ILLINOIS JOURNAL OF MATHEMATICS
Volume 36, Number 2, Summer 1992

$H^1(GL(V), V)$

BY

GARY A. MARTIN AND MARTIN ZIEGLER

Introduction

In this note we prove that for any infinite dimensional vector space V, the first cohomology group $H^1(GL(V), V)$ is trivial. (For us, $GL(V)$ is the group of all linear automorphisms of V, not just those with finite dimensional support.) Thirty years ago, D. G. Higman proved in [3] that $H^1(GL(V), V) = 0$ whenever V is a finite dimensional vector space of dimension at least four over a field K. G. W. Bell gives a proof in [1] that when V is a 3-dimensional space over the 2-element field $\mathbf{F}_2$, the indicated cohomology group is cyclic of order 2. By the easy argument in the next paragraph, and simple calculations in dimensions 1 and 2 over $\mathbf{F}_2$, $H^1(GL(V), V)$ is trivial for all vectors spaces over all fields with the one exception mentioned above.

We begin by observing that the only difficulty arises when $K = \mathbf{F}_2$. Indeed, if $K \neq \mathbf{F}_2$, then let $a \in K$ be different from 0 and 1. Let d: $GL(V) \to V$ be a derivation. We need to find a vector $v \in V$ such that for all $g \in GL(V)$, $dg = (1 - g)v$. Let $v = (1 - a)^{-1}\, da$. Using the centrality of a, it is straightforward to show that for all $g \in GL(V)$, $(1 - g)da = (1 - a)dg$. Then we get that

$$(1 - g)v = (1 - g)(1 - a)^{-1}\, da = (1 - a)^{-1}(1 - g)\, da = dg$$

as desired. (Note that this argument works for any subgroup of $GL(V)$ containing a non-trivial scalar transformation. The referee also points out that a standard spectral sequence argument shows that all higher cohomology vanishes for such a subgroup.)

Yet now, in the infinite dimensional case, although we could assume that $K = \mathbf{F}_2$, there seems to be no advantage in doing so. Since the argument we give works in either case, and is not complicated by allowing arbitrary fields, we present it in that generality.

We use $\operatorname{Hom}(V, W)$ to denote the set of K-linear transformations from V to W. Unless explicitly mentioned, we do not assume anything about the dimensions of vector spaces.

Received September 5, 1990
1980 Mathematics Subject Classification (1985 Revision). Primary 20G10.

1. Lemmas

We begin with a series of lemmas.

LEMMA 1. *Let V be a vector space of dimension at least 2. Then there exist $g, h \in GL(V)$ such that $g + h = 1$.*

Proof. First suppose that $\dim V = n$ is finite. Let p be a monic polynomial of degree n over K such that 0 and 1 are not roots of p. Let g be an endomorphism of V having p as its characteristic polynomial. (For example, let g be represented in some basis by the companion matrix of p.) Let $h = 1 - g$. Then $g, h \in GL(V)$, since neither has 0 as an eigenvalue. Now, if V is infinite-dimensional, decompose it as a direct sum of finite-dimensional subspaces W (each of dimension at least 2). On each summand W, take g_W as above. Let g be the direct sum of the g_W, and let $h = 1 - g$. □

We are grateful to U.M. Kuenzi for the easy proof of the following corollary. When we say that a group of linear transformations acts faithfully on a vector space, we mean no nonzero vector is fixed by all of the transformations in the group. We will use $\hat{V}$ to denote the profinite completion of V. By definition, this is the inverse limit of the finite-dimensional quotients of V. An element $\hat{c}$ of $\hat{V}$ is represented by a coherent system (c_W) of elements of V indexed by the subspaces W of finite codimension in V, where $W' \leq W$ implies $c_{W'} \equiv c_W (\mathrm{mod}\, W)$. $GL(V)$ acts on $\hat{V}$ by $g(\hat{c}) = \hat{d}$ where $d_W = g(c_{W'})$ for $W' = g^{-1}(W)$. This induces an action of $GL(V)$ on $\hat{V}/V$.

COROLLARY 2. *Let V be a vector space of dimension at least 2. Let $\hat{V}$ be the profinite completion of V. Then $GL(V)$ acts faithfully on $\hat{V}/V$.*

Proof. Let $\hat{c} \in \hat{V}$ be such that $\hat{c} + V$ is a fixed point of $GL(V)$. Let $g, h \in GL(V)$ such that $g + h = 1$, as in Lemma 1. Then $g(\hat{c}) \equiv \hat{c} (\mathrm{mod}\, V)$, so that $h(\hat{c}) \in V$, and hence $\hat{c} \in V$. □

Let $T \in \mathrm{Hom}(V, W)$. An automorphism $g \in GL(V)$ is called T-equivariant if $Tg = T$, or equivalently if $\mathrm{Im}(1 - g) \leq \mathrm{Ker}\, T$.

LEMMA 3. *Let $T \in \mathrm{Hom}(V, W)$ be a linear transformation. Let $V_0 = \ker T$, and suppose that $\dim V_0 \geq 2$. Then the group of T-equivariant automorphisms of V acts faithfully on V.*

Proof. Write $V = V_0 \oplus V_1$ and fix a nonzero element $a = a_0 + a_1$ of V. We construct $g \in GL(V)$ such that $\mathrm{Im}(1 - g) \leq \mathrm{Ker}\, T$ and $a \notin \mathrm{Ker}(1 - g)$. Let $f \in GL(V_0)$ and $h \in \mathrm{Hom}(V_1, V_0)$ with additional properties to be specified momentarily. For $v = v_0 + v_1$ with $v_i \in V_i$ we define

$$gv = fv_0 + (1 + h)v_1.$$

Since

$$g^{-1}v = f^{-1}v_0 + (1 - f^{-1}h)v_1,$$

we have $g \in GL(V)$. We also have $\mathrm{Im}(1 - g) \leq \mathrm{Ker}\, T$ because

$$(1 - g)v = (1 - f)v_0 + hv_1 \in V_0.$$

We need only choose f and h so that $(1 - f)a_0 + ha_1 \neq 0$. If $a_1 = 0$ then $a_0 \neq 0$, so we may choose f so that $(1 - f)a_0 \neq 0$, and h may be anything. If $a_1 \neq 0$ then we may choose h and f such that $ha_1 \neq 0$ and $(1 - f)a_0 = 0$. □

Suppose now that $W \leq V$. We will use G_W to denote the subgroup of $GL(V)$ of automorphisms which fix W pointwise. Consider the split exact sequence:

$$(*) \qquad 0 \to W \xrightarrow{i} V \xrightarrow{\pi} V/W \to 0$$

with a splitting $\xi \in \mathrm{Hom}(V/W, V)$ such that $\pi\xi = 1_{V/W}$.

LEMMA 4. *With notation as above, the exact sequence* $(*)$ *induces an exact sequence of groups,*

$$(**) \qquad 1 \to \mathrm{Hom}(V/W, W) \xrightarrow{\tilde{i}} G_W \xrightarrow{\tilde{\pi}} GL(V/W) \to 1,$$

with ξ *inducing an embedding* $\tilde{\xi}\colon GL(V/W) \to G_W$ *such that* $\tilde{\pi}\tilde{\xi} = 1_{GL(V/W)}$.

Proof. We define $\tilde{i}(f) = 1 + if\pi$ for $f \in \mathrm{Hom}(V/W, W)$. We define $\tilde{\pi}(g) = \pi g \xi$ for $g \in G_W$, but note that this is the natural map and does not depend on ξ. We define $\tilde{\xi}(h) = 1 - \xi\pi + \xi h \pi$ for $h \in GL(V/W)$. Verifying the claims is routine, so we leave it to the reader. □

For a splitting ξ of the exact sequence $(*)$, we use G_W^ξ to denote the image of $\tilde{\xi}$ in G_W. In some of the computations left to the reader in what follows, it is worth keeping in mind that $\tilde{\xi}^{-1}$ is $\tilde{\pi}$ restricted to G_W^ξ. For the same reason, note that for $f \in \mathrm{Hom}(V/W, W)$ and for a splitting $\xi \in \mathrm{Hom}(V/W, V)$ of $(*)$, $\tilde{i}(f)\xi = f + \xi$. We denote the kernel of $\tilde{\pi}$ by K_W, and observe that $\mathrm{Hom}(V/W, W)$ and K_W are canonically isomorphic via $\tilde{i}$.

LEMMA 5. *Suppose that W is a subspace of codimension at least 2 in V. As ξ varies over all splittings of the exact sequence* $(*)$, $\bigcup_\xi G_W^\xi$ *generates* G_W.

Proof. Fix a splitting $\eta \in \mathrm{Hom}(V/W, V)$ of $(*)$. By Lemma 4, $G_W = K_W \cdot G_W^\eta$, so it is enough to show that $\bigcup_\xi G_W^\xi$ generates K_W. By Lemma 1, there is a $g \in GL(V/W)$ such that $1 - g \in GL(V/W)$. Fix one. Then every element of $\mathrm{Hom}(V/W, W)$ is of the form $f(1-g)$ for some $f \in \mathrm{Hom}(V/W, W)$. Therefore every element of K_W is of the form $\tilde{i}(f(1-g))$ for some f as above. For each such f, let $\xi = \tilde{i}(f)\eta$. It is easy to see that ξ is also a splitting of $(*)$. But since a straightforward computation shows that $\tilde{i}(f(1-g)) = \tilde{\xi}(g^{-1})\tilde{\eta}(g) \in G_W^\xi \cdot G_W^\eta$, the lemma is proved. □

LEMMA 6. *Let $f \in \mathrm{Hom}(V/W, W)$ with* $\dim \ker f \geq 2$. *Let ξ be a splitting of $(*)$ and set $\eta = \tilde{i}(f)\xi$. Then $G_W^\xi \cap G_W^\eta$ acts faithfully on V/W.*

Proof. By Lemma 3, $\{g \in GL(V/W) | fg = f\}$ acts faithfully on V/W. Next, we note that $G_W^\xi \cap G_W^\eta = \xi(\{g \in GL(V/W) | fg = f\})$. Indeed, if $fg = f$, it is easy to check that $\tilde{\xi}(g) = \tilde{\eta}(g)$. It is also routine to see that any $g \in GL(V/W)$ and its image $\tilde{\xi}(g) \in G_W^\xi$ act in the same way on V/W. Since $\tilde{\xi}$ is injective, $G_W^\xi \cap G_W^\eta$ contains a subgroup which acts faithfully on V/W, and hence acts faithfully itself. □

2. The theorem

THEOREM. *Let V be a vector space over a field K. Then $H^1(GL(V), V) = 0$ unless* $\dim V = 3$ *and* $K = F_2$.

Proof. By Higman [2] we may assume the result for $4 \leq \dim V < \infty$. For $\dim V < 4$, if $K \neq \mathbf{F}_2$ then the cohomology vanishes by the remarks in the introduction. For $\dim V = 1$ and 2 over F_2, the result can be checked quickly by hand. The exceptional case $\dim V = 3$ $K = \mathbf{F}_2$ is treated by Bell in [1].

So assume that $\dim V$ is infinite. Consider the short exact sequence

$$(***) \qquad 0 \to V \to \hat{V} \to \hat{V}/V \to 0.$$

This induces a long exact cohomology sequence, part of which looks like

$$(****) \qquad H^0\big(GL(V), \hat{V}/V\big) \to H^1\big(GL(V), V\big) \to H^1\big(GL(V), \hat{V}\big).$$

As H^0 is the set of fixed-points of the action, by Corollary 2, $H^0(GL(V), \hat{V}/V) = 0$. Suppose that d: $GL(V) \to V$ is a derivation. We must show that the cohomology class of d is 0 in $H^1(GL(V), V)$. By the injectivity of the second map in $(****)$, it suffices to show that the cohomology class of the image of d in $H^1(GL(V), \hat{V})$ is 0. Since the map is induced by the inclusion of V into $\hat{V}$, we need only show that d is an inner derivation from $GL(V)$ into $\hat{V}$. That is to say, there is a $\hat{c} \in \hat{V}$ such that for all $g \in GL(V)$ $dg = (1 - g)\hat{c}$.

The construction of $\hat{c}$ will make use of the following claim.

CLAIM. *If $W \leq V$ and $4 \leq \operatorname{codim} W < \infty$, then there is a unique $c_W \in V/W$ such that for all $g \in G_W$ $dg \equiv (1 - g)c_W (\operatorname{mod} W)$.*

Proof of claim. Let $\xi \in \operatorname{Hom}(V/W, V)$ be a splitting of $(*)$, i.e., a section of the projection π of V onto V/W. Since $GL(V/W)$ acts faithfully on V/W and $\tilde{\xi}$: $GL(V/W) \to G_W$ is an embedding, we have that G_W acts faithfully on V/W. This is easily seen to imply the uniqueness of c_W.

To prove existence, consider $\pi d\tilde{\xi}$: $GL(V/W) \to V/W$. It is routine to check that this is a derivation. By the finite dimensional case, it is an inner derivation, so there is a $c^\xi \in V/W$ such that for all $h \in GL(V/W)$, $\pi d\tilde{\xi}h = (1 - h)c^\xi$. From this it follows that for all $g \in G_W^\xi$, $dg \equiv (1 - g)\xi c^\xi (\operatorname{mod} W)$. Since g fixes W pointwise, $dg \equiv (1 - g)c^\xi (\operatorname{mod} W)$. We now note that c^ξ is independent of the choice of ξ. For given some other splitting η of $(*)$, $f = \eta - \xi \in \operatorname{Hom}(V/W, W)$ so $\eta = \tilde{i}(f)\xi$. Suppose first that $\dim \ker f \geq 2$. Then if $g \in G_W^\xi \cap G_W^\eta$,

$$(1 - g)c^\xi \equiv (1 - g)c^\eta \qquad (\operatorname{mod} W).$$

which implies that g fixes $c^\xi - c^\eta$. So by Lemma 6, $c^\xi = c^\eta$. Now, if $\dim f < 2$, write $f = f_1 + f_2$ where $f_1, f_2 \in \operatorname{Hom}(V/W, W)$ each have kernels of dimension at least 2. (Decompose V/W as $V_1 \oplus V_2$, where each space has dimension at least 2; let $i_1 = 1 \oplus 0$ and $i_2 = 0 \oplus 1$; let $f_1 = fi_1$ and $f_2 = fi_2$.) Now take $\varepsilon = \tilde{i}(f_1)\xi$ and $\eta = \tilde{i}(f_2)\varepsilon$. Then the preceding argument applied twice gives $c^\xi = c^\eta$. So we may set $c_W = c^\xi$ and note that for all $g \in \bigcup_\xi G_W^\xi$, $dg \equiv (1 - g)c_W (\operatorname{mod} W)$. It is easy to see then that this remains true for all g in the group generated by $\bigcup_\xi G_W^\xi$. By Lemma 5, this is G_W. This establishes the claim.

For $4 \leq \operatorname{codim} W < \infty$ define c_W as in the claim. For $\operatorname{codim} W < 4$, pick $W' < W$ with $\operatorname{codim} W \geq 4$ and define $c_W = c_{W'}$. The uniqueness of c_W implies that (c_W) forms a coherent system and that, in the case $\operatorname{codim} W < 4$, c_W is well defined. So (c_W) represents an element $\hat{c}$ of $\hat{V}$. Let $G = \bigcup_W G_W$, where W ranges over subspaces of finite codimension in V. Then for each $g \in G$, $dg = (1 - g)\hat{c}$. To see this, suppose that $g \in G_W$. We must show that for each W' of finite codimension $dg \equiv c_{W'} - gc_{W''} (\operatorname{mod} W')$ where $g(W'') = W'$. If $W' \leq W$, then $W'' = W'$ and this follows from the claim. If not, let $U = W' \cap W$. Then $dg \equiv c_U - gc_U (\operatorname{mod}\ U)$. By coherence, both $dg \equiv c_U - gc_U (\operatorname{mod}\ W')$ and $c_{W'} - gc_{W''} \equiv c_U - gc_U (\operatorname{mod}\ W')$. The result follows.

It is clear that G is a normal subgroup of $GL(V)$. The inclusion map and the natural projection from $GL(V)$ onto $GL(V)/G$ induces the so-called restriction-inflation exact sequence on cohomology:

$$(*****)\qquad 0 \to H^1\big(GL(V)/G, (\hat{V})_G\big) \to H^1\big(GL(V), \hat{V}\big) \to H^1(G, \hat{V})$$

where $(\hat{V})_G$ denotes the elements of $\hat{V}$ that are fixed by all elements of G. We proved in the preceding paragraph that the image of d in $H^1(G, \hat{V})$ is the 0 cohomology class. Thus it suffices to show that $H^1(GL(V)/G, (\hat{V})_G) = 0$. But for this it suffices to show that $(\hat{V})_G = 0$. So suppose that $\hat{v} \in \hat{V}$ is nonzero. Then for some W of finite codimension, $v_W \not\equiv 0 \pmod W$. Since G_W acts faithfully on V/W, there is a $g \in G_W$ such that $gv_W \not\equiv v_W \pmod W$. Then $g\hat{v} \neq \hat{v}$, as desired. $\square$

References

1. Gregory W. Bell, *On the cohomology of the finite special linear groups*, II, J. Algebra, vol. 54 (1978), pp. 239–259.
2. Kenneth S. Brown, *Cohomology of groups*, Springer-Verlag, New York, 1982.
3. D.G. Higman, *Flag-transitive collineation groups of finite projective spaces*, Illinois J. Math., vol. 6 (1962), pp. 434–446.
4. P.J. Hilton and U. Stammbach, *A course in homological algebra*, Springer-Verlag, New York, 1971.

University of Massachusetts Dartmouth
North Dartmouth, Massachusetts

Albert-Ludwigs-Universitat
Freiburg i.Br., Federal Republic of Germany