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L^p AND SOBOLEV SPACE MAPPING PROPERTIES OF THE SZEGÖ OPERATOR FOR THE POLYDISC

BY

PETER M. KNOPF[1]

1. Introduction

Suppose Ω is a domain and $\partial\Omega$ is its boundary. The Szegö operator $\mathscr{S}$ for $\partial\Omega$ is defined to be the orthogonal projection of $L^2(\partial\Omega)$ into $H^2(\partial\Omega)$ where $H^2(\partial\Omega)$ consists of those functions in $L^2(\partial\Omega)$ which are the extensions of holomorphic functions in Ω. It is well known (see [2], p. 55) that the Szegö operator may be expressed as an integral operator of the form

$$\mathscr{S}f(z) = \int_{\partial\Omega} \tilde{S}(z,\zeta)f(\zeta)\,d\sigma(\zeta)$$

where $\tilde{S}$ is the Szegö kernel.

Recently it has been shown (see [1]) that the Szegö operator for the topological boundary of the bidisc in $\mathbf{C}^2$ with respect to Lebesgue surface area measure is bounded on L^p and L^p_α for $1 < p < \infty$ and $\alpha > 0$. In this paper we show that the same results hold for the topological boundary of the polydisc in $\mathbf{C}^n$ for $n \geq 3$. Furthermore one may have arbitrary radii for the polydisc in each dimension and obtain the same results for any n.

The proofs of these results use the Marcinkiewicz Multiplier Theorem in order to reduce the problem to considering a more tractable operator than the Szegö operator. It turns out that the "tractable" operator is simply the composition of $n - 2$ Bergman operators for the disc in $\mathbf{C}$ and of the Szegö operator for the topological boundary of the bidisc in $\mathbf{C}^2$.

We point out to the reader that the mapping properties for the Szegö operator for the distinguished boundary of the polydisc are trivial and should not be confused with the subject of this paper.

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2. The main results

Let D denote the open unit disc in $\mathbf{C}$ and let ∂D be its boundary. The polydisc in $\mathbf{C}^n$ is $D^n \equiv D \times \cdots \times D$ and its closure is $\overline{D}^n = \overline{D} \times \cdots \times \overline{D}$ where we take n copies of D. The topological boundary of D^n clearly is

$$\partial D^n = (\partial D \times \overline{D}^{n-1}) \cup (\overline{D} \times \partial D \times \overline{D}^{n-2}) \cup \cdots \cup (\overline{D}^{n-1} \times \partial D).$$

We let $(\partial D)^n \equiv \partial D \times \cdots \times \partial D$. The operator $\mathscr{S}_n$ will denote the Szegö operator for ∂D^n with respect to Lebesgue measure. If f is in $L^2(\partial D^n)$, it is well known (see [2], p. 55) that

$$(1) \quad \mathscr{S}_n f(z_1, \ldots, z_n) = \lim_{\varepsilon \to 0^+} \int_{\partial D^n} \tilde{S}_n((z_1, \ldots, z_n) - \varepsilon \nu_{(z_1, \ldots, z_n)}, (\zeta_1, \ldots, \zeta_n)) \cdot f(\zeta_1, \ldots, \zeta_n) \, d\sigma(\zeta_1, \ldots, \zeta_n)$$

a.e. on ∂D^n where $d\sigma$ is Lebesgue surface area measure on ∂D^n, $\nu_{(z_1, \ldots, z_n)}$ is the unit outward normal to ∂D^n at $(z_1, \ldots, z_n)$, and $\tilde{S}_n$ is the Szegö kernel.

LEMMA 2.1. *The Szegö kernel for ∂D^n is*

$$\tilde{S}_n((z_1, \ldots, z_n), (\zeta_1, \ldots, \zeta_n)) = S_n(z_1 \bar{\zeta}_1, \ldots, z_n \bar{\zeta}_n)$$

where

$$(2) \quad S_n(x_1, \ldots, x_n) = \sum_{j_1, \ldots, j_n = 0}^{\infty} \frac{\prod_{k=1}^{n} (j_k + 1) x_k^{j_k}}{2\pi^n \sum_{k=1}^{n} (j_k + 1)}.$$

Proof. By the proof of Lemma 2.1 in [6],

$$\left\{ \prod_{k=1}^{n} z_k^{j_k} \right\}_{j_1, \ldots, j_n = 0}^{\infty}$$

forms an orthogonal basis for $H^2(\partial D^n)$. Using polar coordinates $z_k = r_k e^{i\theta_k}$, it is easy to calculate that

$$\left\| \prod_{k=1}^{n} z_k^{j_k} \right\|_{L^2(\partial D^n)}^2 = \frac{2\pi^n \sum_{k=1}^{n} (j_k + 1)}{\prod_{k=1}^{n} (j_k + 1)}.$$

We conclude that

$$\tilde{S}_n((z_1, \ldots, z_n), (\zeta_1, \ldots, \zeta_n)) = \sum_{j_1, \ldots, j_n = 0}^{\infty} \frac{\prod_{k=1}^{n} (j_k + 1) z_k^{j_k} \bar{\zeta}_k^{j_k}}{2\pi^n \sum_{k=1}^{n} (j_k + 1)}$$

as required.

Henceforth we will denote the Szegö kernel for ∂D^n by $S_n(z_1\bar{\zeta}_1, \ldots, z_n\bar{\zeta}_n)$ with S_n defined in the statement of Lemma 2.1.

THEOREM 2.2. *The Szegö operator* $\mathscr{S}_n$ *satisfies* $\|\mathscr{S}_n f\|_{L^p(\partial D^n)} \leq C\|f\|_{L^p(\partial D^n)}$ *for* $1 < p < \infty$ *and* $n \geq 1$.

The constant C both here and in any subsequent use will stand for a constant depending only on n and p.

Proof. The case $n = 1$ is well known and the case $n = 2$ has recently been solved (see [1]). So we assume $n \geq 3$. Fix p with $1 < p < \infty$. By symmetry we may assume that the support of f is contained in $\partial D \times \bar{D}^{n-1}$. Also by symmetry it is enough to show that

$$\|\mathscr{S}_n f\|_{L^p(\partial D^2 \times \bar{D}^{n-2})} \leq C\|f\|_{L^p(\partial D \times \bar{D}^{n-1})}$$

or more generally

$$\|\mathscr{S}_n f\|_{L^p(\partial D^2 \times \bar{D}^{n-2})} \leq C\|f\|_{L^p(\partial D^2 \times \bar{D}^{n-2})}.$$

Since the operator $\mathscr{S}_n$ is difficult to handle directly, consider the operator

$$(3) \quad \mathscr{K}_n f(z_1, \ldots, z_n) \equiv \lim_{\varepsilon \to 0^+} \int_{\partial D^2 \times \bar{D}^{n-2}} K_n(z_1'\bar{\zeta}_1, z_2'\bar{\zeta}_2, z_3\bar{\zeta}_3, \ldots, z_n\bar{\zeta}_n) \cdot f(\zeta_1, \ldots, \zeta_n)\, d\sigma(\zeta_1, \ldots, \zeta_n)$$

where $(z_1', z_2') = (z_1, z_2) - \varepsilon\nu_{(z_1, z_2)}$ and

$$(4) \qquad K_n(x_1, \dots, x_n) \equiv \sum_{j_1, \dots, j_n = 0}^{\infty} \frac{1}{j_1 + j_2 + 2} \prod_{k=1}^{n} (j_k + 1) x_k^{j_k}.$$

We claim that

$$\|\mathscr{S}_n f\|_{L^p(\partial D^2 \times \overline{D}^{n-2})} \leq C \|\mathscr{K}_n f\|_{L^p(\partial D^2 \times \overline{D}^{n-2})}.$$

Since $\partial D^2 = (\partial D \times \overline{D}) \cup (\overline{D} \times \partial D)$, by symmetry it is enough to show that

$$\|\mathscr{S}_n f\|_{L^p(\partial D \times \overline{D}^{n-1})} \leq C \|\mathscr{K}_n f\|_{L^p(\partial D \times \overline{D}^{n-1})}$$

It is obvious that both $\mathscr{K}_n f$ and $\mathscr{S}_n f$ are the boundary values of holomorphic functions in D^2 and hence may be expressed in power series form:

$$(5) \qquad \mathscr{K}_n f(z_1, \dots, z_n) = \lim_{\varepsilon \to 0^+} \sum_{j_1, \dots, j_n = 0}^{\infty} (1 - \varepsilon)^{j_1} a_{j_1 \dots j_n} \prod_{k=1}^{n} z_k^{j_k}$$

and by (1), (2), Lemma 2.1, and (4) we may write

$$(6) \quad \mathscr{S}_n f(z_1, \dots, z_n) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi^n} \sum_{j_1, \dots, j_n = 0}^{\infty} (1 - \varepsilon)^{j_1} \frac{j_1 + j_2 + 2}{\sum_{k=1}^{n} (j_k + 1)} a_{j_1 \dots j_n} \prod_{k=1}^{n} z_k^{j_k}.$$

Introducing the polar coordinates $z_1 = e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}, \dots, z_n = r_n e^{i\theta_n}$ for fixed $r_2, \dots, r_n$ the functions $\mathscr{K}_n f$ and $\mathscr{S}_n f$ have the following multiple Fourier series expansions:

$$\mathscr{K}_n f\left(e^{i\theta_1}, r_2 e^{i\theta_2}, \dots, r_n e^{i\theta_n}\right) \sim \sum_{j_1, \dots, j_n = 0}^{\infty} a_{j_1 \dots j_n} \prod_{k=2}^{n} r_k^{j_k} \prod_{k=1}^{n} e^{i j_k \theta_k}$$

and

$$\mathscr{S}_n f\left(e^{i\theta_1}, r_2 e^{i\theta_2}, \dots, r_n e^{i\theta_n}\right) \sim \frac{1}{2\pi^n} \sum_{j_1, \dots, j_n = 0}^{n} \frac{j_1 + j_2 + 2}{\sum_{k=1}^{n} (j_k + 1)} a_{j_1 \dots j_n} \prod_{k=2}^{n} r_k^{j_k} \prod_{k=1}^{n} e^{i j_k \theta_k}.$$

We will show that

$$\|\mathscr{S}_n f(\cdot, r_2(\cdot), \dots, r_n(\cdot))\|_{L^p((\partial D)^n)} \le C\|\mathscr{K}_n f(\cdot, r_2(\cdot), \dots, r_n(\cdot))\|_{L^p((\partial D)^n)}. \tag{7}$$

This is equivalent to the following lemma.

LEMMA 2.3. *If*

$$f \sim \sum_{j_1, \dots, j_n = 0}^{\infty} a_{j_1 \dots j_n} \prod_{k=1}^{n} e^{i j_k \theta_k}$$

and

$$\tilde{T}f \sim \sum_{j_1, \dots, j_n = 0}^{\infty} \frac{j_1 + j_2 + 2}{\sum_{k=1}^{n} (j_k + 1)} a_{j_1 \dots j_n} \prod_{k=1}^{n} e^{i j_k \theta_k},$$

then

$$\|\tilde{T}f\|_{L^p((\partial D)^n)} \le C\|f\|_{L^p((\partial D)^n)}.$$

Proof. Consider the multiplier on $\mathbf{R}^n$ defined by

$$m(y_1, \dots, y_n) \equiv \frac{y_1 + y_2 + 2}{\sum_{k=1}^{n} (y_k + 1)}.$$

Let $g \in L^p(\mathbf{R}^n)$. Define the operator T_m by

$$\widehat{T_m g}(y_1, \dots, y_n) \equiv m(y_1, \dots, y_n)\hat{g}(y_1, \dots, y_n).$$

Define the infinite rectangle R to be $R \equiv \{(y_1, \dots, y_n) \in \mathbf{R}^n : y_k \ge -\frac{1}{2}$ for all $k\}$ and define the operator S_R by

$$\widehat{S_R g}(y_1, \dots, y_n) \equiv \chi_R(y_1, \dots, y_n)\hat{g}(y_1, \dots, y_n)$$

where χ_R is the characteristic function of the set R. From the proof of the Marcinkiewicz Multiplier Theorem (see [3], Chapter IV, Section 6.3) it can be seen that

$$\|T_m S_R g\|_{L^p(\mathbf{R}^n)} \le C\|g\|_{L^p(\mathbf{R}^n)}.$$

Define ψ such that $\psi \in C^\infty(\mathbf{R})$, ψ is non decreasing, $\psi(x) = 0$ on $(-\infty, -\frac{1}{4}]$, $\psi(x) = 1$ on $[0, \infty)$, and $|\psi^{(n)}(x)| \leq C$ for all $n \geq 1$. Define the operator T_ψ so that

$$\widehat{T_\psi g}(y_1, \ldots, y_n) = \prod_{k=1}^{n} \psi(y_k)\hat{g}(y_1, \ldots, y_n).$$

Since $\Pi_{k=1}^n \psi(y_k)$ is a Marcinkiewicz multiplier on $L^p(\mathbf{R}^n)$ (see [3], Theorem 6′, p. 109) we have

$$\|T_\psi g\|_{L^p(\mathbf{R}^n)} \leq C\|g\|_{L^p(\mathbf{R}^n)}.$$

Note that $T_\psi T_m S_R = T_m T_\psi$ and so $\|T_m T_\psi g\|_{L^p(\mathbf{R}^n)} \leq C\|g\|_{L^p(\mathbf{R}^n)}$. Associated with the operator $T_m T_\psi$ is a unique periodized operator $\widetilde{T_m T_\psi}$ defined in Theorem 3.8 in Chapter VII in [5]. It is obvious that $\widetilde{T_m T_\psi} = \tilde{T}$ and by Theorem 3.8 we have $\|\tilde{T}f\|_{L^p((\partial D)^n)} \leq C\|f\|_{L^p((\partial D)^n)}$ which proves the lemma.

We return to the proof of Theorem 2.2. Raising both sides of (7) to the power p and integrating with respect to the measure $\Pi_{k=2}^n r_k dr_k$ gives

$$\|\mathscr{S}_n f\|^p_{L^p(\partial D \times \bar{D}^{n-1})} \leq C\|\mathscr{K}_n f\|^p_{L^p(\partial D \times \bar{D}^{n-1})}$$

as claimed. We have left to show that

$$\|\mathscr{K}_n f\|_{L^p(\partial D^2 \times \bar{D}^{n-2})} \leq C\|f\|_{L^p(\partial D^2 \times \bar{D}^{n-2})}. \tag{8}$$

Summing up the series in (4) gives

$$K_n(x_1, \ldots, x_n) = 2\pi^n S_2(x_1, x_2) \prod_{k=3}^{n} B(x_k)$$

where S_2 is the Szegö kernel for ∂D^2 and $B(x_k) = \pi^{-1}(1 - x_k)^{-2}$ is the Bergman kernel for D with $k = 3, \ldots, n$. It is well known (see Theorem 3 in [4]) that the Bergman operators $\mathscr{B}^{(k)}$ with associated kernels $B(x_k)$ map $L^p(D)$ into $L^p(D)$. It has been shown (see [1]) that the Szegö operator $\mathscr{S}_2^{(1,2)}$ with the associated kernel $S_2(x_1, x_2)$ maps $L^p(\partial D^2)$ into $L^p(\partial D^2)$. By (3) and (4) we may write

$$\mathscr{K}_n f(z_1, \ldots, z_n) = \mathscr{B}^{(n)} \ldots \mathscr{B}^{(4)}\mathscr{B}^{(3)}\mathscr{S}_2^{(1,2)} f(z_1, \ldots, z_n). \tag{9}$$

Inequality (8) now follows, which concludes the proof of Theorem 2.2.

THEOREM 2.4. *The Szegö operator* $\mathscr{S}_n$ *maps* L^p_α *into* L^p_α *for* $1 < p < \infty$, $\alpha > 0$, *and* $n \geq 1$.

Proof. The cases $n = 1$ and $n = 2$ are known (see [4] and [1]). So we assume $n \geq 3$. By induction and interpolation it is enough to do the case $\alpha = 1$. The inequality

$$\|\mathscr{K}_n f\|_{L_1^p(\partial D^n)} \leq C\|f\|_{L_1^p(\partial D^n)}$$

follows immediately from (9), Theorem 3 in [4], and Theorem 3.5 in [1]. The comparison between $\mathscr{S}_n$ and $\mathscr{K}_n$, which is shown in the proof of theorem 2.2, works for derivatives as well. The theorem now follows.

3. Remarks

(a) The Bergman operator $\mathscr{B}_n$ for the polydisc with respect to Lebesgue surface area measure maps $L_\alpha^p(D^n)$ into $L_\alpha^p(D^n)$ for $1 < p < \infty$, $\alpha > 0$, and any n. This is not difficult to show. Straightforward calculations give

$$\mathscr{B}_n f(z_1, \ldots, z_n) = \mathscr{B}_1^{(n)} \ldots \mathscr{B}_1^{(2)} \mathscr{B}_1^{(1)} f(z_1, \ldots, z_n)$$

where $\mathscr{B}_1^{(k)}$ is the Bergman operator for the disc D in the variable z_k with the associated kernel

$$B(x_k) = \pi^{-1}(1 - x_k)^{-2} \quad \text{for } k = 1, \ldots, n.$$

The same results hold for the domain D_R^n as defined in part (d) below.

(b) $H^p(\partial D^n)$ strictly contains $H^p((\partial D)^n)$ for $1 < p < \infty$ and $n \geq 2$. The set $(\partial D)^n$ is often called the distinguished boundary of the polydisc D^n. In fact it is easy to show that

$$\|f\|_{L^p(\partial D^n)} \leq C\|f\|_{L^p((\partial D)^n)}$$

for any f and if $f(z_1, \ldots, z_n) = (1 - z_1 z_2)^{-3/2p}$ then $f \in H^p(\partial D^n)$ but $f \notin H^p((\partial D)^n)$.

(c) In [1] it is shown that the Szegö operator $\mathscr{S}_2$ for the bidisc is not weak-type $(1, 1)$ nor does it map Λ_γ into Λ_γ (the Lipschitz spaces) for any $0 < \gamma < 1$. It can be shown using the same counterexamples cited in [1] that the Szegö operator $\mathscr{S}_n$ for the polydisc with $n \geq 3$ is not weak-type $(1, 1)$ nor does it map Λ_γ into Λ_γ for any $0 < \gamma < 1$.

(d) Let $D(R_k)$ denote the disc of radius R_k in $\mathbf{C}$. Define

$$D_R^n \equiv D(R_1) \times \cdots \times D(R_n).$$

Let $\mathscr{S}_R$ denote the Szegö operator for ∂D_R^n with respect to Lebesgue surface area measure. The Szegö operator $\mathscr{S}_R$ is bounded on $L^p(\partial D_R^n)$ and $L_\alpha^p(\partial D_R^n)$

for $1 < p < \infty$, $\alpha > 0$, and $n \geq 1$. The proofs are similar to those of Theorems 2.2 and 2.4.

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Pace University
Pleasantville, New York