ILLINOIS JOURNAL OF MATHEMATICS
Volume 35, Number 2, Summer 1991

SEVERAL RESULTS CONCERNING UNCONDITIONALITY IN VECTOR VALUED L^p AND $H^1(\mathscr{F}_n)$ SPACES

BY

PAUL F.X. MÜLLER[1] AND GIDEON SCHECHTMAN

1. Introduction

Recently, vector valued versions of several results concerning basis properties of L^p spaces have been obtained for the spaces $L^p(E)$ where E is a UMD space. In particular, T. Figiel [Fi] has shown that the Haar and Franklin systems are equivalent in $L^p(E)$, $1 < p < \infty$. The main technical result of the present paper, Theorem 2 below, is of a similar nature; one shows that certain "Haar-like" sequences in $L^p(E)$, $1 < p < \infty$, are equivalent to sequences spanning all of an $L^p((\Omega, \mathscr{F}, p), E)$ space. The operator used for this equivalence is closely related to the one used by Maurey in [Ma1] and [Ma2]. An argument of Herz, also used by Maurey, is then used (Theorem 4) to show that a similar equivalence holds in $H^1(\mathscr{F}_n, E)$ spaces (see notations below for the definition of these spaces).

As corollaries, one gets vector valued versions of the Gamlen-Gaudet theorem, characterizing the isomorphic structure of subsequences of the classical Haar functions in L^p. These versions extend also to the finite dimensional case as well as for the H^1 case. The approach here follows the first author's paper [Mü1]. These results are contained in Theorem 3 and Corollary 8.

Another corollary to Theorems 2 and 4 (Corollary 7) is that, if E is UMD then $H^1(\mathscr{F}_n, E)$ has an unconditional decomposition into copies of E and if E has in addition an unconditional basis, then so does $H^1(\mathscr{F}_n, E)$. This extends a result of Maurey stated in [Ma1].

2. The main technical result

Let $(\Omega, \mathscr{F}, |\cdot|)$ be a given probability space. Let E be a Banach space. Then we denote by $L^p(\Omega, \mathscr{F}, |\cdot|, E)$ (or simply by $L^p(E)$) the Banach

Received January 4, 1989.
1980 Mathematics Subject Classification (1985 Revision). Primary 46B20, 42C20, 42B30; Secondary 42B30.
[1]Supported by Erwin Schrödinger-Auslandsstipendium Pr. Nr. J0288P.

space of p-Bochner integrable functions on $(\Omega, \mathcal{F}, |\cdot|)$ with values in E. E is said to have the UMD property if for $1 < p < \infty$ there exists α such that for each E valued martingale difference sequence in $(\Omega, \mathcal{F}, |\cdot|)$, (d_n), and for each sequence $\{\varepsilon_n\} \in \{0, 1\}$ we have

$$(*) \qquad \left\| \sum_k d_k \varepsilon_k \right\|_{L^p(E)} \leq \alpha \left\| \sum_k d_k \right\|_{L^p(E)}.$$

The infimum of all α such that $(*)$ holds is denoted by $\alpha_p(E)$ and will be called "the UMD constant of E (in L^p)".

Given a filtration of finite sub σ-algebras $\{\mathcal{F}_n\}$ of $(\Omega, \mathcal{F}, |\cdot|)$ with $\mathcal{F}_0 = \{\phi, \Omega\}$ we construct another such filtration $\{\mathcal{G}_n\}$ with $\mathcal{F}_{n-1} \subseteq \mathcal{G}_n \subseteq \mathcal{F}_n$, $n = 1, 2, \dots$, in the following way:

Let $\mathcal{A}_n$ be the collection of atoms of $\mathcal{F}_n$. For $A \in \mathcal{A}_{n-1}$ let A^* denote an element of $\mathcal{A}_n$ such that $A^* \subseteq A$ and $|A^*| = \max\{|B|; B \subseteq A, B \in \mathcal{A}_n\}$. Put

$$\mathcal{E}(A) = \{B \in \mathcal{A}_n; B \subseteq A, B \neq A^*\}$$

and

$$\mathcal{E} = \bigcup_{A \in \bigcup_n \mathcal{A}_n} \mathcal{E}(A).$$

LEMMA 1. *Let $A \in \mathcal{A}_{n-1}$ be such that $|A^*| \leq |A|/4$. Then there exist pairwise disjoint collections $\mathcal{B}_1, \dots, \mathcal{B}_m$ in $\mathcal{A}_n \cap A$ such that, putting $F_i = \bigcup_{B \in \mathcal{B}_i} B$, we get*

$$\bigcup_{i=1}^{m} F_i = A \setminus A^*$$

and

$$\frac{1}{4} \leq \frac{|F_1|}{|A|} \leq 1, \quad \frac{1}{4} \leq \frac{|F_{i+1}|}{|F_i|} \leq 1, \quad \frac{1}{4} \leq \frac{|A^*|}{|F_m|} \leq \frac{1}{2}.$$

Proof. Let $A_1, \dots, A_l$ be the atoms in $\mathcal{F}_n \cap (A \setminus A^*)$ and let

$$k_1 = \inf\left\{k; \left|\bigcup_{i=1}^{k} A_i\right| \geq 2|A^*|\right\}.$$

If

$$\left|\bigcup_{i=k+1}^{l} A_i\right| \geq 4|A^*|$$

we put $\mathscr{B}_1' = \{A_1, \dots, A_k\}$ and continue. Otherwise we put $m = 1$, $\mathscr{B}_1' = \{A_1, \dots, A_l\}$ and stop.

Suppose $k_1, \dots, k_{s-1}$ and $\mathscr{B}_1', \dots, \mathscr{B}_{s-1}'$ were already constructed, then we put

$$k_s = \inf\left\{ k; \left| \bigcup_{i=k_{s-1}+1}^{k} A_i \right| \geq 2^s |A^*| \right\}.$$

If

$$\left| \bigcup_{i=k_s}^{l} A_i \right| \geq 2^{s+1} |A^*|$$

we put

$$\mathscr{B}_s' = \{A_{k_{s-1}+1}, \dots, A_{k_s}\}$$

and continue. Otherwise we put

$$\mathscr{B}_s' = \{A_{k_{s-1}+1}, \dots, A_l\}$$

and stop. Finally, assuming we have m steps, put

$$\mathscr{B}_s = \mathscr{B}_{m+1-s}' \qquad \blacksquare$$

The set F_1 constructed above will be recorded under the name $F_1(A)$.

We now introduce the algebras $\mathscr{G}_n$ and some other notations. For $A \in \mathscr{F}_{n-1}$ with $|A^*| < |A|/4$ we let $\mathscr{C}_A$ be the algebra generated by $\{F_1, \dots, F_m\}$. For $A \in \mathscr{F}_{n-1}$ with $|A^*| \geq |A|/4$ we let $\mathscr{C}_A$ be $\mathscr{F}_n \cap A$. $\mathscr{G}_n$ is the algebra generated by $\{\mathscr{C}_A\}_{A \in \mathscr{A}_{n-1}}$.

We wish to define the normalized shift operator D_n for $\mathscr{G}_n$ measurable functions f by

$$D_n f(t) := D_A(f \cdot 1_A)(t) \quad \text{for } t \in A,\ A \in \mathscr{F}_{n-1}$$

where D_A is defined below. For $A \in A_{n-1}$ let

$$X(A) = \operatorname{span}\{1_B x_B;\ B \in \mathscr{E}(A),\ x_B \in E\}$$

and

$$Y(A) = \left\{ f\colon A \to E;\ f \text{ is } \mathscr{F}_n \text{ measurable, and } \int_A f = 0 \right\}$$

and define a linear operator

$$T_A\colon X(A) \to Y(A)$$

according to the following two cases:

Case 1. $|A^*| \geq |A|/4$. Then

$$T_A f = f - \left(\frac{1}{|A^*|}\int_A f\right)1_{A^*}$$

Case 2. $|A^*| < |A|/4$. Then

$$T_A f = f - \mathbf{E}(f|\mathscr{C}_A) + D_A\mathbf{E}(f|\mathscr{C}_A) - \left(\frac{1}{|F_1|}\int_A f\right)1F_1.$$

Here D_A denotes the normalized shift operator defined for $\mathscr{C}_A$ measurable functions supported on A by

$$(D_A f)_{|F_1} = 0,$$

$$(D_A f)_{|F_i} = \frac{|F_{i-1}|}{|F_i|} f_{|F_{i-1}}, \quad 2 \leq i \leq m+1 \;(F_{m+1} = A^*).$$

Note that, by construction, $\int D_A f = \int f$ and that $D_A f$ is $\mathscr{C}_A$ measurable so that T_A maps into $Y(A)$. Lemma 1 implies easily that for all $f \in X(A)$ and all $1 \leq p \leq \infty$

$$\|f\|_{L^p(E)} \leq \|D_A f\|_{L^p(E)} \leq 4\|f\|_{L^p(E)}.$$

From this and the fact that in Case 2,

$$f - \mathbf{E}(f|\mathscr{C}_A) \quad \text{and} \quad D_A\mathbf{E}(f|\mathscr{C}_A) - \left(\frac{1}{|F_1|}\int_A f\right)1_{F_1}$$

form a 2-step martingale difference, it is easy to see (cf. [Mül], Lemma 5, Steps 1, 2) that for some universal C, all $1 \leq p \leq \infty$, and all E,

$$C^{-1}\|f\|_{L^p(E)} \leq \|T_A f\|_{L^p(E)} \leq C\|f\|_{L^p(E)}.$$

This will be used later in the proof of Theorem 4. Right now we would like to extend the $T_A - s$ to an operator on a certain subspace of $L^p(E)$ and prove boundedness.

For $A \in \mathscr{A}_{n-1}$ and $B \in \mathscr{E}(A)$ we let

$$h_B = 1_B \otimes r_n$$

that is, let $\{r_n\}$ be a sequence of independent Rademacher functions on $[0,1]$ and let h_B be defined on $\Omega \times [0,1]$ by $h_B(s,t) = 1_B(s) \cdot r_n(t)$. Let

$$X_{\mathscr{E}}(E) = \operatorname{span}\{h_A x;\ A \in \mathscr{E},\ x \in E\}$$

endowed with the $L^p(E)$ norm, and define

$$T\colon X_{\mathscr{E}}(E) \to L^p(E)$$

by

$$T\Bigl(\sum_{A \in \bigcup \mathscr{A}_n} \sum_{B \in \mathscr{E}(A)} h_B x_B\Bigr) = \sum_{A \in \bigcup \mathscr{A}_n} \sum_{B \in \mathscr{E}(A)} T_A(|h_B|) x_B$$

(for finite sums).

As the operators T_A are surjective, the range of T contains

$$\bigcup_n \left\{f \in L^p(E) \colon f \text{ is } \mathscr{F}_n \text{ measurable, } \int f = 0\right\}$$

Hence the range of T is dense in $\{f \in L^p(E)\colon \int f = 0\}$.

The main technical difficulty of this paper is contained in the following theorem.

THEOREM 2. *Let $1 < p < \infty$ and let E be a UMD space. Then there exists a constant C, depending only on p and the UMD constant $\alpha_p(E)$ of E, such that*

$$\|T\|\,\|T^{-1}\| \le C.$$

Theorem 2 implies that the range of T is closed in $L^p(E)$. Hence, by the previous remark, T maps $X_{\mathscr{E}}(E)$ onto $\{f \in L^p(E)\colon \int f = 0\}$.

In the proof we shall use the following inequality due to E. Stein [St] in the scalar case and J. Bourgain [Bo] in the vector valued case.

For $1 < p < \infty$ and a UMD space E, for all sequences of increasing σ-fields $\{\mathscr{F}_i\} \subseteq (\Omega, \mathscr{F})$ and all sequences of measurable functions $\{f_i\}$,

$$(1) \qquad \mathbf{E}\Bigl\| \sum \varepsilon_i \mathbf{E}(f_i | \mathscr{F}_i)\Bigr\|_{L^p(E)} \le C\mathbf{E}\Bigl\| \sum \varepsilon_i f_i \Bigr\|_{L^p(E)}$$

where $\{\varepsilon_i\}$ is a sequence of Rademacher functions independent of $\mathscr{F}$, and C depends only on p and the UMD constant $\alpha_p(E)$ of E.

We shall also need the so-called *contraction principle* due to Kahane (cf. [Ka] or [M.P.], p. 45, Theorem 4.9 and Remark 4.10).

Let $d_i: \Omega \to E$, $c_i: \Omega \to R$ be measurable with $d_i \in L^p(E)$, $1 \le p \le \infty$ and $c_i \in L^\infty$. Let $\{\varepsilon_i\}$ be an independent Rademacher sequence. Then

$$\mathbf{E}\left\| \sum \varepsilon_i c_i d_i \right\|_{L^p(E)} \le \sup_i \|c_i\|_\infty \cdot \mathbf{E}\left\| \sum \varepsilon_i d_i \right\|_{L^p(E)} \tag{2}$$

Proof of Theorem 2. We first prove that T^{-1} is bounded.

Fix $x_B \in E$, $B \in \mathscr{E}$. For $A \in \bigcup \mathscr{A}_n$ we put

$$f_A = \sum_{B \in \mathscr{E}(A)} h_B x_B, \quad g_A = \sum_{B \in \mathscr{E}(A)} |h_B| x_B.$$

We shall also put

$$\mathscr{K}_n = \{A \in \mathscr{A}_n;\ |A^*| \ge |A|/4\},$$
$$\mathscr{L}_n = \{A \in \mathscr{A}_n;\ |A^*| < |A|/4\}.$$

As the sequence $\{\sum_{A \in \mathscr{A}_n} Tf_A\}_{n=1}^\infty$ forms a martingale difference sequence, we get from the UMD property of E and then from (2) that

$$\left\| \sum_{A \in \bigcup \mathscr{A}_n} Tf_A \right\|_{L^p(E)} \tag{3}$$
$$\ge c\mathbf{E}\left\| \sum_n \varepsilon_n \sum_{A \in \mathscr{A}_n} Tf_A \right\|_{L^p(E)}$$
$$\ge \frac{c}{4}\left(\mathbf{E}\left\| \sum_n \varepsilon_n \sum_{A \in \mathscr{K}_n} Tf_A \right\|_{L^p(E)} + \mathbf{E}\left\| \sum_n \varepsilon_n \sum_{A \in \mathscr{L}_n} Tf_A \right\|_{L^p(E)} \right).$$

We treat separately the first and second terms. For the first, put

$$f_n = \sum_{A \in \mathscr{K}_{n-1}} f_A, \quad g_n = \sum_{A \in \mathscr{K}_{n-1}} g_A.$$

Then

$$Tf_n = g_n - g_n'$$

where g_n and g_n' are disjointly supported; hence by (2) again

$$\mathbf{E}\left\| \sum_n \varepsilon_n Tf_n \right\|_{L^p(E)} \ge \tfrac{1}{2}\mathbf{E}\left\| \sum_n \varepsilon_n g_n \right\|_{L^p(E)} \tag{4}$$
$$= \tfrac{1}{2}\mathbf{E}\left\| \sum_n f_n \right\|_{L^p(E)}.$$

The treatment of the second term in the righthand side of (3) is more complicated. Now let

$$f_n = \sum_{A \in \mathscr{L}_{n-1}} f_A, \quad g_n = \sum_{A \in \mathscr{L}_{n-1}} g_A$$

and let

$$\Delta_1(Tf_n) = g_n - \mathbf{E}(g_n | \mathscr{G}_n),$$

$$\Delta_2(Tf_n) = D_n \mathbf{E}(g_n | \mathscr{G}_n) - \sum_{A \in \mathscr{L}_{n-1}} \left(\frac{1}{|F_1(A)|} \int_A f \right) 1_{F_1(A)}$$

($F_1(A)$, D_n and the algebra $\mathscr{G}_n$ are introduced in Lemma 1 and the discussion following its proof) so that $Tf_n = \Delta_1(Tf_n) + \Delta_2(Tf_n)$. A crucial observation is that the sequence

$$\Delta_2(Tf_1), \Delta_1(Tf_1), \Delta_2(Tf_2), \Delta_1(Tf_2), \ldots$$

is a martingale difference sequence with respect to the sequence $\mathscr{F}_0, \mathscr{G}_1, \mathscr{F}_1, \mathscr{G}_2, \ldots$. Thus we get

$$\text{(5)} \quad \left\| \sum_n Tf_n \right\|_{L^p(E)} \geq c\mathbf{E} \left\| \sum_n \varepsilon_n \Delta_1(Tf_n) + \varepsilon_n' \Delta_2(Tf_n) \right\|_{L^p(E)}$$

$$\geq \frac{c}{4} \left(\mathbf{E} \left\| \sum_n \varepsilon_n \Delta_1(Tf_n) \right\|_{L^p(E)} + \mathbf{E} \left\| \sum_n \varepsilon_n \Delta_2(Tf_n) \right\|_{L^p(E)} \right).$$

We first treat the rightmost term. As

$$\Delta_2(Tf_n) = D_n \mathbf{E}(g_n | \mathscr{G}_n) + \text{a disjoint term}$$

we get from (2) that

$$\text{(6)} \qquad \mathbf{E} \left\| \sum \varepsilon_n \Delta_2(Tf_n) \right\|_{L^p(E)} \geq \mathbf{E} \left\| \sum \varepsilon_n D_n \mathbf{E}(g_n | \mathscr{G}_n) \right\|_{L^p(E)}.$$

Claim. For some constant C depending only on p and the UMD constant of E,

$$\text{(7)} \qquad C^{-1} \mathbf{E} \left\| \sum_n \varepsilon_n \mathbf{E}(g_n | \mathscr{G}_n) \right\|_{L^p(E)} \leq \mathbf{E} \left\| \sum_n \varepsilon_n D_n \mathbf{E}(g_n | \mathscr{G}_n) \right\|_{L^p(E)}$$

$$\leq C\mathbf{E} \left\| \sum_n \varepsilon_n \mathbf{E}(g_n | \mathscr{G}_n) \right\|_{L^p(E)}.$$

Using the claim, (5) and (6) we get

$$\begin{aligned}\Big\|\sum_n Tf_n\Big\|_{L^p(E)} &\ge c'\mathbf{E}\left(\Big\|\sum_n \varepsilon_n(g_n-\mathbf{E}(g_n|\mathscr{G}_n))\Big\|_{L^p(E)}+\Big\|\sum_n \varepsilon_n\mathbf{E}(g_n|\mathscr{G}_n)\Big\|_{L^p(E)}\right)\\ &\ge c'\mathbf{E}\Big\|\sum_n \varepsilon_n g_n\Big\|_{L^p(E)} \ge c'\mathbf{E}\Big\|\sum_n f_n\Big\|_{L^p(E)}\end{aligned}$$

which concludes the proof that T^{-1} is bounded except for the proof of the claim. To prove the claim, fix an $A \in \mathscr{L}_{m-1}$ and (assuming m of Lemma 1 is even), consider the collection of sets $F_1 \cup F_2, F_3 \cup F_4, \dots, F_{m-1} \cup F_m$ (resp. $F_1 \cup F_2, \dots, F_m \cup F_{m+1}$, if m is odd). Let $\mathscr{H}_n^1$ be the algebra generated by these sets where A ranges over $\mathscr{L}_{n-1}$. Using (1) and (2) and, for $A \in \mathscr{L}_{n-1}$, letting $F_A = \bigcup_{i,\,\mathrm{odd}} F_i$, $G_A = \bigcup_{i\ \mathrm{even}} F_i$, we get

$$\begin{aligned}(8)\qquad &\Big\|\sum_n \varepsilon_n\mathbf{E}(g_n|\mathscr{G}_n)\Big\|_{L^p(E)}\\ &\ge c'\max\left\{\Big\|\sum_n \varepsilon_n \sum_{A\in\mathscr{L}_{n-1}} \mathbf{E}(g_n|\mathscr{G}_n)1_{F_A}\Big\|_{L^p(E)},\right.\\ &\qquad\left.\Big\|\sum_n \varepsilon_n \sum_{A\in\mathscr{L}_{n-1}} \mathbf{E}(g_n|\mathscr{G}_n)1_{G_A}\Big\|_{L^p(E)}\right\}.\end{aligned}$$

Now the first term in the max majorizes, by E.M. Stein's inequality (up to constant),

$$\begin{aligned}(9)\qquad &\Big\|\sum_n \varepsilon_n \sum_{A\in\mathscr{L}_{n-1}} \mathbf{E}\big(\mathbf{E}(g_n|\mathscr{G}_n)1_{F_A}|\mathscr{H}_n^1\big)\Big\|_{L^p(E)}\\ &\ge c'\Big\|\sum_n \varepsilon_n(D_A\mathbf{E}(g_n|\mathscr{G}_n))1_{G_A}\Big\|_{L^p(E)}.\end{aligned}$$

In a similar manner, using $F_2 \cup F_3, F_4 \cup F_5, \dots$ as building blocks of $\mathscr{H}_n^2$ we get that the second term in (8) majorizes

$$\Big\|\sum_n \varepsilon_n(D_A\mathbf{E}(g_n|\mathscr{G}_n))1_{F_A\setminus F_1}\Big\|_{L^p(E)}.$$

This, together with (8) proves the righthand side inequality of the claim. The lefthand side inequality is proved similarly. Also the proof that T is bounded is similar to the proof of the scalar case (see [Mül]), and we leave it to the reader. ■

3. A vector valued Gamlen-Gaudet theorem

Here we generalize theorems of Gamlen-Gaudet [G.G.] and the first-named author, to the vector valued case. The proof follows the idea of [Mül].

THEOREM 3. *Let $1 < p < \infty$ and let E be the UMD space. Let $\{n_k\}_{k=1}^{\infty}$ be a (finite or infinite) subsequence of the positive integers and put $H = \overline{\operatorname{span}}\{\chi_{n_k}\}_{k=1}^{\infty}$ where $\{\chi_n\}_{n=1}^{\infty}$ is the Haar system in $L^p[0,1]$. Then*

$$H \otimes_p E = \overline{\left\{ \sum \chi_{n_k} x_k;\ x_k \in E \right\}}$$

is isomorphic to either $L^p(E)$ or $l^p(E)$, or $l_n^p(E)$ for some n.

We may and shall assume that the first two Haar functions are not in $\{\chi_{n_k}\}_{k=1}^{\infty}$.

Proof. Let $\mathscr{F}_n$ be the algebra generated by the supports of all the Haar functions in $\{\chi_{n_k}\}_{k=1}^{\infty}$ which have size $\geq 2^{-n}$, $n = 1, 2, \ldots$ and let $\mathscr{F}_0 = \{\phi, [0,1]\}$. We first assume that given any atom A of $\mathscr{F}_{n-1}$, the supports of the Haar functions in $\{\chi_{n_k}\}_{k=1}^{\infty}$ of size 2^{-n} do not fill up A, i.e., A^*, the largest atom of $\mathscr{F}_n$ contained in A, can be chosen not to be a support of a function from $\{\chi_{n_k}\}_{k=1}^{\infty}$.

Since E is UMD, we get that $\|\sum \chi_{n_k} x_k\|_{L^p(E)}$ is equivalent, with constant depending only on p and the UMD constant of E, to

$$\Big\| \sum_{A \in \cup \mathscr{A}_n} \sum_{B \in \mathscr{E}(A)} h_B x_B \Big\|_{L^p(E)}$$

where $x_B = x_k$ when $|h_B| = |\chi_{n_k}|$. Theorem 2 then implies that $H \otimes_p E$ is isomorphic to $L^p(\vee_n \mathscr{F}_n, E)$ which in turn is isomorphic to $L^p([0,1], E)$, $l^p(E)$ or $l_n^p(E)$ for some n.

To remove the restrictive assumption on the sequence $\{\chi_{n_k}\}_{k=1}^{\infty}$ we remove from the sequence one element of support size 2^{-n} contained in A for each atom of A of $\mathscr{F}_{n-1}$ which the supports of $\{\chi_{n_k}\}$ of size 2^{-n} fill up. In this way we split $\{\chi_{n_k}\}$ into two disjoint subsequences each of which satisfy the additional assumption and the theorem follows using the UMD property again.

4. The H^1 case

We recall that given a filtration of σ-algebras $\mathscr{B}_n \subseteq \mathscr{B}$ on a probability space $(\Omega, B, |\cdot|)$ the space $H^1(\mathscr{B}_n, E)$ is the subspace of $L^1(\Omega, E)$ consisting of all functions for which the norm

$$\|f\|_{H^1} = \int \sup_n \|\mathbf{E}(f|\mathscr{B}_n)\|$$

is finite.

Given a filtration of finite algebras $\mathscr{F}_n$ on $(\Omega, \mathscr{F}, |\cdot|)$, we consider a new filtration $\Omega \times [0,1]$ ($[0,1]$ is equipped with Lebesgue measure) by letting $\mathscr{B}_n$ be the algebra generated by $\mathscr{F}_n$ and the first n Rademacher functions.

The operator T of Section 2 can be viewed as an operator from a subspace of $H^1(\mathscr{B}_n, E)$ into $H^1(\mathscr{F}_n, E)$ (actually onto all mean zero functions in $H^1(\mathscr{F}_n, E)$.

THEOREM 4. *For a constant $0 < C < \infty$ depending only on the UMD constant $\alpha_2(E)$ of E,*

$$\|T\| \, \|T^{-1}\| \leq C$$

where the operator norms are between the appropriate H^1 spaces.

Sketch of Proof. We shall use three facts:

(a) The set $B_n := \operatorname{span}\{x_B h_B : B \in \mathscr{E}(A),\ A \in \mathscr{A}_{n-1},\ x_B \in E\}$ is mapped to the set F_n of functions Tf which are $\mathscr{F}_n$ measurable and $\mathbf{E}(Tf|\mathscr{F}_{n-1}) = 0$. Moreover, if g is a real valued $\mathscr{F}_{n-1}$ measurable function and $f \in \mathscr{F}_n$, then $T^{-1}gf = gT^{-1}f$.

(b) T (resp. T^{-1}) is uniformly bounded in $L^1(E)$ on each of the spaces B_n (resp. F_n).

(c) T when considered in $L^2(E)$ is an isomorphism with constant depending only on the UMD constant of E.

((b) is proved before Theorem 2, (c) is a special case of Theorem 2.)

The proof now follows Section 4 of [Ma2] which in turn is inspired by a result of Herz [He]:

Assume $\|f\|_{H^1(\mathscr{F}_n, E)} = 1$. Substituting absolute values by norms, we define g and h as in Garsia [Ga], p. 92, and we put $f_k' := \mathbf{E}(g|\mathscr{F}_k)$ and $f_k'' := \mathbf{E}(h|\mathscr{F}_k)$, and we let

$$d_k' = f_k' - f_{k-1}', \qquad d_k'' = f_k'' - f_{k-1}'', \qquad \tilde{f}_k = \sum_{j=1}^{k} (f_{j-1}^*)^{-1/2} d_j'.$$

Applying Abel's transform we get

$$\|\tilde{f}_k\| \leq 4(f_{k-1}^*)^{1/2}.$$

It follows that $\tilde{f}_k$ converges in $L^2(E)$ to a function $\tilde{f}$. Applying T^{-1} to $\tilde{f}$, by (a) and (c) we get

$$T^{-1}\tilde{f} = \sum_{j=1}^{\infty} (f_{j-1}^*)^{-1/2} T^{-1} d_j'.$$

By Doob's inequality,

$$\mathbf{E} \sup_k \left\| \sum_{j=1}^{k} (f_{j-1}^*)^{-1/2} T^{-1} d_j' \right\|^2 \leq 4\mathbf{E}\|T^{-1}\tilde{f}\|^2 \leq 16K^2$$

where K is universal

Now, by partial summation,

$$\left\| \sum_{j=1}^{k} T^{-1} d_j' \right\| \leq 2 \cdot \sup_{l \leq k} \left\| \sum_{j=1}^{l} (f_{j-1}^*)^{-1/2} T^{-1} d_j' \right\| (f_{k-1}^*)^{1/2}$$

and

$$\begin{aligned} \left\| T^{-1} \sum d_j' \right\|_{H^1}^2 &\leq C^2 \mathbf{E} \sup_k \left\| \sum_{j=1}^{k} (f_{j-1}^*)^{-1/2} T^{-1} d_j' \right\|^2 \|f\|_{H^1}^2 \\ &\leq 16C^2K^2. \end{aligned}$$

For d_j'' we get

$$\sum_{j=1}^{n} \|d_j''\| \leq 4f_{n-1}^* + 4\sum_{j=1}^{n} \mathbf{E}\left(f_j^* - f_{j-1}^* \mid \mathscr{F}_{j-1}\right)$$

(see Garsia [Ga], p. 93). We thus get from (b) that for a universal constant K

$$\begin{aligned} \mathbf{E} \sup_k \left\| \sum_{j=0}^{k} T^{-1} d_j'' \right\| &\leq \mathbf{E} \sum_{j=0}^{\infty} \|T^{-1} d_j''\| \\ &\leq K\mathbf{E} \sum_{j=0}^{\infty} \|d_j''\| \\ &\leq 8K\mathbf{E} \sup_k \|f_k\| = 8K. \end{aligned}$$

This shows that T^{-1} is bounded. A similar proof shows that T is bounded. ∎

Remark 5. The decomposition of the martingale $\{f_k\}$ into $\{f_k' + f_k''\}$ goes back to B. Davis [Da] in which the equivalence of the L_1-norms of the square function and the maximal function of a real valued martingale is proved. It turns out that B. Davis' inequality generalizes to the appropriate vector valued case. As the proof is basically the same as the proof given in [Bu] for the real case, we do not repeat it here.

THEOREM 6 (B. Davis). *Let E be a UMD space. Then for every E-valued martingale $f = \{f_n\}$,*

$$C^{-1}\|f\|_{H^1(E)} \le \mathbf{E}\int\left\|\sum_{n=1}^{\infty} r_n(f_n - f_{n-1})\right\| \le C\|f\|_{H^1(E)}$$

where $\{r_m\}_{n=1}^{\infty}$ is a sequence of Rademacher functions independent of $\{f_n\}_{n=1}^{\infty}$ and C depends only on the UMD constant of E.

Note also that the righthand side inequality of Theorem 6 follows from the proof of Theorem 4.

We now state two corollaries to Theorems 4 and 6. The first deals with unconditional bases for $H^1(\mathcal{F}_n, E)$.

COROLLARY 1. *Let E be a UMD space and let $\{\mathcal{F}_n\}$ be a finite or infinite filtration of finite algebras. Then $H^1(\mathcal{F}_n, E)$ has an unconditional decomposition into copies of E.*

If in addition, E has an unconditional basis, then so does $H^1(\mathcal{F}_n, E)$.

Proof. The fact that the Rademacher sequence is distributionally invariant under changes of signs together with the contraction principle, (2), imply that

$$\sum_n \sum_{A\in\mathcal{A}_{n-1}} \sum_{B\in\mathcal{E}(A)} \oplus\, h_B \otimes E$$

is an unconditional direct sum in $H^1(\mathcal{B}_n, E)$. Now apply Theorem 4.

If E has an unconditional basis $\{e_i\}_{i=1}^{\infty}$, then by Theorem 6,

$$\left\|\sum_i \sum_n \sum_{A\in\mathcal{A}_{n-1}} \sum_{B\in\mathcal{E}(A)} a_{B,i} h_B \otimes e_i\right\|_{H^1} \approx \mathbf{E}\left\|\sum_i \sum_n \sum_{A\in\mathcal{A}_{n-1}} \sum_{B\in\mathcal{E}(A)} a_{B,i} r_n h_B e_i\right\|.$$

(cf. [LT], p. 50). By the Maurey-Khinchine inequality (E necessarily has cotype $< \infty$), the last expression is equivalent, with constant depending only on the UMD constant of E to

$$\mathbf{E}\left\| \sum_i \left(\sum_n \sum_{A \in \mathscr{A}_{n-1}} \sum_{B \in \mathscr{E}(A)} a_{B,i}^2 h_B^2 \right)^{1/2} e_i \right\|$$

which proves the second assertion of the corollary. ■

We remark that even for $E = R$ the corollary does not seem to be easy. This case, however, was known to Maurey and is stated without proof in [Ma1].

The second corollary is a form of the vector valued Gamlen-Gaudet theorem for $H^1(\delta)$. (The real valued infinite dimensional case was completely resolved in [Mü2].)

COROLLARY 8. *For the dyadic algebras* $\{(\delta_n\}$, *for any (finite or infinite) subsequence H of the Haar function,* $\overline{H \otimes E}$ *(closure in* $H^1(\delta_n, E)$*) is isomorphic to an* $H^1(\mathscr{F}_n, E)$ *space, with constant of isomorphism depending on the UMD constant* $\alpha_2(E)$ *of E only.*

The proof is analogous to that of Theorem 3.

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J. Kepler Universität
Linz, Austria
The Weizmann Institute of Science
Rehovot, Israel