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ITERATED INTEGRALS AND EPSTEIN ZETA FUNCTIONS WITH HARMONIC RATIONAL FUNCTION COEFFICIENTS

BY

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In memory of Kuo-Tsai Chen

1. Introduction

Theta functions (of one complex variable z in the upper half plane) with harmonic polynomial coefficients are well known ([4], [5], see (2.1) below). They satisfy a transformation formula (2.2) under $z \to -z^{-1}$ and their Mellin transforms are Epstein zeta functions of s (see (2.3), (2.4)) which satisfy a corresponding functional equation (2.5) under $s \to k - s$ (k a constant). Using Chen's iterated integrals we find in this paper theta functions with certain harmonic *rational function* coefficients which (when a polynomial coefficient theta function is added) satisfy the same transformation formula (but they are not modular forms). Corresponding Epstein zeta functions satisfy the classical functional equation (2.9 below). We study a particular example related to the Fermat quartic F_4: $X^4 + Y^4 = 1$ and its Jacobian $J(F_4)$ [1]. Here the value at $s = 1$ of the Epstein zeta function with rational function coefficients divided by the product of the L-functions of two elliptic curves (namely $Y^2 = X^3 \pm 4X$) generates the Abel Jacobi image, in $\mathbf{C}/\mathbf{Z}(i)$, of the 1-cycle in $J(F_4)$ given by $[F_4] - [\iota(F_4)]$. (We consider only the Abel-Jacobi image in

$$H^{3,0}(J(F_4))^*/H_3(J(F_4);\mathbf{Z})).$$

Section 2

We recall now the formulas defining the theta and Epstein zeta functions associated to a real symmetric positive definite $h \times h$ matrix Q, two vectors $A, B \in \mathbf{R}^h$, and a (non-zero) homogeneous polynomial $P(X)$ of degree g in h

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variables with complex coefficients, which is *harmonic*:

$$\sum_{i=1}^{h} \frac{\partial^2 P(x_1, \ldots, x_h)}{\partial x_i^2} = 0.$$

The theta function of $z = x + iy$, $y > 0$, is

(2.1)
$$\theta^{P,Q}\begin{bmatrix} A \\ B \end{bmatrix}(z) = \sum_{N \in \mathbf{Z}^h} P(\sqrt{Q}(N + A))\exp(i\pi z Q[N + A] + 2i\pi(N + A)'B)$$

where X' is the transpose of X and $Q[X] = X'QX$. h will always be *even*. Further we assume that as function of y, θ decreases at least like e^{-ky} as $y \to \infty$ and like $e^{-l/y}$ as $y \to 0$ ($k, l > 0$); in other words, the constant term in 2.1 is zero and the same for the transformed series $\theta(-z^{-1})$ (see 2.2): equivalently either P is non-constant or A and B are non-integral. This will assure convergence of all integrals we will write. The transformation formula (equivalent to [5], Prop. 8) is

(2.2)
$$\theta^{P,Q}\begin{bmatrix} A \\ B \end{bmatrix}(-z^{-1}) = (-1)^{g+h/2} i^{h/2} z^{g+h/2} (\det Q)^{-1/2} \exp(2\pi i A'B)\, \theta^{P,Q^{-1}}\begin{bmatrix} -B \\ A \end{bmatrix}(z).$$

By taking the Mellin transform of (2.1) we obtain the Epstein zeta function $\zeta(s, A, B, Q, P)$:

$$\int_0^\infty y^{s-1}\theta^{P,Q}\begin{bmatrix} A \\ B \end{bmatrix}(iy)\, dy = \pi^{-s}\Gamma(s)\zeta(s, A, B, Q, P) \tag{2.3}$$

which we denote as $\xi(s, A, B, Q, P)$. Thus

(2.4)
$$\zeta(s, A, B, Q, P) = \sum_{N \in \mathbf{Z}^h} \frac{P(\sqrt{Q}(N + A))\exp(2\pi i(N + A)'B)}{Q[N + A]^s} \quad \text{for } \operatorname{Re} s > 1.$$

$\xi(s)$ is an entire function of s and by (2.2) satisfies

(2.5)
$$\xi(s, A, B, Q, P) = i^{-g}(\det Q)^{-1/2} \exp(2\pi i A'B)\, \xi(g + \tfrac{1}{2}h - s, -B, A, Q^{-1}, P).$$

We can now state our generalization from polynomials P to certain rational harmonic functions R. In the simplest case, R will be as follows: for $i = 1,2$ let $P_i(X_i)$ be harmonic polynomials in h_i variables, homogeneous of degree g_i and assume

$$g_i + \tfrac{1}{2}h_i = 2 \tag{2.6}$$

(so either $h_i = 2$, $g_i = 1$ or $h_i = 4$, $g_i = 0$). Let $X = (X_1, X_2)$, an h dimensional variable where $h = h_1 + h_2$ and let $g = g_1 + g_2 - 2$. Define R by

$$R(X) = \frac{2P_1(X_1)P_2(X_2)}{X_1'X_1}. \tag{2.7}$$

$R(X)$ and a whole series of similar rational functions will be shown to be harmonic (3.1). $R(X)$ has degree $g = g_1 + g_2 - 2$ and $g + \frac{1}{2}h = 2$ again. Given A_i, B_i, Q_i, P_i for $i = 1,2$, let

$$A = (A_1, A_2), B = (B_1, B_2), Q = Q_1 \oplus Q_2 \quad \text{(block direct sum)}$$

and define $\theta^{R,Q}\begin{bmatrix} A \\ B \end{bmatrix}$ by (2.1) with P replaced by R and similarly $\xi(s, A, B, Q, R)$ by (2.3), (2.4). Note that the denominator of $R(\sqrt{Q}(N + A))$ will only vanish when the numerator vanishes, by our earlier assumption on the "vanishing of the constant term" so in the series (2.1) or (2.4) these terms are to be taken as zero. Let now

(2.8)
$$\begin{aligned} &Z(s, A, B, Q, R) \\ &\quad = \zeta(s, A, B, Q, R) - \zeta(1, A_1, B_1, Q_1, P_1)\zeta(s, A_2, B_2, Q_2, P_2). \end{aligned}$$

Then this Dirichlet series satisfies the functional equation

$$\begin{aligned} \pi^{-s}\Gamma(s)Z(s, A, B, Q, R) &= -i^{-g}(\det Q)^{-1/2}\exp(2\pi i A'B)\pi^{-(2-s)}\Gamma(2-s) \\ &\quad \times Z(2-s, -B, A, Q^{-1}, R). \end{aligned} \tag{2.9}$$

(Recall the main assumptions: P_i are harmonic polynomials of degree g_i in h_i variables with $g_i + \frac{1}{2}h_i = 2$, $R(X) = 2P_1(X_1)P_2(X_2)/X_1'X_1$, A_i, B_i are real vectors, Q_i are real symmetric positive definite $h_i \times h_i$ matrices with h_i even, in fact, $h_i = 2$ or 4 and ζ are defined by (2.3) or (2.4) with $P = P_i$ or $P = R$.) We prove (2.9) in the next section where we also discuss more general rational harmonic functions. Finally we have formulas corresponding to (2.8), (2.9) for the theta functions: with the same notation, define

(2.10)

$$\tilde{\theta}^{R,Q}\begin{bmatrix} A \\ B \end{bmatrix}(z) = \theta^{R,Q}\begin{bmatrix} A \\ B \end{bmatrix}(z) - \tfrac{1}{2}\left(\int_{i\infty}^{0}\theta^{P_1,Q_1}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}(z)\,dz\right)\theta^{P_2,Q_2}\begin{bmatrix} A_2 \\ B_2 \end{bmatrix}(z).$$

Then

(2.11)

$$\tilde{\theta}^{R,Q}\begin{bmatrix} A \\ B \end{bmatrix}\left(-\frac{1}{z}\right) = i^{h/2}(\det Q)^{-1/2}\exp(2\pi iA'B)\,z^2\tilde{\theta}^{R,Q^{-1}}\begin{bmatrix} -B \\ A \end{bmatrix}(z).$$

Section 3

Examples of rational harmonic homogeneous functions of degree g in h variables can be built up as follows:

(3.1) LEMMA. *If $R_i(X_i)$, $i = 1, 2$ are harmonic functions of degree g_i in h_i variables (h_i even) with $g_i + \frac{1}{2}h_i = 2$, then*

$$R(X_1, X_2) = R_1(X_1)R_2(X_2)/X_1'X_1$$

is harmonic of degree $g = g_1 + g_2 - 2$ in $h = h_1 + h_2$ variables, with $g + \frac{1}{2}h = 2$.

Proof. Since the two set of variables are disjoint and R_2 is harmonic, R will be harmonic if and only if R_1/r_1^2 (where $r_1^2 = X_1'X_1$) is harmonic: we will check that this holds (for harmonic R_1) if and only if $g_1 + h_1/2 = 2$.

Denoting by Δ the laplacian and ∇ the gradient, we have the following identities:

$$\Delta(FG) = \Delta(F)G + F\Delta(G) + 2\nabla F\cdot\nabla G,$$

$$\Delta\left(\frac{1}{G}\right) = 2\frac{\nabla G\cdot\nabla G - G\Delta G}{G^3},$$

$$\frac{G}{F}\Delta\left(\frac{F}{G}\right) = \frac{\Delta F}{F} - \frac{\Delta G}{G} + 2\left(\frac{\nabla G}{G}\cdot\frac{\nabla G}{G} - \frac{\nabla F}{F}\cdot\frac{\nabla G}{G}\right).$$

Now suppose $\Delta F = 0$ and $G = r^2 = X'X$; then $\nabla G = 2X$ and $\Delta G = 2n$ (where n = number of variables). Suppose F is homogeneous of degree k, so $\nabla G\cdot\nabla F = 2kF$; then $\Delta(F/G) = 0$ if and only if $\deg F + \frac{1}{2}n = 2$.

The lemma can be used to start an inductive construction by taking R_i as harmonic polynomials P_i. For instance we get

$$R(X) = P_1(X_1)P_2(X_2)\dots P_k(X_k)/r_1^2(r_1^2 + r_2^2)\dots(r_1^2 + \cdots + r_{k-1}^2)$$

where $r_i^2 = X_i'X_i$. Note that $R(X)$ is not continuous on the unit sphere $X'X = 1$.

Next we recall some formulas concerning iterated integrals of 1-forms α_i along paths l. If $\alpha_i = f_i(t)\,dt$ where l is parametrized by $0 \le t \le 1$, then we let

$$(3.2)\qquad \int_l(\alpha_1,\dots,\alpha_k) = \int_{0\le t_1\le\cdots\le t_k\le 1} f_1(t_1)\dots f_k(t_k)\,dt_1\dots dt_k.$$

If $l = l_1 l_2$ (path l_1 followed by path l_2, where the end of l_1 is the beginning of l_2) then

$$(3.3)\qquad \begin{aligned}\int_{l_1 l_2}(\alpha_1,\dots,\alpha_k) &= \sum_{i=0}^{k}\int_{l_1}(\alpha_1,\dots,a_i)\int_{l_2}(\alpha_{i+1},\dots,\alpha_k)\\ &= \int_{l_1}(\alpha_1,\dots,\alpha_k) + \cdots + \int_{l_2}(a_1,\dots,\alpha_k).\end{aligned}$$

If l^{-1} is the path l run backwards, then $\int_{ll^{-1}} = 0$. Combining this with (3.3) we get a series of formulas: define

$$(3.4)\qquad \begin{aligned} I(l;\alpha_1,\alpha_2,\dots,\alpha_k) &= \int_l(\alpha_1,\dots,\alpha_k),\\ \tilde{I}(l;\alpha_1) &= I(l;\alpha_1),\\ \tilde{I}(l;\alpha_1,\alpha_2) &= I(l,\alpha_1,\alpha_2) - \tfrac{1}{2}\tilde{I}(l,\alpha_1)\tilde{I}(l,\alpha_2),\\ \tilde{I}(l;\alpha_1,\alpha_2,\alpha_3) &= I(l;\alpha_1,\alpha_2,\alpha_3) - \tfrac{1}{2}\tilde{I}(l;\alpha_1)\tilde{I}(l;\alpha_2,\alpha_3)\\ &\quad - \tfrac{1}{2}\tilde{I}(l;\alpha_1,\alpha_2)\tilde{I}(l,\alpha_3).\end{aligned}$$

Then using (3.3) with $l_1 = l$, $l_2 = l^{-1}$ we find

$$(3.5)\qquad \tilde{I}(l;\alpha_1,\dots,\alpha_k) = -\tilde{I}(l^{-1};\alpha_1,\dots,\alpha_k)\quad \text{for } k = 1,2,3;$$

e.g., for $k = 2$,

$$\int_l(\alpha_1,\alpha_2) - \tfrac{1}{2}\int_l\alpha_1\int_l\alpha_2 = -\left[\int_{l^{-1}}(\alpha_1,\alpha_2) - \tfrac{1}{2}\int_{l^{-1}}\alpha_1\int_{l^{-1}}\alpha_2\right].$$

Let now

$$\alpha_i = \theta^{P_i,Q_i}\begin{bmatrix}A_i\\B_i\end{bmatrix}(q)\,\frac{dq}{q}$$

where $q = \exp(2\pi i z)$ as before and $\alpha_i = 0$ at $q = 0$ and $q = 1$. Then

$$\left(\int_0^q \alpha_1\right)\alpha_2 = \theta^{R,Q}\begin{bmatrix} A \\ B \end{bmatrix}(q)\frac{dq}{q}$$

where $R = 2P_1(X_1)P_2(X_2)/r_1^2$.

Let $W(q) = \exp(-2\pi i z^{-1})$ and write

$$W^*(\alpha_i) = \theta(W(q))\frac{dW(q)}{W(q)};$$

then (2.2) states $W^*(\alpha_i) = c_i\alpha_i^*$, where

$$\alpha_i^* = \theta^{P_i, Q_i^{-1}}\begin{bmatrix} -B_i \\ A_i \end{bmatrix}(q)\frac{dq}{q},$$

$$c_i = (-1)^{g_i + h_i/2} i^{h_i/2}(\det Q_i)^{-1/2}\exp(2\pi i A_i' B_i).$$

To prove (2.11), we calculate

$$\begin{aligned} W^*\left(\left[\int_0^q \alpha_1\right]\alpha_2\right) &= \left(\int_0^{W(q)}\alpha_1\right)W^*(\alpha_2) \\ &= \left[\int_0^1 \alpha_1 + \int_1^{W(q)}\alpha_1\right]W^*(\alpha_2). \end{aligned}$$

But $W(0) = 1$, $W(1) = 0$ and so

$$\int_0^1 \alpha_1 = -\int_0^1 W^*\alpha_1 = -c_1\int_0^1 \alpha_1^*,$$

$$\int_1^{W(q)}\alpha_1 = \int_{W(0)}^{W(q)}\alpha_1 = \int_0^q W^*\alpha_1 = c_1\int_0^q \alpha_1^*.$$

Then

$$W^*\left(\left[\int_0^q \alpha_1\right]\alpha_2\right) = c_1c_2\left(\int_0^q \alpha_1^*\right)\alpha_2^* - c_1c_2\left(\int_0^1 \alpha_1^*\right)\alpha_2^*$$

and adding

$$-\tfrac{1}{2}\left(\int_0^1 \alpha_1\right)(W^*\alpha_2) = \tfrac{1}{2}c_1c_2\left(\int_0^1 \alpha_1^*\right)\alpha_2^*$$

we have

$$W^*\left(\left[\int_0^q \alpha_1\right]\alpha_2 - \tfrac{1}{2}\left(\int_0^1 \alpha_1\right)\alpha_2\right) = c_1c_2\left(\left[\int_0^q \alpha_1^*\right]\alpha_2^* - \tfrac{1}{2}\left(\int_0^1 \alpha_1^*\right)\alpha_2^*\right)$$

which is just (2.11).

To prove (2.9), let

$$\alpha_1 = \theta^{P_1,Q_1}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}(q)\frac{dq}{q}, \quad \alpha_2 = y^{s-1}\theta^{P_2,Q_2}\begin{bmatrix} A_2 \\ B_2 \end{bmatrix}(q)\frac{dq}{q},$$

$$q = e^{-2\pi y}, \quad \frac{dq}{q} = -2\pi\, dy, \quad W^*\alpha_1 = c_1\alpha_1^*, \quad W^*\alpha_2 = c_2 y^{1-s}\alpha_2^*.$$

Then

$$\xi(s, A, B, Q, R) = \int_{y=0}^{\infty} \theta^{R,Q}\begin{bmatrix} A \\ B \end{bmatrix}(q)\, y^{s-1}\, dy = \frac{1}{2\pi}\int_{q=0}^{1}(\alpha_1, \alpha_2).$$

Similarly

$$\xi(1, A_1, B_1, Q_1, P_1) = \frac{1}{2\pi}\int_0^1 \alpha_1,$$

$$\xi(s, A_2, B_2, Q_2, P_2) = \frac{1}{2\pi}\int_0^1 \alpha_2,$$

$$\begin{aligned}
&\xi(s, A, B, Q, R) - \pi\xi(1, A_1, B_1, Q_1, P_1)\xi(s, A_2, B_2, Q_2, P_2) \\
&\quad = \frac{1}{2\pi}\left[\int_0^1(\alpha_1, \alpha_2) - \tfrac{1}{2}\int_0^1\alpha_1\int_0^1\alpha_2\right] \\
&\quad = -\frac{1}{2\pi}\left[\int_0^1(W^*\alpha_1, W^*\alpha_2) - \tfrac{1}{2}\int_0^1 W^*\alpha_1\int_0^1 W^*\alpha_2\right] \quad \text{(by 3.5)} \\
&\quad = -c_1c_2\big(\xi(2-s, -B, A, Q^{-1}, R) \\
&\qquad\qquad -\pi\xi(1, -B_1, A_1, Q_1^{-1}, P_1)\xi(2-s, -B_2, A_2, Q_2^{-1}, P_2)\big)
\end{aligned}$$

since $W^*\alpha_1 = c_1\alpha_1$, $W^*\alpha_2 = c_2y^{1-s}\alpha_2^*$.

Now using $\xi(s) = \pi^{-s}\Gamma(s)Z(s)$, we get (2.9).

It is clear that one can get further formulas of this type using three (or more) differentials.

Section 4

Example. The degree four Fermat curve $X^4 + Y^4 = 1$ and some related elliptic curves.

This Fermat curve has the classical uniformization by Jacobi's theta functions: in our previous notation, let the 1×1 matrix Q be 1, and the polynomial P be 1, and let

$$\theta_2(z) = \theta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(z), \theta_3(z) = \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(z), \theta_4(z) = \theta\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}(z).$$

Then let

$$X = \frac{\theta_2(8z)}{\theta_3(8z)}, \quad Y = \frac{\theta_4(8z)}{\theta_3(8z)}$$

Then X, Y satisfy the Fermat curve equation and furthermore the subgroup $\Gamma_0(64)$ (all integral unimodular matrices $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with $c \equiv 0 \ (64)$) is their stabilizer in the full modular group. The map (X, Y) of the upper half plane to the Fermat curve F_4 maps the positive y-axis oriented from $y = \infty$ to $y = 0$ onto the segment l: $X = 0$ to 1, Y real on F_4. The upper half plane modulo $\Gamma_0(64)$, denoted $X_0(64)$ (with cusps adjoined) is mapped 1-1 onto F_4.

A basis for the holomorphic 1-forms on F_4 is given by

$$\alpha_1 = \frac{1}{2} Y^{-2} dX, \quad \alpha_2 = \frac{1}{2} Y^{-3} dX, \quad \alpha_3 = \tfrac{1}{2} X Y^{-3} dX.$$

The corresponding 1-forms on the upper half plane will be denoted by

$$\alpha_i = f_i(q) \frac{dq}{q}$$

(the $f_i(q)$ are cusp forms) where $q = \exp(2\pi i z)$. In fact α_1 is a pull back from a 1-form on $X_0(32)$ which uniformizes the elliptic curve $Y^2 = 1 - X^4$ (this last is isogenous over Q to the elliptic curve $Y^2 = X^3 - X$). A good reference for these cusp forms $f_i(q)$ is Koblitz's book [6].

To express $f_1(q)$ as a theta function, let Q_1 be the symmetric 2×2 matrix $16I$ so $Q_1[N] = 16(n_1^2 + n_1^2)$, let $P_1(X)$ be the homogeneous linear polynomial

$$P_1(X) = 2WX' \quad \text{where } W = \left[\frac{1+i}{4}, \frac{-1+i}{4}\right] \in \mathbf{C}^2,$$

let $A_1 = \operatorname{Re} W = [\frac{1}{4}, -\frac{1}{4}]$, $B_1 = [0, 0]$; then

$$f_1(q) = \theta^{P_1, Q_1}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = q - 2q^5 - 3q^9 + 6q^{13} \cdots . \tag{4.1}$$

To obtain f_2 we twist f_1 by the Dirichlet character

$$\chi(n) = \left(\frac{2}{n}\right)$$

(quadratic residue symbol which is zero for n even and $(-1)^{(n^2-1)/8}$ for n odd). f_2 is the unique newform for $\Gamma_0(64)$ given by

$$f_2 = \sum_{n=1}^{\infty} \chi(n) c(n) q^n = q + 2q^5 - 3q^9 - 6q^{13} + \cdots \quad \text{if } f_1 = \sum_{1}^{\infty} c(n) q^n.$$

(f_2 corresponds to the elliptic curve $Y^2 = X^3 - 4X$ while f_1 corresponds to $Y^2 = X^3 + 4X$ as well as to $Y^2 = X^3 - X$).

Let $W_N(z) = -N^{-1}z^{-1}$, $(f\,|\,W_N)(z) = f(-N^{-1}z^{-1})N^{-1}z^{-2}$. The action of W_{64} on $X_0(64)$ is the same thing as the interchange of X and Y on the Fermat curve. It follows that

$$f_2|\,W_{64} = -f_2 \quad \text{and} \quad f_1|\,W_{64} = -f_3.$$

Also,

$$f_1|\,W_{32} = -f_1 \text{ so } (f_1|\,W_{64})(z) = -2f_1(2z).$$

Finally $f_3(z) = 2f_1(2z)$, or $f_3(q) = 2f_1(q^2)$, (where we write $f_i(z)$ for $f_i(\exp 2\pi iz)$).

To write f_2 as a theta function, we rewrite f_1 as the infinite series consisting of a summation over all Gaussian integers a congruent to 1 modulo $2 + 2i$ (the conductor), i.e., over all $a = 1 + 2(n_1 - n_2) + 2(n_1 + n_2)i$ where $n_1, n_2 \in \mathbf{Z}$. Then

$$f_1 = \sum_{a\in\mathbf{Z}[i]} aq^{a\bar{a}}, \quad a \equiv 1 \ (\mathrm{mod}\, 2 + 2i)$$

$$f_2 = \sum \chi(a\bar{a})\, aq^{a\bar{a}}.$$

Since $a\bar{a} = 1 + 4(n_1 - n_2) + 8(n_1^2 + n_2^2)$ we have

$$\chi(n) = (-1)^{(n^2-1)/8},$$

$$\chi(a\bar{a}) = (-1)^{n_1 - n_2} = e^{i\pi(n_1 - n_2)} = e^{2i\pi N' B_2}$$

where $B_2 = [\frac{1}{2}, -\frac{1}{2}]$. Let $A_2 = [\frac{1}{4}, -\frac{1}{4}]$ and $Q_2 = Q_1$, $P_2 = P_1$ (as in (4.1) and the lines just before it). Then

$$f_2 = -i\theta^{P_2, Q_2}\begin{bmatrix} A_2 \\ B_2 \end{bmatrix}(z) \tag{4.2}$$

(since $\exp(2\pi i A_2' B_2) = i$).

Finally, let $Q = Q_1 \oplus Q_2 = 16 I_4$,

$$R(x_1, x_2, x_3, x_4) = P_1(x_1, x_2)P_2(x_3, x_4)/(x_1^2 + x_2^2)$$

$$A = A_1 \oplus A_2 = \left[\tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{1}{4}, -\tfrac{1}{4}\right], \quad B = \left[0, 0, \tfrac{1}{2}, -\tfrac{1}{2}\right].$$

Then we will find a geometric meaning for the Mellin transform evaluated at $s = 1$, i.e., the "iterated period"

$$\int_{y=0}^{\infty} \theta^{R,Q}\begin{bmatrix} A \\ B \end{bmatrix} dy = \frac{1}{2\pi}\int_{q=0}^{1} \theta^{R,Q}\begin{bmatrix} A \\ B \end{bmatrix}\frac{dq}{q} = \frac{1}{\pi}\zeta(1, A, B, Q, R).$$

We recall [1], [2] that for any compact Riemann surface S embedded in its Jacobian J, the cycles S and $\iota(S)$ (or S^-) are homologous, where ι is the inverse in the group J. Thus $S - \iota(S)$ is the boundary of a 3-chain C_3, C_3 unique up to a 3-cycle. Let now β_i, $i = 1, 2, 3$ be real harmonic 1-forms with periods in $\mathbf{Z}$ and satisfying

$$\int_S \beta_i \wedge \beta_j = 0 \quad (i, j = 1,2,3).$$

$\int_C \beta_1 \wedge \beta_2 \wedge \beta_3$ is only defined $\operatorname{mod} \mathbf{Z}$ and can be calculated as follows (see also [3] for the simplest proof).

First, suppose that β represents the cohomology class dual to the homology class of a simple closed curve l_3; let S_3 be the surface with boundary obtained by cutting S along l_3 and let $\beta_3 = dB_3$, B_3 a differentiable function on S_3. If l_3 has basepoint x_0, let $\int_{l_3}(\beta_1, \beta_2)$ be the iterated integral. Then

$$(4.3) \qquad \int_C \beta_1 \wedge \beta_2 \wedge \beta_3 = 2\left(\int_{l_3} (\beta_1, \beta_2) + \int_{S_3} B_3 \beta_1 \wedge \beta_2 \right) \quad \text{in } \mathbf{R}/\mathbf{Z}.$$

For general β_i (still satisfying the hypotheses on the β_i) (4.3) can be replaced by the corresponding $\mathbf{Z}$-linear combination.

Suppose now the β_i are holomorphic instead of real harmonic and have periods in $\mathbf{Z}(i)$ instead of $\mathbf{Z}$ (this in fact is the case for F_4), then (4.3) remains valid in $\mathbf{C}/\mathbf{Z}(i)$ instead of $\mathbf{R}/\mathbf{Z}$, and furthermore the second term on the right hand side of (4.3) vanishes since $\beta_i \wedge \beta_j = 0$.

Thus $\int_C \beta_1 \wedge \beta_2 \wedge \beta_3$ reduces to the iterated integral $\int_{l_3}(\beta_1, \beta_2)$ $(\operatorname{mod} \mathbf{Z}(i))$. For the Fermat curves it turns out that all these iterated integrals reduce to iterated integrals over a single non-closed path: $0 \le X \le 1$, Y real. For F_4, of genus 3, the $\mathbf{Z}(i)$ submodule of $\mathbf{C}/\mathbf{Z}(i)$ generated by $\int_C \beta_1 \wedge \beta_2 \wedge \beta_3$, where the β_i are any $\mathbf{Z}(i)$ basis of holomorphic 1-forms with $\mathbf{Z}(i)$ periods, is generated by (the real number)

$$(4.4) \qquad \frac{\left[\int_0^1 (\alpha_1, \alpha_2) - \frac{1}{2} \int_0^1 \alpha_1 \int_0^1 \alpha_2 \right]}{\frac{1}{2} \int_0^1 \alpha_1 \int_0^1 \alpha_2}$$

where $\alpha_1 = \frac{1}{2} Y^{-2} dX$, $\alpha_2 = \frac{1}{2} Y^{-3} dX$ as before. Now replacing α_1 by

$$\theta^{P_1, Q_1} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \frac{dq}{q},$$

α_2 by

$$-i\theta^{P_2, Q_2} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} \frac{dq}{q}$$

and (α_1, α_2) by

$$-i\theta^{R,Q}\begin{bmatrix} A \\ B \end{bmatrix}\frac{dq}{q}$$

and their integrals over $q = 0$ to 1 by

$$2\zeta(1, A_1, B_1, Q_1, P_1), \quad -2i\zeta(1, A_2, B_2, Q_2, P_2) \quad \text{and} \quad -2i\zeta(1, A, B, Q, R)$$

respectively we can state our result as follows, using the notation (2.8) for

$$\begin{aligned} Z(s, A, B, Q, R) &= \zeta(s, A, B, Q, R) \\ &\quad - \zeta(1, A_1, B_1, Q_1, P_1)\zeta(s, A_2, B_2, Q_2, P_2). \end{aligned}$$

(4.5) THEOREM. *Consider the three Dirichlet series*

$$\zeta(s, A_i, B_i, Q_i, R_i) = \sum_{N \in \mathbf{Z}^{h_i}} \frac{R_i\left(\sqrt{Q_i}(N + A_i)\right)\exp(2\pi i(N + A_i)'B_i)}{Q_i(N + A_i)^s},$$

where

$$\begin{gathered} h_1 = h_2 = 2, \quad R_1(X) = R_2(X) = 2XW', \\ W = \left[\frac{1+i}{4}, \frac{-1+i}{4}\right], \quad Q_1 = Q_2 = 16I_2, \\ A_1 = A_2 = \left[\tfrac{1}{4}, -\tfrac{1}{4}\right], \quad B_1 = [0,0], \quad B_2 = \left[\tfrac{1}{2}, -\tfrac{1}{2}\right], \\ h_3 = 4, \quad R_3(X_1, X_2) = \frac{2R_1(X_1)R_2(X_2)}{X_1'X_1}, \quad Q_3 = 16I_4 \\ A_3 = [A_1, A_2], \quad B_3 = [B_1, B_2]. \end{gathered}$$

Let

$$\begin{aligned} Z(s, A_3, B_3, Q_3, R_3) &= \zeta(s, A_3, B_3, Q_3, P_3) \\ &\quad - \zeta(1, A_1, B_1, Q_1, R_1)\zeta(s, A_2, B_2, Q_2, P_2). \end{aligned}$$

Then Z is entire in s, satisfies (2.9), *and*

$$\frac{Z(1, A_3, B_3, Q_3, R_3)}{\zeta(1, A_1, B_1, Q_1, R_1)\zeta(1, A_2, B_2, Q_2, R_2)}$$

considered in $\mathbf{C}/\mathbf{Z}(i)$ generates the image of the cycle $[F_4] - \iota[F_4]$ (homologous to zero in the Jacobian $J(F_4)$), under the Abel-Jacobi map, in the 1*-dimensional complex torus $H^{3,0}(J(F_4))^*/H_3(J; \mathbf{Z})$ (where $*$ indicates the dual space).*

REFERENCES

1. B. HARRIS, *Homological versus algebraic equivalence in a Jacobian*, Proc. Nat. Acad. of Sciences USA, vol. 80, 1983, pp. 1157–1158.
2. ______, *Harmonic volumes*, Acta Math., vol. 150 (1983), pp. 91–123.
3. ______, "Differential characters and the Abel-Jacobi map" in *Algebraic K-theory: connections with geometry and topology*; J. F. Jardine and V. P. Snaith, Eds., Kluwer Academic Publishers, 1989, pp. 69–86.
4. D. MUMFORD, *Tata Lectures on Theta*, *I*, Chapter 2, Birkhauser; Boston, 1983.
5. C. L. SIEGEL, *Lectures on advanced analytic number theory*; Tata Institute, Bombay, 1961.
6. N. KOBLITZ, *Elliptic Curves and modular functions*, Springer-Verlag, New York.

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