$ and $\operatorname{supp} \varphi \subseteq K$. Then

$$|\langle u_{\alpha_0}, \varphi\rangle| \le \|L\varphi\|_{cv} \|u_{\alpha_0}\|_{A(G)} \le M\|L\varphi\|_{cv},$$

where $\|L\varphi\|_{cv}$ is the norm of φ as a left convolution operator on $L^2(G)$. However

$$\operatorname{Re}\langle u_{\alpha_0}, \varphi\rangle = \int_G \left(\operatorname{Re} u_{\alpha_0}(x)\right) \varphi(x)\, dx \ge (1 - \varepsilon)\|\varphi\|_1$$

and

$$\|\varphi\|_1 \le M\|L\varphi\|_{cv}.$$

As K was arbitrary,

$$\|\psi\|_1 \le M\|L\psi\|_{cv} \quad \text{for every } \psi \in C_{00}(G),\ \psi \ge 0.$$

Given $\psi \in C_{00}(G)$, $\psi \ge 0$, we have

$$\|\psi\|_1^n = \|\psi^{*n}\|_1 \le M\|L^n_{\psi^*}\|_{cv} \le M\|L\psi\|^n_{cv}.$$

Therefore,

$$\|\psi\|_1 \leq M^{1/n}\|L\psi\|_{cv} \quad \text{for every } n.$$

It follows that $\|\psi\|_1 = \|L\psi\|_{cv}$ for every $\psi \in C_{00}(G)$, $\psi \geq 0$. This implies that G is amenable (cf. [27, p. 85]). □

It appears that the Fourier algebra of an amenable group is rich in closed ideals with bounded approximate identities while for non-amenable groups many potential candidates are eliminated. Indeed, for non-amenable groups, if $I(A)$ has a bounded approximate identity, then A must be "topologically large."

We cannot remove completely the topological restrictions imposed on A in the statement of Proposition 3.11. For example, if G is any discrete group, $1_{\{e\}}$ is an identity for $I(G \setminus \{e\})$.

Leptin's theorem [21] shows that if G is amenable, then $A(G)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ with $\|u_\alpha\|_{A(G)} \leq 1$ for each $\alpha \in \mathfrak{A}$. For ideals in $A(G)$, this is seldom true. In fact, we have:

PROPOSITION 3.12. *Let A be a closed subset of G. Then $I(A)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ with $\|u_\alpha\|_{A(G)} \leq 1$ for every $\alpha \in \mathfrak{A}$ if and only if $G \setminus A = xH$ for some $x \in G$ and some open amenable subgroup H of G.*

Proof. Suppose that $G \setminus A = xH$, where H is an open amenable subgroup of G. Since H is an open subgroup, $I(x^{-1}A)$ can be identified with $A(H)$. As H is amenable, $A(H)$ has a bounded approximate identity $\{v_\alpha\}_{\alpha \in \mathfrak{A}}$ with $\|v_\alpha\|_{A(H)} \leq 1$.

Conversely, assume that $I(A)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ with $\|u_\alpha\|_{A(G)} \leq 1$ for every $\alpha \in \mathfrak{A}$. By Lemma 3.3, $1_{G \setminus A} \in B(G_d)$ and $\|1_{G \setminus A}\|_{B(G_d)} = 1$. Let $x \in G \setminus A$. Then $G \setminus A = xH$, where H is a subgroup of G [cf. 9, p. 377]. Since A is closed, H is open. Again, we identify $A(H)$ with $I(x^{-1}A)$. As $I(x^{-1}A)$ has a bounded approximate identity, H must be amenable. □

In case $A = H$ is a closed subgroup, Proposition 3.12 shows that $I(H)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ with $\|u_\alpha\|_{A(G)} \leq 1$ if and only if H is an amenable group and H has index two in G. In particular, $I(\{e\})$ has such an approximate identity if and only if $G = \{e, x\}$.

4. Bounded approximate identities in weak-* closed ideals

Recall that $B(G)$ is the dual of $C^*(G)$. The purpose of this section is to characterize the ideals in $A(G)$ which are closed in the relative weak-*

topology that $A(G)$ inherits as a closed subspace of $B(G)$ and at the same time possess bounded approximate identities.

DEFINITION 4.1. Let $K \subset G$ be compact. Define

$$A_K(G) = \{u \in A(G), \operatorname{supp} u \subseteq K\}.$$

It is easy to see that $A_K(G)$ is a closed ideal in $A(G)$.

It is also easy to show that $A_K(G)$ is closed in the weak-$*$ topology on $B(G)$ [cf. 38, p. 464].

LEMMA 4.2. *Let $K \subset G$ be compact. If K is open, then K is the union of finitely many cosets of an open compact subgroup H of G.*

Proof. For every $x \in K$, let V_x be a symmetric neighborhood of e such that $xV_x^2 \subseteq K$. As K is compact, there exists $\{x_1, \dots, x_n\}$ such that

$$K = \bigcup_{i=1}^{n} x_i V_{x_i}.$$

Let $W = \bigcap_{i=1}^{n} V_{x_i}$. Let $y \in K$. Then

$$y = x_{i_0} v_{i_0} \quad \text{for some } x_{i_0} \in K \text{ and some } v_{i_0} \in V_{x_{i_0}}.$$

Hence

$$yW \subseteq x_{i_0} V_{x_{i_0}} W \subseteq x_{i_0} V_{x_{i_0}}^2 \subseteq K.$$

It follows that W generates an open compact subgroup H with $KH = K$. As H is open and K is compact, K is the union of finitely many cosets of H. □

We now state the main result of this section.

THEOREM 4.3. *Let I be a closed non-zero ideal in $A(G)$ which is weak-$*$ closed in $B(G)$. Then the following are equivalent:*

(i) *I has an identity.*
(ii) *I has a bounded approximate identity.*
(iii) *$I = A_K(G)$ for some compact open subset K of G.*

Furthermore, if any of the above holds, then

$$I = \bigoplus_{i=1}^{n} L_{x_i} A(H)$$

for some compact open subgroup H of G and $\{x_1, \dots, x_n\} \subseteq G$.

Proof. Clearly (i) implies (ii).

Assume that I has a bounded approximate identity $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$. We may assume that $\{u_\alpha\}$ converges in the weak-$*$ topology to some $u \in I$. Let $A = Z(I)$. Since $u \in I$, $u \in I(A)$. As I is non-zero, $G \setminus A$ is non-empty.

If $x \in G \setminus A$, there exists $v \in I$ and an open neighborhood U of x such that $|v(y)| > 0$ for every $y \in U$. Since $\lim_\alpha \|u_\alpha v - v\|_{A(G)} = 0$, $\{u_\alpha\}$ converges uniformly to 1 on U. It follows that $u = 1_{G \setminus A}$. Let $K = G \setminus A$. Then K is open. But $1_K \in A(G) \subseteq C_0(G)$, so K is also compact. It is clear that $I = A_K(G) = 1_K A(G)$. Therefore (ii) $\Rightarrow$ (iii).

Assume that K is compact and open and that $I = A_K(G)$. By Lemma 3.2.2,

$$K = \bigcup_{i=1}^{n} x_i H$$

for some open subgroup H of G. It follows that $K \in \mathscr{R}(G)$ and hence that $1_K \in A(G)$ [15]. Therefore 1_K is an identity for I so (iii) implies (i).

In each case

$$K = \bigcup_{i=1}^{n} x_i H$$

is the disjoint union of finitely many cosets of a compact open subgroup H of G. It is easy to see that

$$A_K(G) = \oplus L_{x_i} A(H). \qquad \square$$

If G is compact, then $A(G) = B(G)$ and $A(G)$ is itself a dual Banach space. In [34], K. Taylor showed that if G is a separable group with a completely reducible left regular representation, then $A(G)$ is a dual space. The "$ax + b$" group, which consists of matrices of the form

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix};\ a \in \mathbf{R}^+,\ b \in \mathbf{R} \right\},$$

is separable and has a completely reducible left regular representation, but it is not compact (cf. [16]). Therefore, if G is the "$ax + b$" group, then $A(G)$ is a dual space. As G is also amenable, we shall see that this implies that multiplication on $A(G)$ is not weak-$*$ to weak-$*$ separately continuous on bounded spheres.

PROPOSITION 4.4. *Let G be a locally compact group for which $A(G)$ is the dual of a Banach space $A_*(G)$. Let A be a closed subset of G for which $I(A)$ has a bounded approximate identity. If for each $u \in A(G)$ the map $v \mapsto uv$ is* *weak-$*$ to weak-$*$ continuous on bounded spheres, then*

$$G \setminus A^0 = \bigcup_{i=1}^{n} x_i H$$

for some open compact subgroup H of G.

Proof. Let $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ be a bounded approximate identity in $I(A)$. We may assume that $w^* - \lim_\alpha u_\alpha = u$ for some $u \in A(G)$.

Let $T \in A_*(G)$. Let $v \in I(A)$. Then

$$(*)\quad |\langle uv, T\rangle - \langle v, T\rangle| \leq |\langle uv, T\rangle - \langle u_\alpha v, T\rangle| + |\langle u_\alpha v, T\rangle - \langle v, T\rangle|.$$

As $\lim_\alpha \|u_\alpha v - v\| = 0$ and $w^* - \lim_\alpha u_\alpha v = uv$, $(*)$ can be made as small as we like. Hence

$$\langle uv, T\rangle = \langle v, T\rangle \quad \text{for every } T \in A_*(G) \text{ and every } v \in I(A).$$

It follows that $u(G \setminus A) = 1$.

Let $x \in A^0$. There exists $v \in A(G)$ with $\operatorname{supp} v \subseteq A^0$ and $v(x) = 1$. Let $T \in A_*(G)$. Then

$$\langle uv, T\rangle = \lim_\alpha \langle u_\alpha v, T\rangle = 0.$$

Therefore, $uv = 0$ and $u = 1_{G \setminus A^0}$. As $u \in A(G) \subseteq C_0(G)$, $G \setminus A^0$ is compact and open. Now apply Lemma 4.2. □

COROLLARY 4.5. *Let G be an amenabale locally compact group. Assume that $A(G)$ is a dual Banach space. Then the multiplication on $A(G)$ is weak-$*$ to weak-$*$ separately continuous on bounded spheres if and only if G is compact.*

Proof. If G is compact, $A(G) = B(G)$. It is an easy task to verify that multiplication on $B(G)$ is always weak-$*$ to weak-$*$ separately continuous on bounded spheres.

Conversely, since G is amenable, $I(\{e\})$ has a bounded approximate identity (Corollary 3.10). If multiplication on $A(G)$ is weak-$*$ to weak-$*$ separately continuous on bounded spheres, then $G \setminus \{e\}^0$ is compact by Proposition 4.4. Hence G is compact. □

COROLLARY 4.6. *Let G be the "$ax + b$" group. Let $A_*(G)$ be a predual of $A(G)$. Then multiplication on $A(G)$ is not weak-$*$ to weak-$*$ separately continuous on bounded spheres.*

5. Cofinite ideals in $A(G)$

DEFINITION 5.1. Let $\mathscr{A}$ be a Banach algebra. An ideal I in $\mathscr{A}$ is called *cofinite* if the dimension of $\mathscr{A}/I$ is finite. The dimension of $\mathscr{A}/I$ is called the *codimension* of I.

Let $I^2 = \{\sum_{i=1}^{n} u_i v_i;\ u_i, v_i \in I\}$. Then I^2 is an ideal of $\mathscr{A}$ contained in I. I is said to *factorize weakly* if $I^2 = I$. Such ideals are also called *idempotents*.

Our interest in cofinite ideals in $A(G)$ was motivated by three papers of G. Willis [35], [36] and [37]. Willis succeeded in showing that if G is non-amenable, then no closed cofinite left ideal in $L^1(G)$ has a bounded approximate identity [35]. In contrast to this result, he proved in [36] and [37] that for every locally compact group, every closed codimension one ideal is idempotent and for a large class of groups every codimension two ideal is idempotent. In this section, we will show that while the analogue of Willis' first result holds true for $A(G)$, if G is non-amenable, no closed cofinite ideal in $A(G)$ is idempotent.

We begin with a lemma that may be part of folklore.

LEMMA 5.2. *Let G be an amenable locally compact group. Then $A(G)$ satisfies Ditkin's condition.*

Proof. Condition (i) follows immediately from Lemma 3.6 and the proof of Proposition 3.2. For non-compact G, condition (ii) follows from Leptin's theorem [21]. □

PROPOSITION 5.3. *Let G be an amenable group. Let A be a closed subset of G. If* bdy (A) *contains no non-empty perfect set, then A is an S-set.*

Proof. The proposition follows immediately from Lemma 5.2 and Ditkin's theorem (cf. [14, p. 497]). □

COROLLARY 5.4. *Let G be an amenable locally compact group. Let A be a closed discrete subset of G. Then A is an S-set. In particular, every finite subset of G is an S-set and if G is discrete, every subset is an S-set.*

Proof. This is immediate from Proposition 5.3. □

COROLLARY 5.5. *Let G be an amenable discrete group. Let $A \subset G$. Then $s(A) < \infty$ if and only if $1_A \in B(G)$.*

Proof. If $1_A \in B(G)$, then $s(A) = \|1_A\|_{B(G)}$.

If $S(A) < \infty$, then $I(A)$ has a bounded approximate identity (Proposition 3.2) and $1_A \in B(G)$ by Lemma 3.3.

COROLLARY 5.6. *Let G be an amenable locally compact group. Let I be a closed cofinite ideal in $A(G)$. Then $I = I(A)$ for some finite set $A = \{x_1, \ldots, x_n\}$, where n is the codimension of I. Furthermore $I^2 = I$.*

Proof. Let $A = Z(I)$. Since I is cofinite, A must be finite. Therefore A is an S-set by Corollary 5.4 and $I = I(A)$. If $A = \{x_1, \ldots, x_n\}$, let $u_i \in A(G)$ be such that $u_i(x_i) = 1$, $u_i(x_j) = 0$ if $i \neq j$. Then $\{u_i + I(A)\}$ is a basis for $A(G)/I(A)$.

Proposition 3.2 implies that $I(A)$ has a bounded approximate identity. By Cohen's factorization theorem [14, p. 268], $I^2(A) = I(A)$. □

LEMMA 5.7. *Let G be a non-amenable locally compact group. Let $I = I(\{e\})$. Then I^2 is not closed in $A(G)$.*

Proof. $\{e\}$ is an S-set [14, p. 229] and $Z(I^2) = \{e\}$. Therefore if I^2 is closed, $I^2 = I$. Assume that $I^2 = I$. Let $v \in A(G)$. Let $u \in A(G) \cap C_{00}(G)$ with $u(e) = 1$. Then $v = uv + (v - uv)$ with $v = uv \in I$. Hence

$$v - uv = \sum_{i=1}^{n} w_i v_i \quad \text{for } w_i, v_i \in I.$$

As G is non-compact, there exists $x \in G \setminus \operatorname{supp} u$. Since $uv \in I(\{x\})$ and $I^2(\{x\}) = I(\{x\})$,

$$uv = \sum_{j=1}^{m} t_j m_j \quad \text{for } t_j, m_j \in I(\{x\}).$$

Thus

$$v = \sum_{i=1}^{n} w_i v_i + \sum_{j=1}^{m} t_j m_j \in A^2(G)$$

which is impossible by a result of Losert [23, p. 139]. □

The proof of this lemma can be easily modified to show that if G is a non-amenable locally compact group, no ideal of the form $I(\{x_1, \ldots, x_n\})$ can be idempotent.

THEOREM 5.8. *Let G be a locally compact group. Then G is amenable if and only if every cofinite ideal is of the form $I(A)$ where A is a finite subset of G.*

Proof. If G is amenable, then every closed cofinite ideal is idempotent by Corollary 5.6. By [4, Theorem 2.3], every cofinite ideal is closed and hence is of the form $I(A)$ for some finite subset A of G.

Conversely, if every cofinite ideal is closed, [4, Theorem 2.3] implies that $I^2(\{e\})$ is closed. Therefore G is amenable by Lemma 5.7. □

We can now answer the analogue of Willis and Dales' "weak" automatic continuity question [4, p. 397].

THEOREM 5.9. *Let G be a locally compact group. Then the following are equivalent*:

(i) *G is amenable.*
(ii) *Each homomorphism from $A(G)$ with finite dimensional range is continuous.*

Proof. This follows immediately from Theorem 6.8 and from [4, Theorem 2.3]. □

6. Banach modules

DEFINITION 6.1. Let $\mathscr{A}$ be a Banach algebra. By a *left Banach-$\mathscr{A}$-module* (resp. *right Banach-$\mathscr{A}$-module*, *Banach-$\mathscr{A}$-bimodule*) we will mean an algebraic left-module (resp. right-module, bimodule) X which is itself a Banach space and is such that

$$\|a \cdot x\| \le \|a\| \, \|x\|$$

$$(\text{resp. } \|x \cdot a\| \le \|x\| \, \|a\|, \quad \|a \cdot x\| \le \|a\| \, \|x\| \quad \text{and} \quad \|x \cdot a\| \le \|x\| \, \|a\|)$$

for every $x \in X$, $a \in \mathscr{A}$.

Let X and Y be left (resp. right) Banach $\mathscr{A}$-modules. A linear map Γ: $X \to Y$ is called a *left* (resp. *right*) *module homomorphism* if

$$\Gamma(u \cdot x) = u \cdot \Gamma(x) \quad (\text{resp. } \Gamma(x \cdot u) = \Gamma(x) \cdot u)$$

for every $u \in \mathscr{A}$, $x \in X$.

Let $\mathrm{Hom}_L^{\mathscr{A}}(X, Y)$ (resp. $\mathrm{Hom}_R^{\mathscr{A}}(X, Y)$) denote the continuous left (resp. right) module homomorphisms of X into Y. With respect to the usual operator norm, $\mathrm{Hom}_L^{\mathscr{A}}(X, Y)$ (resp. $\mathrm{Hom}_R^{\mathscr{A}}(X, Y)$) is a Banach space. If X and Y are Banach $\mathscr{A}$-bimodules, then we denote

$$\mathrm{Hom}_L^{\mathscr{A}}(X, Y) \cap \mathrm{Hom}_R^{\mathscr{A}}(X, Y)$$

by $\mathrm{Hom}^{\mathscr{A}}(X, Y)$. In case $X = \mathscr{A}$, $\mathrm{Hom}_L^{\mathscr{A}}(\mathscr{A}, Y)$ (resp. $\mathrm{Hom}_R^{\mathscr{A}}(\mathscr{A}, Y)$) is the space of *left* (resp. *right*) *$(\mathscr{A}, Y)$-multipliers*. If Y is a Banach $\mathscr{A}$-bimodule, then $\mathrm{Hom}^{\mathscr{A}}(\mathscr{A}, Y)$ is the space of *$(\mathscr{A}, Y)$-multipliers*.

Let X be a left (resp. right) Banach $\mathscr{A}$-module. Then X^* becomes a right (resp. left) Banach $\mathscr{A}$-module as follows:

$$\langle T \cdot u, x\rangle = \langle T, u \cdot x\rangle \quad \text{for every } u \in \mathscr{A},\ x \in X,\ T \in X^*$$
$$(\text{resp. } \langle u \cdot T, x\rangle = \langle T, x \cdot u\rangle \quad \text{for every } u \in \mathscr{A},\ x \in X,\ T \in X^*).$$

Furthermore, a simple calculation shows that if

$$\Gamma \in \operatorname{Hom}_L^{\mathscr{A}}(X, Y) \quad \big(\text{resp. } \Gamma \in \operatorname{Hom}_R^{\mathscr{A}}(X, Y)\big),$$

then

$$\Gamma^* \in \operatorname{Hom}_R^{\mathscr{A}}(Y^*, X^*) \quad \big(\text{resp. } \Gamma^* \in \operatorname{Hom}_L^{\mathscr{A}}(X^*, Y^*)\big).$$

PROPOSITION 6.2. *Let $\mathscr{A}$ be a Banach algebra with a bounded right (resp. left) approximate identity $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$. Let X be a right (resp. left) Banach $\mathscr{A}$-module. Let*

$$\Gamma \in \operatorname{Hom}_L^{\mathscr{A}}(\mathscr{A}, X^*) \quad \big(\textit{resp. } \Gamma \in \operatorname{Hom}_R^{\mathscr{A}}(\mathscr{A}, X^*)\big).$$

Then there exists $T \in X^*$ such that

$$\Gamma(u) = u \cdot T \quad (\textit{resp. } \Gamma(u) = T \cdot u)$$

for every $u \in \mathscr{A}$.

Proof. $\Gamma(u) = \lim_\alpha \Gamma(uu_\alpha)$ for every $u \in \mathscr{A}$. As $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ is bounded, we may assume that $\Gamma(u_\alpha)$ converges in the weak-$*$ topology to some $T \in X^*$. Let $x \in X$. Then

$$\begin{aligned}
\langle \Gamma(u), x\rangle &= \langle \lim_\alpha \Gamma(uu_\alpha), x\rangle \\
&= \lim_\alpha \langle \Gamma(uu_\alpha), x\rangle \\
&= \lim_\alpha \langle u \cdot \Gamma(u_\alpha), x\rangle \\
&= \lim_\alpha \langle \Gamma(u_\alpha), x \cdot u\rangle \\
&= \langle T, x \cdot u\rangle \\
&= \langle u \cdot T, x\rangle.
\end{aligned}$$

Therefore, $\Gamma(u) = u \cdot T$. The proof of the second statement is identical. □

DEFINITION 6.3. Let X be a left (resp. right) Banach $\mathscr{A}$-module. Let Y be a left (resp. right) Banach $\mathscr{A}$-submodule of X. We say that Y is left (resp. right) *invariantly complemented* if there exists a projection P from X onto Y such that

$$P \in \operatorname{Hom}_L^{\mathscr{A}}(X, Y) \quad \big(\text{resp. } P \in \operatorname{Hom}_R^{\mathscr{A}}(X, Y)\big).$$

If X and Y are both Banach $\mathscr{A}$-bimodules, Y is called *invariantly complemented* in X if there exists a projection P from X onto Y with $P \in \operatorname{Hom}^{\mathscr{A}}(X, Y)$.

Let I be a closed subspace of $\mathscr{A}$. Let

$$I^{\perp} = \{\varphi \in \mathscr{A}^*;\ \varphi(u) = 0 \quad \text{for every } u \in I\}.$$

Let X be a closed subspace of $\mathscr{A}^*$. Let

$${}^{\perp}X = \{u \in \mathscr{A};\ \varphi(u) = 0 \quad \text{for every } \varphi \in X\}.$$

If I is a closed left [resp. right] ideal in $\mathscr{A}$, then $I^{\perp}$ is a weak-$*$ closed right (resp. left) submodule of $\mathscr{A}^*$ and conversely.

PROPOSITION 6.4. *Let $\mathscr{A}$ be a Banach algebra with a bounded right (resp. left) approximate identity. Let I be a closed left (resp. right) ideal in $\mathscr{A}$. Then I has a bounded right (resp. left) approximate identity if and only if $I^{\perp}$ is right (resp. left) invariantly complemented.*

Proof. Let $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ be a bounded right approximate identity in I. We may assume that $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ converges in the weak-$*$ topology of $\mathscr{A}^{**}$. Define

$$\langle PT, u \rangle = \langle T, u \rangle - \lim_{\alpha} \langle u_\alpha, T \cdot u \rangle \quad \text{for } T \in \mathscr{A}^*,\ u \in A.$$

P is a continuous operator on $\mathscr{A}^*$ with

$$\|P\| \le 1 + \sup_{\alpha \in \mathfrak{A}} \{\|u_\alpha\|\}.$$

If $u \in I$, $\lim_\alpha \langle u_\alpha, T \cdot u \rangle = \langle T, u \rangle$, so $PT \in I^{\perp}$.

Suppose $T \in I^{\perp}$. Then, if $u \in \mathscr{A}$, $\langle uu_\alpha, T \rangle = 0$ for every $\alpha \in \mathfrak{A}$. Hence

$$\langle PT, u \rangle = \langle T, u \rangle \quad \text{for every } u \in \mathscr{A}$$

and $PT = T$. Therefore, P is a projection of $\mathscr{A}^*$ onto $I^{\perp}$.

Finally, if $u, v \in \mathscr{A}$ and $T \in \mathscr{A}^*$, then

$$\begin{aligned}
\langle (PT) \cdot u, v \rangle &= \langle PT, uv \rangle \\
&= \langle T, uv \rangle - \lim_{\alpha} \langle u_\alpha, T \cdot uv \rangle \\
&= \langle T \cdot u, v \rangle - \lim_{\alpha} \langle u_\alpha, (T \cdot u) v \rangle \\
&= \langle P(T \cdot u), v \rangle.
\end{aligned}$$

Therefore, $P \in \operatorname{Hom}_R^{\mathscr{A}}(\mathscr{A}^*, I^{\perp})$.

Conversely, assume that $I^\perp$ is right invariantly complemented and that $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ is a bounded right approximate identity for $\mathscr{A}$. Let $P \in \mathrm{Hom}_R^{\mathscr{A}}(\mathscr{A}^*, I^\perp)$ be a projection of $\mathscr{A}^*$ onto $I^\perp$. Then

$$(1 - P) \in \mathrm{Hom}_R^{\mathscr{A}}(\mathscr{A}^*, \mathscr{A}^*),$$

where 1 denotes the identity operator on $\mathscr{A}^*$. We have

$$(1 - P)^* \in \mathrm{Hom}_L^{\mathscr{A}}(\mathscr{A}^{**}, \mathscr{A}^{**})$$

and $(1 - P)^*$ is a projection of $\mathscr{A}^{**}$ onto $(I^\perp)^\perp = I^{-w^*}$, the weak-$*$ closure of I in $\mathscr{A}^{**}$. Since

$$(1 - P)^* \in \mathrm{Hom}_L^{\mathscr{A}}(\mathscr{A}, \mathscr{A}^{**}),$$

by Proposition 7.2, there exists $\Gamma_0 \in A^{**}$ such that

$$(1 - P)^*(u) = u \cdot \Gamma_0 \quad \text{for every } u \in \mathscr{A}.$$

Furthermore, we may assume that $\Gamma_0 = w^* - \lim_\alpha(1 - P)^*(u_\alpha)$ and $\Gamma_0 \in (I^\perp)^\perp$. Let $u \in I$. If $T \in \mathscr{A}^*$, then

$$\begin{aligned}\langle u \cdot \Gamma_0, T\rangle &= \langle \Gamma_0, T \cdot u\rangle \\ &= \lim_\alpha \langle (1 - P)^*(u_\alpha), T \cdot u\rangle \\ &= \lim_\alpha \langle u_\alpha, (1 - P)(T \cdot u)\rangle \\ &= \lim_\alpha \langle uu_\alpha, (1 - P)T\rangle \\ &= \langle u, (1 - P)T\rangle \\ &= \langle u, T\rangle.\end{aligned}$$

Therefore, Γ_0 is a right identity for I^{-w^*}.

There exists a bounded net $\{v_\beta\}_{\beta\in B}$ which converges in the weak-$*$ topology of Γ_0. Therefore $\{v_\beta\}_{\beta\in B}$ is a bounded weak right approximate identity in I. Hence I must also have a bounded approximate identity (cf. [2, p. 58]). □

LEMMA 6.5. *Let $\mathscr{A}$ be a Banach algebra with a right (resp. left) bounded approximate identity $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$. Let $\Gamma \in \mathrm{Hom}_L^{\mathscr{A}}(\mathscr{A}, X)$ (resp. $\Gamma \in \mathrm{Hom}_R^{\mathscr{A}}(\mathscr{A}, X)$). Let i be a weak-$*$ limit point of $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ in $\mathscr{A}^{**}$. If π: $X \to X^{**}$ is the canonical embedding, then*

$$\pi(\Gamma(u)) = u \cdot \Gamma^{**}(i) \quad (\textit{resp. } \pi(\Gamma(u)) = \Gamma^{**}(i) \cdot u)$$

for every $u \in \mathscr{A}$.

Proof. Let $\Gamma \in \mathrm{Hom}_L^{\mathscr{A}}(\mathscr{A}, X)$. Then

$$w^* - \lim_\alpha \Gamma^{**}(\pi(u_\alpha)) = \Gamma^{**}(i).$$

Hence, for every $u \in \mathscr{A}$ and $T \in X^*$,

$$\langle u \cdot \Gamma^{**}(i), T \rangle = \lim_\alpha \langle u \cdot \Gamma^{**}(\pi(u_\alpha)), T \rangle.$$

Therefore,

$$\pi(\Gamma(u)) = w^* - \lim_\alpha \pi(\Gamma(u \cdot u_\alpha)) = w^* - \lim_\alpha \pi(u \cdot \Gamma(u_\alpha)) = u \cdot \Gamma^{**}(i).$$

□

The next proposition is due to Gulick, Liu and van Rooij [11, p. 142] for $\mathscr{A} = L^1(G)$. It is easy to see that their proof carries over to any Banach algebra $\mathscr{A}$ with a bounded approximate identity.

PROPOSITION 6.6. *Let $\mathscr{A}$ be a Banach algebra with a right (resp. left) approximate identity $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ such that $\|u_\alpha\|_{\alpha \in \mathfrak{A}} \le C$ for every $\alpha \in \mathfrak{A}$. Then there exists a linear map $\mathscr{M}$: $\mathrm{Hom}_L^{\mathscr{A}}(\mathscr{A}, X) \to (X^* \cdot \mathscr{A})^*$ (resp. $(\mathscr{A} \cdot X^*)^*$) such that*

$$\|\Gamma\| \le \|\mathscr{M}\Gamma\| \le C\|\Gamma\|$$

for every $\Gamma \in \mathrm{Hom}_L^{\mathscr{A}}(\mathscr{A}, X)$. Furthermore, $\mathscr{M}$ is onto if and only if $\mathscr{A} \cdot X^{**} \subseteq \pi(X)$ (resp. $X^{**}\mathscr{A} \subseteq \pi(X)$).

7. Invariant projections on $VN(G)$

We now apply the results of Section 6 to the algebra $A(G)$.

PROPOSITION 7.1. *Let $A \subset G$ be closed. Suppose that $I(A)$ has a bounded approximate identity. Then there exists a projection P of $VN(G)$ onto $I(A)^\perp$ such that $u \cdot P(T) = P(u \cdot T)$ for every $u \in A(G)$, $T \in VN(G)$.*

Proof. This is simply Proposition 6.4, if we observe that the existence of a bounded approximate identity for $\mathscr{A}$ is not used in the "only if" direction of the proof. □

M. Bekka showed that if G is any locally compact group and I is any closed ideal in $L^1(G)$, then I has a bounded approximate identity if and only if

$$I^\perp = \left\{ g \in L^\infty(G);\ \int_G g(x) f(x)\, d\mu_G(x) = 0 \quad \text{for every } f \in I \right\}$$

is the range of a continuous projection on $L^\infty(G)$ which commutes with the left module action of $L^1(G)$ on $L^\infty(G)$ [1]. We show that the analogue of Bekka's theorem holds for $A(G)$ when G is an amenable group. Moreover, the class of amenable groups can be characterized by the equivalence of these two statements.

THEOREM 7.2. *Let G be an amenable locally compact group. Let X be a weak-* closed $A(G)$-submodule of $VN(G)$. Then the following are equivalent:*
(i) *X is invariantly complemented.*
(ii) *${}^\perp X$ has a bounded approximate identity.*
Furthermore, if G is any locally compact group for which ${}^\perp X$ has a bounded approximate identity whenever X is a weak- closed invariantly complemented submodule of $VN(G)$, then G is amenable.*

Proof. The first statement is Proposition 6.4. The second statement follows from the observation that $X = \{0\}$ is weak-* closed and invariantly complemented, while $A(G) = {}^\perp X$ has a bounded approximate identity if and only if G is amenable. □

DEFINITION 7.3. We denote $\langle A(G) \cdot VN(G)\rangle^-$ by $UCB(\hat{G})$. The C^*-algebra $UCB(\hat{G})$ was introduced by E. Granirer, who studied its properties in [10]. If G is amenable, Cohen's factorization theorem implies that $UCB(\hat{G}) = A(G) \cdot VN(G)$.

PROPOSITION 7.4. *Let G be amenable. Let*

$$\Gamma \in \mathrm{Hom}^{A(G)}\big(UCB(\hat{G}), UCB(\hat{G})\big).$$

Then there exists $\Gamma_0 \in \mathrm{Hom}^{A(G)}(VN(G), VN(G))$ such that

$$\Gamma_0|_{UCB(\hat{G})} = \Gamma \quad \textit{and} \quad \|\Gamma\| = \|\Gamma_0\|.$$

Proof. Let $\{u_\alpha\}_{\alpha\in\mathfrak{A}}$ be a bounded approximate identity for $A(G)$ with $\|u_\alpha\|_{A(G)} \leq 1$ for each α. Given $\alpha \in \mathfrak{A}$ define a bilinear form

$$\Lambda_\alpha\colon VN(G) \times A(G) \to \mathbf{C}$$

by

$$\Lambda_\alpha(T, u) = \langle \Gamma(u_\alpha \cdot T), u\rangle.$$

Then $\|\Lambda_\alpha\| \leq \|\Gamma\|$ for each $\alpha \in \mathfrak{A}$. It follows from an argument similar to the proof of the Banach-Alaoglu theorem (cf. [31, p. 66]) that there exists a subnet

$\{\Lambda_{\alpha_k}\}$ of $\{\Lambda_\alpha\}$ and a bilinear form Λ_0: $VN(G) \times A(G) \to \mathbf{C}$ such that $\|\Lambda_0\| \leq \|\Gamma\|$ and Λ_{α_k} converges pointwise to Λ_0.

Define Γ_0: $VN(G) \to VN(G)$ by

$$\langle \Gamma_0(T), u \rangle = \Lambda_0(T, u) \quad \text{for every } T \in VN(G),\ u \in A(G).$$

Then $\|\Gamma_0\| \leq \|\Gamma\|$. If $T \in UCB(\hat{G})$, then $\|u_{\alpha_k} \cdot T - T\|_{VN(G)} \to 0$. Hence,

$$\lim_{\alpha} \langle \Gamma(u_{\alpha_k} \cdot T), u \rangle = \langle \Gamma(T), u \rangle,$$

so $\Gamma_0|_{UCB(\hat{G})} = \Gamma$. □

PROPOSITION 7.5. *Let G be an amenable locally compact group. Let X be a weak-* closed $A(G)$-submodule of $VN(G)$. Then X is invariantly complemented in $VN(G)$ if and only if $X \cap UCB(\hat{G})$ is invariantly complemented in $UCB(\hat{G})$.*

Proof. Let P be an invariant projection of $VN(G)$ onto X. Let

$$T = u \cdot T_1 \in UCB(\hat{G}).$$

Then

$$P(T) = P(u \cdot T_1) = u \cdot P(T_1) \in (A(G) \cdot VN(G)) \cap X$$

and hence $P|_{UCB(\hat{G})}$ is an invariant projection of $UCB(\hat{G})$ onto $UCB(\hat{G}) \cap X$.

Conversely, let P be an invariant projection of $UCB(\hat{G})$ onto $UCB(\hat{G}) \cap X$. Let P_0 be the extension of P to $VN(G)$ constructed in the proof of Proposition 7.4 with respect to the bounded approximate identity $\{u_\alpha\}_{\alpha \in \mathfrak{A}}$ of $A(G)$.

Let $u \in {}^{\perp}X$ and $T \in VN(G)$. Then $P(u_\alpha T) \in X$ and

$$\langle P_0(T), u \rangle = \lim_{\alpha} \langle P(u_\alpha \cdot T), u \rangle = 0.$$

Therefore, $P_0 T \in ({}^{\perp}X)^{\perp} = X$.

If $T \in X$, then $u_\alpha T \in UCB(\hat{G}) \cap X$. Therefore,

$$\begin{aligned} \langle P_0(T), u \rangle &= \lim_{\alpha} \langle P(u_\alpha \cdot T), u \rangle \\ &= \lim_{\alpha} \langle u_\alpha \cdot T, u \rangle \\ &= \langle T, u \rangle \quad \text{for every } u \in A(G). \end{aligned}$$

Hence P_0 is a projection of $VN(G)$ onto X. □

We do not know whether the assumption that G be amenable is necessary in either Proposition 7.4 or 7.5.

8. Applications to discrete groups

We close this chapter with some applications to discrete groups. The first result is an analogue of Lau and Losert's [19, Corollary 4].

PROPOSITION 8.1. *Let G be a discrete amenable group. Then G has the following property.*

($*$) *If X is a weak-$*$ closed invariantly complemented subspace of $VN(G)$, then there exists a weak-$*$ to weak-$*$ continuous projetion P from $VN(G)$ onto X such that*

$$P(u \cdot T) = u \cdot P(T) \quad \textit{for every } u \in A(G),\ T \in VN(G).$$

Conversely, if G is a locally compact group with property ($*$), then G is discrete.

Proof. Let G be discrete and amenable. Let X be a weak-$*$ closed invariantly complemented subspace of $VN(G)$. By Theorem 7.2, ${}^{\perp}X$ has a bounded approximate identity, so ${}^{\perp}X = I(A)$ for some $A \in \mathscr{R}(G)$ and $1_A \in B(G)$. Define P: $VN(G) \to X$ by

$$P(T) = 1_A \cdot T.$$

P is indeed the desired projection.

Suppose that G has property ($*$). As $\langle L_e \rangle$, the 1-dimensional linear span of $\{L_e\}$, is invariantly complemented [19, Theorem 2], property ($*$) implies that there exists a weak-$*$ to weak-$*$ continuous invariant projection P_0 of $VN(G)$ onto $\langle L_e \rangle$. If $u, v \in A(G)$, $P_0^*(u) \in A(G)$ and

$$\langle P_0^*(uv), T\rangle = \langle P_0(T), uv\rangle = \langle P_0(u \cdot T), v\rangle = \langle u \cdot P_0^*(v), T\rangle.$$

Therefore, $P_0^*|_{A(G)} \in \mathrm{Hom}^{A(G)}(A(G), A(G))$. There exists a continuous function u_0 on G such that

$$P_0^*(u) = u_0 u \quad \text{for every } u \in A(G).$$

Let $x_0 \in G$. Then

$$\begin{aligned}\langle P_0(L_{x_0}), u\rangle &= \langle P_0^*(u), L_{x_0}\rangle \\ &= \langle uu_0, L_{x_0}\rangle \\ &= u_0(x_0)u(x_0) \\ &= u_0(x_0)\langle L_{x_0}, u\rangle.\end{aligned}$$

As P_0 is a projection onto $\langle L_e \rangle$ and $L_{x_0} \notin \langle L_e \rangle$, we have $u_0(x_0) = 0$. Therefore $u_0 = 1_{\{e\}}$ and G is discrete. □

LEMMA 8.2. *Let G be an amenable discrete group and let*

$$\Gamma \in \mathrm{Hom}^{A(G)}(VN(G), VN(G)).$$

Then Γ *is weak-* to weak-* continuous.*

Proof. Let $u \in A(G)$. Since G is amenable, $u = wv$ for some $w, v \in A(G)$. Let $T \in VN(G)$. Then

$$\begin{aligned} \langle \Gamma^*(u), T \rangle &= \langle \Gamma^*(wv), T \rangle \\ &= \langle w, v\Gamma(T) \rangle \\ &= \langle v \cdot \Gamma^*(w), T \rangle. \end{aligned}$$

By [17, Theorem 3.7], $\Gamma^*(u) = v \cdot \Gamma^*(w) \in A(G)$. □

PROPOSITION 8.3. *Let G be an amenable discrete group. Let P be a continuous projection of* $VN(G)$ *onto a weak-* closed* $A(G)$*-submodule X of* $VN(G)$. *If*

$$P \in \mathrm{Hom}^{A(G)}(VN(G), VN(G)),$$

then $X^{\perp} = I(A)$ *for some* $A \in R(G)$. *Furthermore,*

$$P(T) = 1_A \cdot T \quad \textit{for every } T \in VN(G).$$

Proof. Since G is amenable and discrete, P is weak-* to weak-* continuous by Lemma 8.2. Therefore, there exists a function u_0 on G such that

$$P^*(u) = u_0 u \quad \text{for every } u \in A(G).$$

Let $u \in A(G)$, $T \in VN(G)$. Then

$$\langle P(T), u \rangle = \langle P^*(u), T \rangle = \langle u_0 u, T \rangle = \langle u, u_0 \cdot T \rangle,$$

so $P(T) = u_0 \cdot T$. Since P is a projection, $u_0 = 1_A$ for some A and $A = Z({}^{\perp}X)$. As X is invariantly complemented and G is amenable, Theorem 7.2 shows that $I(A)$ has a bounded approximate identity. By Proposition 3.5, $A \in \mathscr{R}(G)$. □

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