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COPIES OF l_∞ IN KÖTHE SPACES OF VECTOR VALUED FUNCTIONS

BY

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Let (S, Σ, μ) be a σ-finite complete measure space and X be a Banach space. Recently the following result has appeared

THEOREM 1 [7]. *Let $1 \leq p < \infty$. Then l_∞ embeds into $L^p(\mu, X)$ if and only if it embeds into X.*

The purpose of this note is to extend Theorem 1 to a more general class of vector-valued functions; namely, Köthe spaces $E(X)$ of vector-valued functions. Specifically, we show that l_∞ embeds into $E(X)$ if and only if it embeds into either E or X. We recall that $L^p(\mu, X)$ spaces as well as Orlicz or Musielak-Orlicz spaces of vector-valued functions are special cases of Köthe spaces.

Before giving our result, we need some definitions and results. Let $\mathscr{M}(S) = \mathscr{M}$ be the space of Σ-measurable real valued functions with functions equal μ-almost everywhere identified. A Köthe space E [6] is a Banach subspace of $\mathscr{M}$ consisting of locally integrable functions such that (i) if $|u| \leq |v|$ μ. a.e., with $u \in \mathscr{M}$, $v \in E$ then $u \in E$ and $\|u\|_E \leq \|v\|_E$, (ii) for each $A \in \Sigma$, $\mu(A) < \infty$, the characteristic function χ_A of A is in E. Köthe spaces are Banach lattices if we put $u \geq 0$ when $u(s) \geq 0$ μ. a.e. Furthermore, Köthe spaces are σ-complete Banach lattices. The following theorems will be utilized in the sequel.

THEOREM 2 [5]. *Given a Köthe space E, there exists an increasing sequence (S_n) in Σ with $\mu(S_n) < \infty$ and $\mu(S \setminus \bigcup_{n \in N} S_n) = 0$ for which the following chain of continuous inclusions holds*:

$$L^\infty(S_n) \subset E(S_n) \subset L^1(S_n).$$

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We recall that a Banach lattice has an *order continuous norm* if, for every downward directed net $\{x_\alpha\}$ with $\inf_\alpha\{x_\alpha\} = 0$, $\lim_\alpha \|x_\alpha\| = 0$.

THEOREM 3 [6, p. 7]. *Let E be a σ-complete Banach lattice not containing l_∞. Then E has an order continuous norm.*

In this paper, we consider, for a real Banach space X, the family of all strongly measurable functions F: $S \to X$ (identifying functions which are μ.a.e. equal) such that $\|f(\cdot)\|_X \in E$, where E is a Köthe space. Such a space, denoted by $E(X)$, is a Banach space under the norm $\|f\|_{E(X)} = \| \|f(\cdot)\|_X\|_E$.
We need something more.

THEOREM 4 [8]. *Let T: $l_\infty \to F$, F a Banach space, be an operator with $T(e_n) \to 0$. Then there is an infinite subset M of N with $T_{|l^\infty(M)}$ an isomorphism.*

A (bounded) subset H of a Banach space F is *limited* if for any w^*-null sequence $(x_n^*) \subset F^*$ one has

$$\lim_n \sup_H |x_n^*(x)| = 0.$$

The following result can be found in [3] and [9].

THEOREM 5 [3], [9]. *Let (x_n) be a copy of the unit vector basis of c_0 in a Banach space F. If (x_n) is not limited, then a subsequence $(x_{k(n)})$ of (x_n) spans a complemented copy of c_0 inside F.*

THEOREM 6 [2]. *If H is a limited subset of a Banach space F, then*

$$\lim_k \sup_H \|T_k(x)\|_Z = 0$$

for every sequence (T_k) of operators from F into an arbitrary Banach space Z such that $\lim_k \|T_k(x)\|_Z = 0$ for all $x \in F$.

We are now ready to prove our result. Using general principles, we are able to embed $E(X)$ "locally" into a suitable $L^1(\mu, X)$-space; then we can use Mendoza's theorem to reach our goal.

THEOREM 7. *l_∞ embeds into $E(X)$ if and only if it embeds into either E or X.*

Proof. We need to show the "only if" part. Let us assume l_∞ does not embed into E. We show that l_∞ must embed into X. First of all, we prove it is possible to suppose $\mu(S) < \infty$. Let j denote the isomorphism of l_∞ onto a closed subspace of $E(X)$. We observe that $(j(e_n))$ is a limited sequence, otherwise by virtue of Theorem 5 we should have a copy of c_0 contained in $j(l_\infty)$ and complemented in $E(X)$; this would give the existence of a projection of l_∞ onto c_0; a contradiction, because such a projection would be weakly compact [1, p. 150]. Now, let (S_n) be the sequence of elements of Σ considered in Theorem 2. For all $k \in \mathbf{N}$, we consider $E(S_k, X)$, the Köthe space of vector valued functions defined on S_k. It is clear that $E(S_k, X)$ can be isometrically embedded into $E(X)$ (identifying it with $\{f\chi_{S_k}: f \in E(X)\}$); hence we assume that $E(S_k, X)$ is a closed subspace of $E(X)$. It is very simple to see that the linear operator P_k: $E(X) \to E(S_k, X)$ defined by $P_k(f) = f\chi_{S_k}$ is continuous and that $\|P_k(f) - f\|_{E(X)} \to 0$, for all $f \in E(X)$, because, thanks to Theorem 3, E is an order continuous Banach lattice.

Now, recall the following well known result: If T_k are operators from l_∞ converging in the strong operator topology to T and no T_k preserves a copy of l_∞, then T does not preserve a copy of l_∞. (The proof of this result can be easily obtained using some facts contained in [1], Chapters I and VI; see, for instance, the proof of Corollary 5 on p. 150 of [1].) Since (P_k) converges in the strong operator topology to the identity on $E(X)$ and l_∞ embeds into $E(X)$, one of the operators P_k must preserve a copy of l_∞. So there is $k^* \in \mathbf{N}$ such that $E(S_{k^*}, X)$ contains a copy of l^∞. Since $\mu(S_{k^*}) < \infty$, our claim is proved. So let us assume $\mu(S) < \infty$ in the sequel.

Let $f \in E(X)$. f is strongly measurable and $u(\cdot) = \|f(\cdot)\|_X$ is in E; by virtue of Theorem 2, $u \in L^1(S)$ and this gives that $f \in L^1(S, X)$. Furthermore, the inclusion j_1: $E(X) \to L^1(S, X)$ is continuous. Indeed, if $f \in E(X)$, $\bar{u}(\cdot) = \|f(\cdot)\|_X \in E$; by virtue of Theorem 2 there is $c_2 > 0$ such that $\|u\|_{L^1(S)} \leq c_2\|u\|_E$ for all $u \in E$ and, applying this last inequality to $\bar{u}$, we get $\|f\|_{L^1(S,X)} \leq c_2\|f\|_{E(X)}$ for all $f \in E(X)$. The existence of this continuous embedding, Theorem 4 and Mendoza's theorem 1 imply that $\lim_n \|j(e_n)\|_{L^1(S,X)} = 0$. Now, we need to show that

$$\lim_{\mu(A)\to 0} \sup_n \| j(e_n)\chi_A\|_{E(X)} = 0. \tag{1}$$

If (1) were false, we could find a sequence $(A_k) \subset \Sigma$, $\mu(A_k) < 1/2^k$, and a subsequence $(j(e_{n(k)}))$ of $j(e_n)$ such that

$$\inf_k \left\| j(e_{n(k)})\chi_{A_k} \right\|_{E(X)} > 0. \tag{2}$$

Now, let $B_h = \bigcup_{k=h}^{\infty} A_k$. It is clear that $B_h \supset A_h$, $B_h \supset B_{h+1}$ and $\mu(B_h) \to 0$. Let $f \in E(X)$. This means that $u(\cdot) = \|f(\cdot)\|_X \in E$. We have that $\{u\chi_{B_h}\}$ is a downward directed sequence in E with $\inf_h\{u\chi_{B_h}\} = u\chi_{\bigcap_{h\in\mathbf{N}} B_h} = 0$, because $\chi_{\bigcap_{h\in\mathbf{N}} B_h}$ is surely equal to zero everywhere outside of $\bigcap_{h\in\mathbf{N}} B_h$, a set of measure zero. Since E is an order continuous Banach lattice, $\lim_h \|u\chi_{B_h}\|_E =$

0. Since $u(s)\chi_{B_h}(s) \geq u(s)\chi_{A_h}(s)$ on S and E is a Köthe space, we have $\lim_h \|u\chi_{A_h}\|_E = 0$. Now observe that

$$u(\cdot)\chi_{A_h}(\cdot) = \|f(\cdot)\chi_{A_h}(\cdot)\|_X;$$

hence $\lim_h \|f\chi_{A_h}\|_{E(X)} = 0$. This means that the operators T_h: $E(X) \to E(X)$ defined by $T_h(f) = f\chi_{A_h}$ verify the limit relation

$$\lim_h \|T_h f\|_{E(X)} = 0$$

for all $f \in E(X)$. On the other hand, $(j(e_n))$ is limited and so, by virtue of Theorem 6, we get

$$\lim_k \sup_n \|j(e_n)\chi_{A_k}\|_{E(X)} = 0,$$

a fact that contradicts (2). Hence (1) is true.

This means that $\{\|j(e_n)\|_X\}$ is equi-integrable in E [4, pp. 135–136]. Now, let us recall the following well known result: If E is an order continuous Köthe space and $\{x_n\}$ is a sequence which is equi-integrable in E and $\lim_n \|x_n\|_{L^1} = 0$, then $\lim_n \|x_n\|_E = 0$ (for a proof, use also the existence of a continuous embedding of $L^\infty(S)$ into E). From this result it follows that $\lim_n \|j(e_n)\|_{E(X)} = 0$, a contradiction that finishes our proof.

COROLLARY. *$E(X)$ is an order continuous Banach lattice if and only if E and X are, provided X is a σ-complete Banach lattice.*

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