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A HECKE CORRESPONDENCE THEOREM FOR AUTOMORPHIC INTEGRALS WITH RATIONAL PERIOD FUNCTIONS

BY

JOHN H. HAWKINS AND MARVIN I. KNOPP[1]

I. Introduction

1. Following Riemann, Hecke in his celebrated work [4], [5] uncovered—using the Mellin transform and its inverse—the systematic theory relating automorphic (in particular, modular) forms to Dirichlet series satisfying a functional equation invariant under a transformation of the form $s \to \alpha - s$, with α real. Knopp, in [9], demonstrated that the Mellin transform of a modular integral, with rational period function, on the full modular group $\Gamma(1) = SL(2, \mathbf{Z})$ satisfies precisely the same functional equation occurring in Hecke, provided that the poles of the period function in question lie in $\mathbf{Q}$. (By Theorem 1 of [9], this means the poles are either 0 or ∞.) Moreover, he proved a converse theorem, as in Hecke, whence the simple functional equation discovered by Riemann and Hecke can no longer obtain when the period function has poles outside of $\mathbf{Q}$. Nevertheless, Theorem 2 in Knopp's earlier article [8] suggests the possibility, in the latter case, of a functional equation with a more complex structure (but still under a transformation of the form $s \to \alpha - s$, α real).

We have discovered just such a functional equation for the Mellin transform of a modular integral, with arbitrary rational period function, of any (integral) weight; this is the main object of the present article. Theorem 2 (§III) describes this result and Theorem 4 (§IV) the expected converse; these results include as special cases Theorem 3 of [9] and its converse, Theorem 4 of [9], when the poles of the rational period function are 0 or ∞. It is curious that while the ordinary hypergeometric functions figure prominently in the proof of Theorem 2, they drop out in the calculation of the functional equation (and hence do not appear in the statement of Theorem 4).

It is essential to emphasize that we formulate the results of §§III, IV not for the full modular group $\Gamma(1)$, but instead for the subgroup Γ_θ, of index 3 in $\Gamma(1)$. This choice is not merely a matter of convenience or a desire for

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generality; it goes to the heart of our method of proof. We began this work with entire modular integrals (with rational period functions) on the full modular group explicitly in mind, but we found that the much larger class of entire modular integrals on Γ_θ is the appropriate context for our results. For, as we gradually realized, the methods we employ in no way depend upon the stringent conditions met by rational period functions on $\Gamma(1)$. On the contrary, it suffices to invoke the much weaker restrictions satisfied by rational period functions for Γ_θ. Presumably, there is an alternative approach that exploits these stricter conditions to obtain a form of the functional equation that reflects the extra structure imposed by the relations in $\Gamma(1)$. (See the speculation of §V).

We thank Dr. Richard Cavaliere for a number of stimulating conversations at the beginning of our work on this article.

2. Definitions and notations. $\Gamma(1)$, the full *modular group*, is the set of matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $a, b, c, d \in \mathbf{Z}$ and $ad - bc = 1$. For the most part, we shall consider $\Gamma(1)$ as a group of linear fractional transformations acting on the Riemann sphere:

$$Mz = \frac{az + b}{cz + d}.$$

While this interpretation negates the distinction between M and $-M$, we must maintain this distinction in the discussion of multiplier systems below. Similarly for subgroups of $\Gamma(1)$.

$\Gamma(1)$ is generated by the two transformations

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \tag{1.1}$$

as matrices, $T^2 = (ST)^3 = -I$, but in keeping with the identification of M and $-M$ in $\Gamma(1)$ we write

$$T^2 = (ST)^3 = I. \tag{1.2}$$

In fact, these are the only relations in the generators S, T of $\Gamma(1)$. (See, for example, [7, Chapter 1] for these and the following facts about Γ_θ).

The *theta-subgroup* Γ_θ is the subgroup of $\Gamma(1)$ generated by S^2 and T (where $S^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, of course). By the remark following (1.2), $T^2 = I$ is the only relation in the generators S^2, T of Γ_θ, for our purposes, the crucial difference between Γ_θ and $\Gamma(1)$.

Generalizing the definition of $\Gamma(1)$, Hecke [4], [5] considered the class of groups $G(\lambda_n)$—now called the *Hecke* groups—where $\lambda_n = 2\cos(\pi/n)$, $n \in \mathbf{Z}$, $n \geq 3$ and $G(\lambda_n)$ is generated by $S_n = \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix}$ and T; with the identification of $\pm I$, the relations in this case are

$$(1.3) \qquad T^2 = (S_n T)^n = I.$$

Note that in the Hecke notation $\Gamma(1) = G(\lambda_3)$, while Γ_θ may be written $G(\lambda_\infty)$, since $\lambda_\infty = 2\cos 0 = 2$. (Hecke also considered the discrete groups $G(\lambda)$, generated by $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ and T, for arbitrary $\lambda > 2$; for $\lambda < 2$ he restricted himself to $\lambda = 2\cos(\pi/n)$, since these are the only $\lambda < 2$ for which $G(\lambda)$ is discrete [4, Chapter 3], [5, §5].)

Suppose $2k$ is an integer (not necessarily even) and $F \not\equiv 0$ is a function meromorphic in $\mathscr{H}$, the upper half-plane, such that

$$(1.4) \quad \bar{v}(M)(cz+d)^{-2k} F(Mz) = F(z), \quad z \in \mathscr{H}, \quad M = \begin{pmatrix} * & * \\ c & d \end{pmatrix},$$

for all $M \in \Gamma$, a discrete subgroup of $SL(2, \mathbf{R})$, where $v(M)$ is a complex number depending upon M (not upon z) and $|v(M)| = 1$, for all $M \in \Gamma$. The set $\{v(M) | M \in \Gamma\}$ is called a *multiplier system for* Γ *and the weight* $2k$. We observe that, in general, $v(M)$ is a function on the matrix group Γ, but not on the linear fractional group Γ. To see why this is so, recall that $(-M)z = Mz$, since both equal $(az+b)/(cz+d)$. Thus, since $F \not\equiv 0$, (1.4) implies that

$$\bar{v}(M)(cz+d)^{-2k} = \bar{v}(-M)(-cz-d)^{-2k},$$

or

$$(1.5) \qquad v(-M) = (-1)^{2k} v(M).$$

(To put this another way, $v(M)$ is a function on the linear fractional group Γ only if $2k$ is even.) It follows from (1.4) that a multiplier system v is a character on the matrix group Γ (on the linear fractional group Γ as well, when $2k$ is even).

If the function F satisfying (1.4) is *holomorphic* in $\mathscr{H}$ and satisfies the growth restriction

$$(1.6) \qquad |F(z)| \leq K(|z|^\alpha + y^{-\beta}), \quad y = \operatorname{Im} z > 0,$$

for some $K, \alpha, \beta > 0$, then F is called an *entire automorphic* (*modular*, if $\Gamma \subset \Gamma(1)$) *form on* Γ, *of weight* $2k$, *with multiplier system* v.

Here, we are interested in the generalization to *entire modular* (or *automorphic*) *integrals*, in which the characteristic functional equation (1.4) is replaced by

(1.7)
$$\bar{v}(M)(cz+d)^{-2k}F(Mz) = F(z) + q_M(z), \quad z \in \mathscr{H}, \quad M = \begin{pmatrix} * & * \\ c & d \end{pmatrix},$$

for all $M \in \Gamma$, where again F is *holomorphic* in $\mathscr{H}$ and satisfies (1.6), v is a multiplier system on Γ, as before, and q_M is a rational function of z, the *rational period function of F corresponding to* $M \in \Gamma$. Once again, $2k$ is called the *weight* of F.

Introducing the customary *stroke operator* $F|_{2k}^{v}M$ (or simply $F|M$) to represent the left-hand side of (1.7), we may rewrite (1.7) as

$$F|M = F + q_M, \quad M \in \Gamma. \tag{1.8}$$

For any function F defined on the Riemann sphere, the fact that v is a character on Γ implies

$$F|M_1M_2 = (F|M_1)|M_2, \quad M_1, M_2 \in \Gamma,$$

so that

$$q_{M_1M_2} = q_{M_1}|M_2 + q_{M_2}, \quad M_1, M_2 \in \Gamma, \tag{1.9}$$

follows from (1.8).

For present purposes it is sufficient to consider only the Hecke groups $\Gamma = G(\lambda_n)$, $n \geq 3$, and we include Γ_θ by allowing $n = \infty$. Since $G(\lambda_n)$ is generated by S_n and T, (1.8) is equivalent in this case to

$$F|S_n = F + q_{S_n}, \quad F|T = F + q_T. \tag{1.10}$$

The entire automorphic integrals for Γ_θ are precisely the class for which we may expect a correspondence theorem analogous to the one Hecke found for automorphic forms, provided $q_{S_n} \equiv 0$. For then (1.10) takes the form

$$F|_{2k}^{v}S_n = F, \quad F|_{2k}^{v}T = F + q_T \tag{1.11}$$

and $F|S_n = F$, together with (1.6), yields the expansion

$$F(z) = \sum_{m+\kappa \geq 0} a_{m+\kappa} \exp[2\pi i(m+\kappa)z/\lambda_n], \quad z \in \mathscr{H}, \tag{1.12}$$

with κ defined by

$$\mathscr{v}(S_n) = \exp(2\pi i\kappa), \quad 0 \le \kappa < 1. \tag{1.13}$$

In particular, the system of period functions of an automorphic integral F for Γ_θ is effectively generated by the single period function q_T. We shall call q_T *the* period function of F; we generally write q instead of q_T, also.

Since F satisfies (1.6) it is relatively easy to check that

$$a_{m+\kappa} = O(m^\gamma), \quad m \to +\infty, \tag{1.14}$$

for some $\gamma > 0$, in fact, that (1.14) is equivalent to (1.6). Thus, the *Dirichlet series associated with* F, defined formally by its Mellin transform,

$$\phi(s) = \phi_F(s) = \sum_{m+\kappa>0} a_{m+\kappa}/(m+\kappa)^s, \tag{1.15}$$

actually converges absolutely in the right half-plane $\sigma > 1 + \gamma$ and uniformly on compact subsets thereof.

Now, for the groups $G(\lambda_n)$, including $\Gamma(1)$ $(n = 3)$, but *excluding* Γ_θ $(n = \infty)$, the defining relations (1.3) impose upon the rational period function $q = q_T$ of F in (1.11) the necessary conditions

$$q|_{2k}^{\mathscr{v}} T + q = 0 \tag{1.16}$$

and

$$q|_{2k}^{\mathscr{v}}(S_nT)^{n-1} + q|_{2k}^{\mathscr{v}}(S_nT)^{n-2} + \cdots + q = 0. \tag{1.17}$$

(Obviously, nontrivial rational solutions exist only if the weight $2k$ is an integer—which is why we consider only integral weights.) In contrast, for $\Gamma = \Gamma_\theta$ only the condition (1.16) is imposed upon q_T by (1.11). This is the distinction between Γ_θ and the other Hecke groups we can exploit: it turns out that construction of q satisfying (1.16) lies fairly close at hand (§II), while the determination of q satisfying (1.16) and (1.17) simultaneously is a difficult problem [3]. In any event, these necessary conditions upon q ((1.16) alone or (1.16) and (1.17)) are sufficient as well, in the sense that if they hold for q, then there exists F *meromorphic* in $\mathscr{H}$ and satisfying (1.11). (See [10].) If the weight $2k \ge 2$, the construction of [10] can be arranged so that F is *holomorphic* in $\mathscr{H}$ and has growth restriction (1.6) (i.e., F is an *entire* automorphic—or modular—integral). If $2k \le 0$, we can take F to be holomorphic in $\mathscr{H}$, but not necessarily fulfilling the condition (1.6). (See §III.1) If $2k = 1$, we may not assume that F is holomorphic in $\mathscr{H}$.

As we infer from the discussion above of the relations (1.16) and (1.17), the class of rational period functions q for Γ_θ is a much wider one than that for any of the other Hecke groups. In fact, any rational period function for any Hecke group is necessarily a rational period function for Γ_θ, but not conversely. Thus, a result for rational period functions connected with Γ_θ necessarily has far broader applicability than the corresponding result for any other Hecke group. In part for this reason and in part because our present method of proof does not appear well suited to exploiting the relation (1.17) to strengthen our conclusions, we restrict our attention in this paper almost exclusively to the group Γ_θ and its rational period functions.

3. Outline. In §II we determine all rational period functions for Γ_θ and in fact we give there a basis for the vector space of these rational period functions that is relatively easy to apply in treating the Mellin transform Φ_F of an entire modular integral F on Γ_θ. In §III we turn to the Mellin transforms themselves—the focus of our attention—and we derive the principal result, Theorem 2 (§III.2), which contains the functional equation (3.9), (3.25) of Φ_F. In §III.3 we describe in detail the meromorphic continuation to the entire complex s-plane of $\Phi_F(s)$, which has at worst simple poles at integral values of s. Section IV presents the statement and proof of Theorem 4, the converse of Theorem 2.

II. Rational period functions for Γ_θ

1. In this section, we describe the general solution of the functional equation

$$q|_{2k}^{v}T + q = 0 \tag{2.1}$$

in the space $\mathscr{C}(z)$ of rational functions. The proof is given in a paper of Hawkins [3]. The solution allows us to express the Mellin transform of an entire automorphic integral for Γ_θ in a form that readily leads to a formulation and proof of the Hecke correspondence theorem referred to in the title of this paper (see Theorems 2, 4 in §§III, IV).

As we pointed out earlier the weight $2k$ must be integral, so the consistency condition for multiplier systems implies in particular that

$$v(T) = \pm i^{2k}. \tag{2.2}$$

We define $\delta = \delta(v, k)$ by

$$\delta = \tfrac{1}{2}\left[1 + v(T)i^{2k}\right]. \tag{2.3}$$

Next, for any integer r, we define $f_r(\alpha)$ by

$$f_r(\alpha)=\frac{1}{(z-\alpha)^r}\Big|(I-T) = \frac{1}{(z-\alpha)^r}-\frac{\bar{v}(T)\left(-\frac{1}{\alpha}\right)^r}{z^{2k-r}\left(z+\frac{1}{\alpha}\right)^r}, \tag{2.4}$$

when $\alpha \neq 0$, and, when $\alpha = 0$, by

$$f_r(0) = z^{-r}|(I-T) = \frac{1}{z^r}-\frac{\bar{v}(T)(-1)^r}{z^{2k-r}}. \tag{2.5}$$

(For convenience we have written $|$ for $|_{2k}^{v}$.) Henceforth, when we write $f_r(\alpha)$, we shall tacitly assume $\alpha \neq 0, \pm i$ for convenience.

THEOREM 1. *Let $q(z)$ be a rational solution of* (2.1),

$$q|_{2k}^{v}T+q=0.$$

Then, in the notation above, $q(z)$ has the unique representation

$$q(z)=\sum_{k\le r\le L} c_r f_r(0)+\sum_{j=1}^{p}\sum_{r=1}^{M(j)} c_{rj} f_r(\alpha_j)+\sum_{\substack{r=1\\ r\equiv 2k+\delta(2)}}^{M} d_r f_r(i) + \sum_{\substack{r=1\\ r\equiv\delta(2)}}^{M'} d'_r f_r(-i), \tag{2.6}$$

where c_r, c_{rj}, d_r, and d'_r are complex constants, with $c_k=(1-\delta)c_k$ for even $2k$. Conversely, any rational function q of the form (2.6) *is a solution of* (2.1).

Remarks. (i). For even $2k$ and $\delta=1$, $f_k(0)\equiv 0$; this is the reason for the requirement $c_k=(1-\delta)c_k$ for even $2k$ (otherwise, the representation (2.6) is not unique).

(ii). The functions $f_r(0)$, for $r \geq k$ ($r>k$, except when $2k$ is even and $\delta=0$), $f_r(\alpha_j)$, for $r\geq 1$, $f_r(i)$, for $r\equiv 2k+\delta(2)$ (and $r\geq 1$), and $f_r(-i)$, for $r\equiv\delta(2)$ (and $r\geq 1$), form a basis over $\mathbf{C}$ for the linear space of solutions of (2.1) in $\mathbf{C}(z)$.

Theorem 1 is actually more general than we require; we need only those solutions of (2.1) holomorphic in $\mathscr{H}$, because only those can occur as rational period functions of *entire* modular integrals. Furthermore we make no use of the uniqueness of the representation (2.6), that is, of the linear independence of the various functions f_r that appear in (2.6). What we use in §§III, IV is the following consequence of Theorem 1.

COROLLARY. *The rational period function $q(z)$ of an entire automorphic integral of integral weight $2k$ and multiplier system $\mathscr{v}$ for Γ_θ has the representation*

$$q(z) = \sum_{k \le r \le L} c_r f_r(0) + \sum_{j=1}^{p} \sum_{r=1}^{M(j)} c_{rj} f_r(\alpha_j) + \sum_{r=1}^{M} d_r f_r(-i), \tag{2.6$'$}$$

where c_r, c_{rj}, and d_r are complex constants, with $c_k = (1-\delta)c_k$ for even $2k$. Here, $\operatorname{Im} \alpha_j \le 0$ *and $\alpha_j \ne -i$, for $1 \le j \le p$.*

2. Since the proof of Theorem 2 depends in an essential way upon the corollary to Theorem 1, for the sake of completeness we give here a detailed, independent proof of the corollary.

To begin, all of the rational functions $f_r(0)$, $f_r(\alpha_j)$, $f_r(-i)$ arising in the corollary satisfy the condition (2.1) since they have the form $\varphi | T - \varphi$. Furthermore, because

$$\operatorname{Re} Tz = \operatorname{Re}(-1/z) = -\operatorname{Re} z/|z|^2,$$

a function satisfying (2.1), and without poles in $\operatorname{Re} z > 0$, likewise has no poles in $\operatorname{Re} z < 0$. From these two observations we conclude: if $\alpha_1, \ldots, \alpha_l$ are the poles of q with positive real part, there exist complex numbers c_{rj}, $1 \le j \le l$, such that the poles of

$$q_1(z) = q(z) - \sum_{j=1}^{l} \sum_{r=1}^{M(j)} c_{rj} f_r(\alpha_j)$$

are restricted to the imaginary axis and the point $i\infty$. (Here $M(j)$ is the order of the pole α_j.)

Since $q(z)$ has no poles in $\mathscr{H}$, neither does $q_1(z)$. This leaves the points $z = iy$, $y \le 0$, and $i\infty$ to consider. The points $-i$ and 0 (the latter possibly appearing together with $i\infty$) are exceptional when they occur as poles of q. Putting aside discussion of these points for now, we let $\alpha_{l+1}, \ldots, \alpha_p$ be the poles of q in $(-i, 0)$. Since T: $z \to -1/z$ interchanges the two intervals $(-i, 0)$ and $(-i\infty, -i)$, a function satisfying (2.1) and without poles in $(-i, 0)$

has no poles in $(-i\infty, -i)$. Hence, there are complex numbers c_{rj}, $l+1 \le j \le p$, such that

$$\begin{aligned} q_2(z) &= q_1(z) - \sum_{j=l+1}^{p} \sum_{r=1}^{M(j)} c_{rj} f_r(\alpha_j) \\ &= q(z) - \sum_{j=1}^{p} \sum_{r=1}^{M(j)} c_{rj} f_r(\alpha_j) \end{aligned}$$

has poles, if any, only at 0, $i\infty$ and $-i$.

The point $-i$ must be treated with more care since it is fixed by T. In this case a simple calculation reduces (2.4) to

$$\begin{aligned} f_r(-i) &= \{1 - \bar{v}(T)(-i)^{2r} z^{r-2k}\}(z+i)^{-r} \\ &= \{1 - \bar{v}(T)(-1)^{r} i^{2k}\}(z+i)^{-r} + \text{higher powers of } z+i, \end{aligned}$$

where the second line follows from the first after replacement of z^{r-2k} by its Taylor expansion at $-i$. We note that by (2.2), $\bar{v}(T)(-1)^r i^{2k} = \pm 1$. Thus, at $-i$, $f_r(-i)$ has the Taylor expansion

$$f_r(-i) = \begin{cases} 2(z+i)^{-r} & + \text{higher powers of } z+i, \\ \gamma(z+i)^{-r+1} & + \text{higher powers of } z+i, \end{cases}$$

for $\bar{v}(T)(-1)^r i^{2k} = -1, +1$, respectively.

On the other hand, suppose the rational period function $q(z)$ has the Taylor expansion

$$q(z) = \alpha(z+i)^{-r} + \text{higher powers of } z+i,$$

at $z = -i$. Then the Taylor expansion at $-i$ of $q|T + q$ has the form

$$\alpha\{1 + \bar{v}(T)(-1)^r i^{2k}\}(z+i)^{-r} + \text{higher powers of } z+i.$$

Since $q|T + q = 0$, the assumption $\alpha \neq 0$ implies that $\bar{v}(T)(-1)^r i^{2k} = -1$. This shows that $\bar{v}(T)(-1)^r i^{2k} = -1$ is a necessary condition for a rational period function q to have the term $(z+i)^{-r}$ as the smallest power of $z+i$ in its principal part at $-i$. It follows that there exist complex numbers d_r such that the rational period function

$$\begin{aligned} q_3(z) &= q_2(z) - \sum_{r=1}^{M} d_r f_r(-i) \\ &= q(z) - \sum_{j=1}^{p} \sum_{r=1}^{M(j)} c_{rj} f_r(\alpha_j) - \sum_{r=1}^{M} d_r f_r(-i) \end{aligned}$$

has poles, if any, only at 0 and $i\infty$. (Here M is the order of the pole $-i$; by the calculation above, the parity of M is determined by the condition $(-1)^M = -v(T)i^{-2k}$.)

We have

$$q(z) = q_3(z) + \sum_{j=1}^{p} \sum_{r=1}^{M(j)} c_{rj} f_r(\alpha_j) + \sum_{r=1}^{M} d_r f_r(-i),$$

where $q_3(z) = \Sigma_m z^m$, a Laurent polynomial, satisfies the relation (2.1). Observe that

$$z^{-r}|T = \bar{v}(T)(-1)^r z^{r-2k}, \quad \text{or} \quad z^{r-2k}|T = \bar{v}(T)(-1)^{2k-r} z^{-r};$$

thus a Laurent polynomial satisfying (2.1) and with no term of the form z^{-r}, $r > k$, likewise has no term of the form z^{r-2k}. It follows that there exist complex numbers $c_{k+1}, \dots, c_L, \beta$ such that

$$q_3(z) - \sum_{k+1 \le r \le L} c_r f_r(0) = \begin{cases} \beta z^{-k} & \text{if } k \in Z, \\ 0 & \text{if } k \notin Z. \end{cases}$$

(Recall that $2k \in Z$, but k may not be in Z.)

If $k \notin Z$, then $q_3(z) = \Sigma_{k+1 \le r \le L} c_r f_r(0)$. Assume $k \in Z$. From (2.5), we have

$$f_k(0) = \left(1 - \bar{v}(T)(-1)^k\right) z^{-k}.$$

If $\bar{v}(T)(-1)^k \neq 1$, choose $c_k = \beta/(1 - \bar{v}(T)(-1)^k)$. Suppose, on the other hand, that $\bar{v}(T)(-1)^k = 1$, so that $f_k(0) = 0$. Applying (2.1) to βz^{-k}, we find that $0 = \beta z^{-k}(1 + \bar{v}(T)(-1)^k)$, so that $\beta = 0$; in this case the choice of c_k is arbitrary. In all cases we have shown that $q_3(z) = \Sigma_{k \le r \le L} c_r f_r(0)$, so that (2.6′) holds and the proof of the corollary is complete.

III. The direct Hecke theorem

The main result of this section, Theorem 2, is an explicit form, based on the representation (2.6), of the functional equation for the Dirichlet series associated with an entire automorphic integral F of weight $2k$ and multiplier system v for Γ_θ.

1. By (1.14) the Dirichlet series associated with F, namely

$$\phi(s) = \phi_F(s) = \sum_{m+\kappa>0} a_{m+\kappa}/(m+\kappa)^s, \tag{3.1}$$

converges absolutely in some right half-plane, say, $\sigma > 1 + \gamma$ (where $s = \sigma + it$, as usual).

If F is an (entire) automorphic integral of integral weight $2k$ and multiplier system v for Γ_θ, with rational period function q, for convenience we may always assume $F(iy) \to 0$ as $y \to +\infty$ because (see (2.5)) $f_0(0) = 1 - \bar{v}(T)z^{-2k}$ is the period function for the integral (of weight $2k$) $F(z) \equiv -1$. Thus, in all cases, by (1.12) and (1.14), as $y \to +\infty$,

$$F(x+iy) = O(e^{-\pi\varepsilon y}) \tag{3.2}$$

uniformly in x, for some $\varepsilon > 0$.

By (1.6) and (3.2), it makes sense to consider the *Mellin transform of* F, defined by

$$\Phi(s) = \Phi_F(s) = \int_0^\infty F(iy)y^s \frac{dy}{y}. \tag{3.3}$$

for $\sigma > \beta$. By absolute convergence, we have, for $\sigma > 1 + \beta$,

$$\Phi(s) = \pi^{-s}\Gamma(s)\phi(s), \tag{3.4}$$

where $\phi(s)$ is given by (3.1); in fact, by (1.6), we can take $\gamma = \beta$ in (1.14) so for $\sigma > 1 + \beta$,

$$\int_0^\infty \sum_{m+\kappa>0} a_{m+\kappa} e^{-\pi(m+\kappa)y} y^s \frac{dy}{y} = \sum_{m+\kappa>0} a_{m+\kappa} \int_0^\infty e^{-\pi(m+\kappa)y} y^s \frac{dy}{y}$$

because, by (1.14)

$$\left| a_{m+\kappa} \int_0^\infty e^{-\pi(m+\kappa)y} y^s \frac{dy}{y} \right| \le \frac{|a_{m+\kappa}|}{(m+\kappa)^\sigma}\Gamma(\sigma) = O\big(\Gamma(\sigma)/m^{\sigma-\beta}\big)$$

and $1/m^{\sigma-\beta}$ is a term of a convergent series.

Now, following Hecke (and Riemann) we have, for $\sigma > \beta$, by (1.11),

$$\begin{aligned}\int_0^1 F(iy)y^s \frac{dy}{y} &= \int_1^\infty F(-1/iy)y^{-s} \frac{dy}{y} \\ &= v(T)i^{2k}\int_1^\infty F(iy)y^{2k-s}\frac{dy}{y} + v(T)i^{2k}\int_1^\infty q(iy)y^{2k-s}\frac{dy}{y}.\end{aligned}$$

Thus, we have, for $\sigma > \beta$,

$$\Phi(s) = D_k(s) + E_k(s), \tag{3.5}$$

where, by (3.2),

$$D_k(s) = \int_1^{\infty} F(iy)\left[y^s + v(T)i^{2k}y^{2k-s}\right]\frac{dy}{y} \tag{3.6}$$

is an entire function of s, satisfying the functional equation (see (2.2)),

$$D_k(2k-s) - v(T)i^{2k}D_k(s) = 0; \tag{3.7}$$

and

$$E_k(s) = v(T)i^{2k}\int_1^{\infty} q(iy)y^{2k-s}\frac{dy}{y} \tag{3.8}$$

is analytic in $\sigma > \beta$.

Now, (3.7) already tells us that the appropriate form of the functional equation for $\Phi(s)$ is

$$\Phi(2k-s) - v(T)i^{2k}\Phi(s) = R_k(s), \tag{3.9}$$

say. But, at this point, (3.9) is merely formal, because we do not know that $\Phi(s)$ has a meromorphic continuation to the whole s-plane; moreover, even if we knew $\Phi(s)$ to be meromorphic in the s-plane, we would still want to have a canonical form for $R_k(s)$ in order to find a converse Hecke theorem for functions $\Phi(s)$ satisfying (3.4) (for $\sigma > \beta$, some $\beta > 0$) and the functional equation (3.9).

2. We solve both the problem of continuation and the problem of a canonical form for $R_k(s)$ by evaluation of $E_k(s)$, using the representation (2.6′), in §II, for rational period functions for Γ_θ. The evaluation of $E_k(s)$ reduces to that of the two integrals (see (2.4) and (2.5))

$$\int_1^{\infty} f_r(0)y^{2k-s}\frac{dy}{y} = \int_1^{\infty}\left\{\frac{1}{(iy)^r} - \frac{\bar{v}(T)(-1)^r}{(iy)^{2k-r}}\right\}y^{2k-s}\frac{dy}{y}, \tag{3.10}$$

for $k \le r \le L$, and

(3.11)

$$\int_1^{\infty} f_r(\alpha)y^{2k-s}\frac{dy}{y} = \int_1^{\infty}\left\{\frac{1}{(iy-\alpha)^r} - \frac{\bar{v}(T)(-1/\alpha)^r}{(iy)^{2k-r}(iy+1/\alpha)^r}\right\}y^{2k-s}\frac{dy}{y},$$

for $1 \le r \le M$ (and $\alpha \neq 0$). We note, in passing, that (1.11) and (3.2) imply that, as $y \to 0^+$,

$$F(iy) \sim -q(iy),$$

so, in general, it must be true that $\beta \geq 1$; in particular, the integrals (3.10) and (3.11) converge for $\sigma > \beta$. In fact, (3.10) converges for

$$\sigma > \max_{k \leq r \leq L} \{2k - r, r\} = L \tag{3.12}$$

and (3.11) converges for

$$\sigma > \max_{r \geq 1} \{2k - r, 0\} = \max\{2k - 1, 0\}. \tag{3.13}$$

Thus, the poles of $\Phi(s)$, for $\sigma \geq 2k$, can come only from the terms of $E_k(s)$ involving (3.10).

The first integral (3.10) is trivial, of course: we have, for the part of $E_k(s)$ corresponding to the first sum in (2.6′) (call it $E_k^0(s)$) just the rational function

$$E_k^0(s) = \sum_{k \leq r \leq L} c_r(-i)^r \left\{ \frac{1}{r - s} + \frac{\varepsilon(T) i^{2k}}{r - (2k - s)} \right\}, \tag{3.14}$$

which clearly satisfies the functional equation,

$$E_k^0(2k - s) - \varepsilon(T) i^{2k} E_k^0(s) = 0, \tag{3.15}$$

the same as that for $D_k(s)$, (3.7).

For the second integral, (3.11), we use the well-known integral representation for the ordinary hypergeometric function (see, for example, Lebedev [11, Chapter 9]),

(3.16)
$$ {}_2F_1[\alpha, \beta; \gamma; z] = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 y^{\alpha - 1}(1 - y)^{\gamma - \alpha - 1}(1 - zy)^{-\beta}\, dy,$$

valid for $\operatorname{Re} \alpha$, $\operatorname{Re}(\gamma - \alpha) > 0$, and z in $\mathbf{C} \setminus [1, \infty)$, with the principal branches taken for all the exponentials. We have, for c in $\mathbf{C} \setminus (-\infty, -1]$ and $\sigma < r$,

$$\int_1^\infty \frac{y^s}{(c + y)^r} \frac{dy}{y} = \frac{1}{r - s}\, {}_2F_1[r - s, r; 1 + r - s; -c]; \tag{3.17}$$

(3.17) already proves that $\Phi(s)$ has a meromorphic continuation to the whole

s-plane, but we shall postpone the investigation of the poles and residues (see §III.3). We actually use another form of (3.17) obtained by first applying the identity, valid for z in $\mathbf{C} \setminus [1, \infty)$, with the principal branch taken for $(1-z)^{-\beta}$ (see Lebedev [11, Chapter 9]),

$$ {}_2F_1[\alpha, \beta; \gamma; z] = (1-z)^{-\beta}{}_2F_1\left[\gamma - \alpha, \beta; \gamma; \frac{-z}{1-z}\right], $$

to get, for c in $\mathbf{C} \setminus (-\infty, -1]$ and $\sigma < r$,

$$ \text{(3.18)} \qquad \int_1^\infty \frac{y^s}{(c+y)^r}\frac{dy}{y} = \frac{(c+1)^{-r}}{r-s}{}_2F_1\left[1, r; 1+r-s; \frac{c}{c+1}\right]. $$

(Incidentally, this corrects an error in Oberhettinger [12], formula 2.22.) Thus, we have, assuming (3.13) on σ, with $\tau = i\alpha$ (so $\operatorname{Re}\tau \geq 0$),

(3.19)

$$ \int_1^\infty \left\{ \frac{1}{(iy-\alpha)^r} - \frac{\bar{v}(T)\left(-\frac{1}{\alpha}\right)^r}{(iy)^{2k-r}\left(iy + \frac{1}{\alpha}\right)^r} \right\} y^{2k-s}\frac{dy}{y} $$
$$ = \frac{(-i)^r}{(\tau+1)^r}\left\{ \frac{1}{r-(2k-s)}{}_2F_1\left[1, r; 1+r-(2k-s); \frac{\tau}{\tau+1}\right] \right. $$
$$ \left. - \frac{\bar{v}(T)i^{-2k}}{s}{}_2F_1\left[1, r; 1+s; \frac{1}{\tau+1}\right] \right\}. $$

Now, it is convenient to have functions with the same argument; since we want to consider $\Phi(s)$ under the transformation $s \to 2k - s$, we should find an expression for the first function in (3.19), with argument $\tau/(\tau+1)$, in terms of hypergeometric functions with arguments $1/(\tau+1)$. Since

$$ \frac{1}{\tau+1} = 1 - \frac{\tau}{\tau+1} $$

we use the identity (see Lebedev [11, Chapter 9]),

$$ {}_2F_1[\alpha, \beta; \gamma; z] = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}{}_2F_1[\alpha, \beta; \alpha+\beta+1-\gamma; 1-z] $$
$$ + \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta} $$
$$ \times {}_2F_1[\gamma-\beta, \gamma-\alpha; 1+\gamma-\alpha-\beta; 1-z], $$

valid, at any rate, for $|z|$, $|1-z|<1$ with the principal branch taken for $(1-z)^{\gamma-\alpha-\beta}$, to get (with $\alpha=1$, $\beta=r$, $\gamma=1+s-(2k-r)$, and $z=\tau/(\tau+1)$)

$$\begin{aligned}
&{}_2F_1\left[1, r; 1+s-(2k-r); \frac{\tau}{\tau+1}\right] \\
&\quad= \frac{(2k-s)-r}{2k-s}\,{}_2F_1\left[1, r; 1+(2k-s); \frac{1}{\tau+1}\right] \\
&\qquad -[(2k-s)-r]B(2k-s, r-(2k-s))\left(\frac{1}{\tau+1}\right)^{s-2k}\left(\frac{\tau+1}{\tau}\right)^{s-2k+r}
\end{aligned}$$

($B(x, y)$ denotes the beta function, as usual); we used, here,

$$\begin{aligned}
&{}_2F_1\left[1+s-2k, s-(2k-r); 1+s-2k; \frac{1}{\tau+1}\right] \\
&\quad= \left(1-\frac{1}{\tau+1}\right)^{-(s-(2k-r))} = \left(\frac{\tau+1}{\tau}\right)^{s-2k+r}.
\end{aligned}$$

Thus, (3.19) becomes (still under the assumption (3.13) on σ)

(3.20)

$$\begin{aligned}
&\int_1^\infty\left\{\frac{1}{(iy-\alpha)^r}-\frac{\bar{v}(T)\left(-\frac{1}{\alpha}\right)^r}{(iy)^{2k-r}\left(iy+\frac{1}{\alpha}\right)^r}\right\}y^{2k-s}\frac{dy}{y} \\
&\quad= -\frac{(-i)^r}{(\tau+1)^r}\left\{\frac{\bar{v}(T)i^{-2k}}{s}\,{}_2F_1\left[1, r; 1+s; \frac{1}{\tau+1}\right]\right. \\
&\qquad\qquad \left.+\frac{1}{2k-s}\,{}_2F_1\left[1, r; 1+(2k-s); \frac{1}{\tau+1}\right]\right\} \\
&\qquad +\frac{(-i)^r}{\tau^r}B(2k-s, r-(2k-s))\tau^{2k-s};
\end{aligned}$$

principal branches being understood, as above, we may also write

$$\tau^{2k-s} = \exp\left(\frac{\pi i}{2}(2k-s)\right)\alpha^{2k-s} = i^{2k}\exp\left(-\frac{\pi i s}{2}\right)\alpha^{2k-s}.$$

Note that, when $\alpha = -i$, $\tau = 1$ and (3.20) reduces to

$$(3.20') \qquad \int_1^\infty \left\{ \frac{1}{(iy+i)^r} - \frac{\bar{v}(T) i^r}{(iy)^{2k-r}(iy+i)^r} \right\} y^{2k-s} \frac{dy}{y}$$
$$= -\frac{(-i)^r}{2^r} \left\{ \frac{\bar{v}(T) i^{-2k}}{s} {}_2F_1\left[1, r; 1+s; \frac{1}{2}\right] \right.$$
$$\left. + \frac{1}{2k-s} {}_2F_1\left[1, r; 1+(2k-s); \frac{1}{2}\right] \right.$$
$$+ (-i)^r B(2k-s, r-(2k-s)).$$

We denote by $E_k^h(s)$ that part of $E_k(s)$ involving the hypergeometric functions appearing in (3.20) and (3.20'), namely

(3.21)

$$E_k^h(s) = -\sum_{j=1}^{p} \sum_{r=1}^{M(j)} c_{rj} \frac{(-i)^r}{(\tau_j+1)^r} \left\{ \frac{1}{s} {}_2F_1\left[1, r; 1+s; \frac{1}{\tau_j+1}\right] \right.$$
$$\left. + \frac{v(T) i^{2k}}{2k-s} {}_2F_1\left[1, r; 1+(2k-s); \frac{1}{\tau_j+1}\right] \right\}$$
$$- \sum_{\substack{r=1 \\ r \equiv \delta(2)}}^{M} d_r \frac{(-i)^r}{2^r} \left\{ \frac{1}{s} {}_2F_1\left[1, r; 1+s; \frac{1}{2}\right] \right.$$
$$\left. + \frac{v(T) i^{2k}}{2k-s} {}_2F_1\left[1, r; 1+(2k-s); \frac{1}{2}\right] \right\},$$

where, of course, $\tau_j = i\alpha_j$, $1 \le j \le p$; in fact, $E_k^h(s)$ is meromorphic in the whole s-plane and, clearly, satisfies the functional equation.

$$(3.22) \qquad E_k^h(2k-s) - v(T) i^{2k} E_k^h(s) = 0$$

as for $E_k^0(s)$ and $D_k(s)$.

Finally, we denote by $E_k^e(s)$ the rest of $E_k(s)$, namely

(3.23)
$$
\begin{aligned}
E_k^e(s) &= E_k(s) - E_k^0(s) - E_k^h(s) \\
&= v(T)i^{2k} \sum_{j=1}^{p} \sum_{r=1}^{M(j)} c_{rj} \frac{(-i)^r}{\tau_j^r} B(2k-s, r-(2k-s))\tau_j^{2k-s} \\
&\quad + v(T)i^{2k} \sum_{\substack{r=1 \\ r \equiv \delta(2)}}^{M} d_r(-i)^r B(2k-s, r-(2k-s)) \\
&= v(T)(-1)^{2k} \sum_{j=1}^{p} \sum_{r=1}^{M(j)} c_{rj} \left(-\frac{1}{\alpha_j}\right)^r B(2k-s, r-(2k-s)) e^{-\pi i s/2} \alpha_j^{2k-s} \\
&\quad + v(T)i^{2k} \sum_{\substack{r=1 \\ r \equiv \delta(2)}}^{M} d_r(-i)^r B(2k-s, r-(2k-s));
\end{aligned}
$$

clearly, $E_k^e(s)$ is meromorphic in the s-plane.

We summarize our results.

THEOREM 2. *Let F be an entire automorphic integral of integral weight $2k$ and multiplier system v for Γ_θ, with period function q given by* (2.6′)*; suppose that F has the Fourier expansion* (1.12), *with zero constant term (in other words,* (3.2) *holds in all cases). Let $\Phi(s)$ be the Mellin transform of F, given by* (3.3) *for $\sigma > \beta$ and by* (3.4) *for $\sigma > \beta + 1$. Then $\Phi(s)$ has a meromorphic continuation to the whole s-plane, being represented, for $\sigma > \beta$, by*

$$\Phi(s) = D_k(s) + E_k^0(s) + E_k^h(s) + E_k^e(s), \tag{3.24}$$

where $D_k(s)$ is entire and $E_k^0(s)$, $E_k^h(s)$, and $E_k^e(s)$ are meromorphic in the s-plane (these functions being given by (3.6′), (3.14), (3.21), *and* (3.23)); *moreover, $\Phi(s)$ satisfies the functional equation*

$$\Phi(2k-s) - v(T)i^{2k}\Phi(s) = R_k(s), \tag{3.9}$$

where

(3.25)

$$
\begin{aligned}
R_k(s) &= E_k^e(2k-s) - v(T)i^{2k}E_k^e(s) \\
&= \sum_{j=1}^{p}\sum_{r=1}^{M(j)} c_{rj}\frac{(-i)^r}{\tau_j^r}\big\{v(T)i^{2k}B(s,r-s)\tau_j^s \\
&\qquad\qquad -B(2k-s,r-(2k-s))\tau_j^{2k-s}\big\} \\
&\quad + \sum_{\substack{r=1\\ r\equiv\delta(2)}}^{M} d_r(-i)^r\big\{v(T)i^{2k}B(s,r-s) - B(2k-s,r-(2k-s))\big\} \\
&= i^{2k}\sum_{j=1}^{p}\sum_{r=1}^{M(j)} c_{rj}\left(-\frac{1}{\alpha_j}\right)^r\big\{v(T)B(s,r-s)e^{\pi is/2}\alpha_j^s \\
&\qquad\qquad -B(2k-s,r-(2k-s))e^{-\pi is/2}\alpha_j^{2k-s}\big\} \\
&\quad + \sum_{\substack{r=1\\ r\equiv\delta(2)}}^{M} d_r(-i)^r\big\{v(T)i^{2k}B(s,r-s) \\
&\qquad\qquad -B(2k-s,r-(2k-s))\big\}.
\end{aligned}
$$

Remarks. (i) To obtain (3.25), we have used the functional equations (3.7), (3.15), and (3.22), of course. Note also, that $R_k(s)$ obviously satisfies the functional equation,

$$R_k(2k-s) + v(T)i^{2k}R_k(s) = 0, \tag{3.26}$$

without reference to (3.9).

(ii) The expression (3.25) depends on only the *nonzero* poles α_j, $1 \le j \le p$, and $-i$ of the period function, $q(z)$; in particular, the pole terms at 0 and ∞ do not affect $R_k(s)$. Thus, if $q(z)$ does not have finite nonzero poles, the functional equation (3.9) reduces to the one occurring in Hecke [4], [5] and in Knopp [9], as we mentioned in the introduction.

(iii) For the Hecke groups, $G(\lambda_n)$, for $3 \le n < \infty$, the major modifications needed are to (3.25), from which the last sum should be omitted, because the nonzero poles in these cases are real (see Knopp [9]). We now have, for

$\sigma > \beta + 1$,

$$\Phi(s) = \int_0^\infty F(iy) y^s \frac{dy}{y} = \left(\frac{2\pi}{\lambda_n}\right)^{-s} \Gamma(s)\phi(s), \tag{3.4'}$$

instead of (3.4); we have the same representation (3.24) for $\Phi(s)$ and the same functional equation (3.9), with (3.25) modified as indicated.

3. We must now consider poles and residues, as well as growth properties in vertical strips, of $\Phi(s)$, in preparation for the converse of Theorem 2 in §IV. In fact, we shall show that $\Phi(s)$ has, at worst, simple poles at certain integer points—infinitely many of them, in general—and that $\Phi(s)$ is bounded in lacunary vertical strips of the form

$$S = S(\sigma_1, \sigma_2; t_0): \sigma_1 \le \sigma \le \sigma_2, \quad |t| \ge t_0 > 0. \tag{3.27}$$

It is clear from (3.24) that it is sufficient to prove these facts for $D_k(s)$, $E_k^h(s)$, and $E_k^e(s)$. By (3.6), $D_k(s)$ is clearly bounded in vertical strips; $D_k(s)$ is entire, as we have observed. By (3.14), $E_k^0(s)$ is also clearly bounded in lacunary strips (3.27). $E_k^0(s)$ is a rational function with simple poles at the integers

$$2k - L, 2k - L + 1, \ldots, \frac{2k-1}{2}, \frac{2k+1}{2}, \ldots, L - 1, L,$$

for odd weight $2k$; at the integers

$$2k - L, 2k - L + 1, \ldots, k - 1, k + 1, \ldots, L - 1, L,$$

for even weight $2k$ and $\delta = 1$; at the integers

$$2k - L, 2k - L + 1, \ldots, k - 1, k, k + 1, \ldots, L - 1, L,$$

for even weight $2k$ and $\delta = 0$. It is clear that the residue of $E_k^0(s)$ at $s = m$, $k < m \le L$, is

$$-c_m(-i)^m; \tag{3.28}$$

at $s = m$, $2k - L \le m < k$, it is

$$v(T) c_{2k-m} i^m; \tag{3.28'}$$

and at $s = k$, when $2k$ is even and $\delta = 0$, it is

$$-2c_k(-i)^k. \tag{3.28''}$$

For $E_k^e(s)$, we first recall Stirling's formula for $|\Gamma(\sigma + it)|$ as $|t| \to \infty$,

$$\left|\Gamma(\sigma + it)\right| \sim \sqrt{2\pi}\,|t|^{\sigma - 1/2} e^{-\pi|t|/2}.$$

(See Lebedev [11, Chapter 1].) Then, since

$$|\arg \tau_j| \leq \frac{\pi}{2},$$

we have, as $|t| \to \infty$ in lacunary strips (3.27),

$$E_k^e(\sigma + it) = O\left(\exp\left(-\left(\frac{\pi}{2} - \varepsilon\right)|t|\right)\right), \tag{3.29}$$

for any $\varepsilon > 0$, from (3.23). It is clear that $E_k^e(s)$ has, at worst, simple poles at all the integers; since

$$E_k^h(s) + E_k^e(s)$$

is analytic in the half-plane $\sigma > \max\{2k - 1, 0\}$ (see the discussion relating to (3.11) and (3.13)), we need to consider the poles only at integers—for either function $E_k^h(s)$ or $E_k^e(s)$—less than or equal to $\max(2k - 1, 0\}$. If $2k \geq 1$, then the residue of $E_k^e(s)$ at $s = m$, for $m \leq 2k - 1$, is

$$\begin{aligned}
&v(T) i^{2k} \sum_{j=1}^{p} \sum_{r=1}^{M(j)} c_{rj} \binom{2k - 1 - m}{2k - r - m} (-i)^r (-\tau_j)^{2k-r-m} \\
&\qquad + v(T) i^{2k} \sum_{\substack{r=1 \\ r \equiv \delta(2)}}^{M} d_r \binom{2k - 1 - m}{2k - r - m} i^r (-1)^{2k-m} \\
&= v(T) \sum_{j=1}^{p} \sum_{r=1}^{M(j)} c_{rj} \binom{2k - 1 - m}{2k - r - m} i^m \alpha_j^{2k-r-m} \\
&\qquad + v(T) i^{2k} \sum_{\substack{r=1 \\ r \equiv \delta(2)}}^{M} d_r \binom{2k - 1 - m}{2k - r - m} i^r (-1)^{2k-m},
\end{aligned} \tag{3.30}$$

with the usual convention on the binomial coefficient $\binom{a}{b}$ for $b > a$. On the other hand, if $2k \le 0$, then the residue of $E_k^e(s)$ at $s = -m$, for $0 \le m$, is

$$(3.30')\quad \varepsilon(T) i^{2k} \sum_{j=1}^{p} \sum_{r=1}^{M(j)} c_{rj} \binom{r-m-2k-1}{r-1} i^r (-\tau_j)^{2k-r+m}$$
$$+ \varepsilon(T) i^{2k} \sum_{\substack{r=1 \\ r \equiv \delta(2)}}^{M} d_r \binom{r-m-2k-1}{r-1} (-i)^r (-1)^{2k+m}$$
$$= \varepsilon(T) \sum_{j=1}^{p} \sum_{r=1}^{M(j)} c_{rj} \binom{r-m-2k-1}{r-1} (-1)^r (-i)^m \alpha_j^{2k-r+m}$$
$$+ \varepsilon(T) i^{2k} \sum_{\substack{r=1 \\ r \equiv \delta(2)}}^{M} d_r \binom{r-m-2k-1}{r-1} (-i)^r (-1)^{2k+m}.$$

Finally, we have $E_k^h(s)$ to consider. Returning to the integral representation (3.16), we have, for $\sigma > 0$ (and $\operatorname{Re} \tau \ge 0$, $\tau \ne 0$),

(3.31)
$$\frac{1}{s}\, {}_2F_1\left[1, r; 1+s; \frac{1}{\tau+1}\right] = \frac{\Gamma(1+s)}{\Gamma(1)\Gamma(s)s} \int_0^1 (1-y)^{s-1} \left(1 - \frac{y}{\tau+1}\right)^{-r} dy$$
$$= \left(\frac{\tau+1}{\tau}\right)^r \int_0^1 y^{s-1} \left(1 + \frac{y}{\tau}\right)^{-r} dy.$$

If $|\tau| > 1$, then we have

$$(3.32)\quad \frac{1}{s}\, {}_2F_1\left[1, r; 1+s; \frac{1}{\tau+1}\right] = \left(\frac{\tau+1}{\tau}\right)^r \sum_{m=0}^{\infty} \frac{(r)_m}{m!} \left(-\frac{1}{\tau}\right)^m \frac{1}{s+m},$$

where as usual,

$$(r)_m = \frac{\Gamma(r+m)}{\Gamma(r)}.$$

In (3.32), it is clear that the right side is meromorphic in the s-plane with simple poles at $s = -m$, $m \ge 0$, the residue at $s = -m$ being

$$(3.33)\quad \left(\frac{\tau+1}{\tau}\right)^r \frac{(r)_m}{m!} \left(-\frac{1}{\tau}\right)^m;$$

and that, moreover, it is bounded in lacunary strips (3.27). On the other hand, if $|\tau| \le 1$ (and $\operatorname{Re} \tau \ge 0$), by using (3.31) and an integral representation for the beta function (see Lebedev [11, Chapter 1]) we have, for $0 < \sigma < r$,

$$(3.31') \quad \frac{1}{s}\,{}_2F_1\left[1, r; 1 + s; \frac{1}{\tau + 1}\right] = \frac{(\tau + 1)}{\tau}\tau^s B(s, r - s) \\ -(\tau + 1)^r \int_0^1 y^{r-s-1}(1 + \tau y)^{-r}\, dy,$$

from which we obtain the same properties as before, including formula (3.33) for the residue at $s = -m$, $m \geq 0$, for the left side of (3.31′).

We now summarize those results pertinent to our converse of Theorem 2.

THEOREM 3. *The function* $\Phi(s)$, *defined by* (3.3) *for* $\sigma > \beta$, *has a meromorphic continuation to the s-plane with, at worst, simple poles at all integer points* $m \leq L$, *and is bounded on every lacunary vertical strip*, S, *described in* (3.27).

Remarks. (i) We might point out we have proved, in fact, that boundedness in lacunary strips (3.27) of each function $D_k(s)$, $E_k^0(s)$, $E_k^h(s)$, and $E_k^e(s)$ is *uniform* in σ. The same is therefore true of $\Phi(s)$.

(ii) By (3.4), the meromorphic continuation of $\Phi(s)$ implies that of $\phi(s)$ (and conversely). Since $\Gamma(s)$ has simple poles at $s = -m$, $m \geq 0$, the residues of $\Phi(s)$ at these points give the *values* of $\phi(s)$ at these points:

$$\phi(-m) = (-1)^m \pi^m m! \operatorname*{res}_{s=-m} \Phi(s).$$

The poles of $\Phi(s)$ at $s = 1, 2, \ldots, L$, if they exist, are also poles of $\phi(s)$.

(iii) The main point of Theorem 3 is that it suggests there *is* a converse of Theorem 2 exactly as for the automorphic forms originally considered by Hecke. The proof we give for the converse is the same one Hecke gave, with appropriate modifications.

IV. The converse Hecke theorem

We now prove, in some detail, the following converse to Theorem 2. (We renumber the relevant equations and expressions of §III, for convenience.)

THEOREM 4. *Suppose the Dirichlet series*

$$(4.1) \qquad \phi(s) = \sum_{m+\kappa>0} \frac{a_{m+\kappa}}{(m + \kappa)^s}$$

converges absolutely in a half-plane $\sigma > \beta > 0$ and has a meromorphic continuation to the s-plane, also denoted by $\phi(s)$, such that:

(a) *The only possible singularities of $\phi(s)$ are simple poles at the integers $s = m$, $0 < m \leq L$;*

(b) *The function*

$$\Phi(s) = \pi^{-s}\Gamma(s)\phi(s) \tag{4.2}$$

is bounded in every lacunary vertical strip $S = S(\sigma_1, \sigma_2; t_0)$ described by

$$S\colon -\infty < \sigma_1 \leq \sigma \leq \sigma_2 < \infty, \qquad |t| \geq t_0 > 0; \tag{4.3}$$

(c) *The function $\Phi(s)$, defined by* (4.2), *satisfies the functional equation*

$$\Phi(2k - s) - v(T)i^{2k}\Phi(s) = R_k(s), \tag{4.4}$$

where $2k$ is an integer, v is a multiplier system of weight $2k$ for Γ_θ, and

(4.5)
$$R_k(s) = \sum_{j=0}^{p}\sum_{r=1}^{M}\left\{A_{rj}B(s, r - s)\tau_j^s - B_{rj}B(2k - s, r - (2k - s))\tau_j^{2k-s}\right\},$$

for some integers $p, M \geq 1$, complex numbers A_{rj}, B_{rj} and complex numbers $\tau_j = i\alpha_j$ with $\alpha_j \neq 0$ and

$$\frac{-3\pi}{2} < \arg \alpha_j < \frac{\pi}{2}, \tag{4.6}$$

for $0 \leq j \leq p$.

Then $\phi(s)$ is the Dirichlet series associated with an entire automorphic integral F of weight $2k$ and multiplier system v for Γ_θ whose period function q has nonzero poles only at $\alpha_0, \ldots, \alpha_p, -1/\alpha_0, \ldots, 1/\alpha_p$.

To prove this we use, of course, the inverse Mellin transform. Since, for $\sigma \geq c > \beta$,

$$|\phi(s)| \leq \sum_{m+\kappa>0} \frac{|a_{m+\kappa}|}{(m + \kappa)^\sigma} \leq \sum_{m+\kappa>0} \frac{|a_{m+\kappa}|}{(m + \kappa)^c} < \infty \tag{4.7}$$

and, as $|t| \to \infty$,

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi}|t|^{\sigma - 1/2}e^{-\pi|t|/2}, \tag{4.8}$$

by absolute convergence we have

$$(4.9)\qquad F(iy) = \sum_{m+\kappa>0} a_{m+\kappa} e^{-\pi(m+\kappa)y} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}\Phi(s)\,ds,$$

for $\operatorname{Re} y > 0$ (convergence is uniform for $\operatorname{Re} y > 0$ and $|\arg y| \leq \pi/2 - \varepsilon$, each $\varepsilon > 0$); we shall assume $y > 0$ throughout the proof, for convenience.

We want to move the line of integration to $\sigma = 2k - c$, of course. Note that, by (a), $c > L$, in any case; we may also *assume* that $L \geq 2k$, for convenience, and that $L < c < L + 1$. That we pick up only the residues of $\Phi(s)$ at

$$2k - L, 2k - L + 1, \ldots, L - 1, L$$

is a consequence of property (b), the boundedness of $\Phi(s)$ on lacunary vertical strips. Indeed, by (4.8), $\Phi(s)$ *vanishes exponentially* as $|t| \to \infty$ in lacunary vertical strips (4.3). Because $\phi(s)$ is *bounded* for $\sigma \geq c$ by (4.7), it follows that $\Phi(s)$ vanishes exponentially on $\sigma = c$ by (4.8). Since (4.6) and (4.8) imply that, for some $\varepsilon > 0$,

$$R_k(s) = O(e^{-\varepsilon|t|})$$

as $|t| \to \infty$ in lacunary strips (4.3), by the functional equation (4.4) it follows that $\Phi(s)$ vanishes exponentially also on $\sigma = 2k - c$. By the Phragmén-Lindelöf principle, then, $\Phi(s)$ vanishes exponentially as $|t| \to \infty$ in the lacunary strip $S(2k - c, c; t_0)$. Therefore,

$$\lim_{|T|\to\infty} \int_{2k-c+iT}^{c+iT} y^{-s}\Phi(s)\,ds = 0,$$

which proves our assertion.

Thus, we now have

$$(4.10)\qquad F(iy) = \frac{1}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} y^{-s}\Phi(s)\,ds + \sum_{2k-L\leq r\leq L} \operatorname*{res}_{s=r}\{y^{-s}\Phi(s)\},$$

whence, by (4.4) and (4.9),

$$(4.10')\qquad (F|_{2k}^{\nu}T)(iy) - F(iy) = -\sum_{2k-L\leq r\leq L} \operatorname*{res}_{s=r}\{y^{-s}\Phi(s)\} + \frac{\bar{\nu}(T)i^{-2k}}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} y^{-s}R_k(s)\,ds.$$

To finish the proof, it is necessary only to show that the right side of (4.10′) is a rational function in iy.

The sum on the right in (4.10′) is obviously a rational function of iy, being of the form

$$\sum_{r=2k-L}^{L} A_r y^{-r}, \tag{4.11}$$

because the poles of $\Phi(s)$ are simple; the poles (if any) of (4.11) are at 0 and ∞.

For the integral in (4.10′), by (4.5) we need to consider only the integrals (for $y > 0$)

$$\frac{1}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} y^{-s}B(s, r-s)e^{\pi is/2}\alpha^s\,ds \tag{4.12}$$

and

$$\begin{aligned} &\frac{1}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} y^{-s}B(2k-s, r-(2k-s))e^{-\pi is/2}\alpha^{2k-s}\,ds \\ &\quad = \frac{i^{-2k}}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-(2k-s)}B(s, r-s)e^{\pi is/2}\alpha^s\,ds, \end{aligned} \tag{4.13}$$

where r is an integer, $1 \leq r \leq M$, $\alpha \neq 0$ satisfies (4.6), and, of course,

$$\alpha^s = |\alpha|^s e^{(i \arg \alpha)s}; \tag{4.14}$$

note that the integrals (4.12) and (4.13) converge absolutely (uniformly for $\operatorname{Re} y > 0$ and $|\arg y| \leq \pi/2 - \varepsilon$, each $\varepsilon > 0$) by (4.8) and (4.6).

Recall (see Lebedev [11, Chapter 1]) that, for $0 < \sigma < r$ and b in $\mathbf{C}\setminus(-\infty, 0]$,

$$\int_0^\infty \frac{y^s}{(b+y)^r}\frac{dy}{y} = b^{s-r}B(s, r-s),$$

the principal branch being taken for b^{s-r}. Thus, for $0 < c < r$, since $B(s, r-s)$ vanishes exponentially on any vertical line, we have, in particular,

$$\frac{b^r}{(b+y)^r} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{y}{b}\right)^{-s} B(s, r-s)\,ds, \tag{4.15}$$

for $y > 0$ and b in $\mathbf{C}\setminus(-\infty, 0]$. Since, as we just pointed out, $B(s, r-s)$ vanishes exponentially on any vertical line, it follows that in moving the line of integration we pick up only the residues of the integrand in (4.15) at the appropriate points.

Thus, if we move the lines of integration in (4.12) and (4.13) to $\sigma = \frac{1}{2}$, we obtain (recall that, by assumption, $2k \leq L < c < L + 1$), by (4.15),

$$(4.12') \qquad \frac{1}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} y^{-s}B(s, r-s)e^{\pi i s/2}\alpha^s\, ds$$
$$= -\sum_{m=0}^{L-2k}\binom{r+m-1}{r-1}\alpha^{-m}(iy)^m + \frac{(-\alpha)^r}{(iy-\alpha)^r};$$

and, similarly,

(4.13′)
$$\frac{1}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} y^{-s}B(2k-s, r-(2k-s))e^{-\pi i s/2}\alpha^{2k-s}\, ds$$
$$= -\sum_{m=r}^{L}\binom{m-1}{r-1}(-1)^r(-\alpha)^m(iy)^{m-2k} + \frac{1}{(iy)^{2k-r}\left(iy+\frac{1}{\alpha}\right)^r}.$$

Since both (4.12′) and (4.13′) are rational functions of iy, whose nonzero poles are α and $-1/\alpha$, we are done, by (4.5).

Remarks. (i) What we have proved, of course, is that, for $y > 0$,

$$(4.16) \qquad (F|_{2k}^{v}T)(iy) = F(iy) + q(iy),$$

where $q(z)$ is a rational function whose nonzero poles can be only at $\alpha_0, \ldots \alpha_p$, $-1/\alpha_0, \ldots, -1/\alpha_p$. By the identity theorem, (4.16) holds for $\operatorname{Re} y > 0$—that is, for all $z = iy$ in $\mathscr{H}$. That $q(z)$ has the form (2.6′) is a consequence of (the corollary to) Theorem 1. (Note that $F(z)$, defined by (4.9), satisfies the growth condition (1.6)—see §I.2—hence must be an *entire* modular integral.)

(ii) The only modification in the statement of Theorem 4 needed to obtain the converse of Theorem 2 for any Hecke group $G(\lambda_n)$, $3 \leq n \leq \infty$, is in the definition of $\Phi(s)$: we would use, instead,

$$(4.2') \qquad \Phi(s) = \left(\frac{2\pi}{\lambda_n}\right)^{-s}\Gamma(s)\phi(s).$$

(iii) It may be of interest to note that if we ignore Dirichlet series and require only that (for example)

$$F(iy) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}\phi(s)\,ds$$

be analytic for $z = iy$ in $\mathscr{H}$, for some $c > 0$, and that, as $y \to +\infty$,

$$F(iy) \to 0,$$

then the existence of a Fourier expansion, as in (4.9), is equivalent to the existence of a Dirichlet series expansion (4.1) for the function

$$\phi(s) = \frac{\Phi(s)}{\pi^{-s}\Gamma(s)},$$

absolutely convergent in some half-plane $\sigma > \beta > 0$. The same remark applies to the case of any Hecke group, by Remark (ii).

(iv) Finally, we should point out that the Phragmén-Lindelöf principle still applies when (b) is replaced by an apparently much weaker condition:

(b′) The function

$$\Phi(s) = \pi^{-s}\Gamma(s)\phi(s) \tag{4.2}$$

is of *finite order* in every lacunary vertical strip $S = S(\sigma_1, \sigma_2; t_0)$. (In fact, this can be replaced by an even weaker condition. Incidentally, a function $f(s)$, holomorphic in some region S, is said to be of *finite order* in S if

$$f(s) = O(e^{|s|^\rho})$$

for some $\rho > 0$ and all s in S.)

V. Conclusion

We return to the observation that a rational function q_T is a rational period function for Γ_θ (in the sense of (1.11), with $n = \infty$) if and only if q_T satisfies (1.16), while rational period functions on $G(\lambda_n)$ must satisfy (1.16) and (1.17) simultaneously. Once again letting $G(\lambda)$ denote the discrete (Hecke) group generated by T and $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, with arbitrary $\lambda > 2$, we observe that $G(\lambda)$, like $\Gamma_\theta = G(\lambda_\infty)$, has the single relation $T^2 = I$. (Abstractly, $G(\lambda)$ is isomorphic to Γ_θ, of course.) For this reason, the rational period functions q_T on $G(\lambda)$, in the sense of (1.11), with S_n replaced by $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $\lambda > 2$, are precisely the same as the rational period functions on Γ_θ. Furthermore, any

q_T connected with any $G(\lambda_n)$ (Hecke group with translation $\lambda_n < 2$) is *a fortiori* connected as well with Γ_θ and with $G(\lambda)$, for arbitrary $\lambda > 2$.

Consequently, from the point of view of rational period functions Γ_θ cannot be distinguished from the groups $G(\lambda)$, $\lambda > 2$. From the perspective of function theory (including the theory of automorphic forms), however, Γ_θ parallel the groups $G(\lambda_n)$ and not the $G(\lambda)$, with $\lambda > 2$.

For, since the fundamental region of Γ_Θ, like those of the $G(\lambda_n)$, has finite hyperbolic area, it follows that Γ_Θ—again like the groups $G(\lambda_n)$—has a finite-dimensional vector space of entire automorphic forms. On the other hand, the space of entire forms has infinite dimension for the group $G(\lambda)$, because $G(\lambda)$ has infinite hyperbolic area. (The distinction is between compact and open Riemann surfaces.)

For these concluding remarks, then, we include Γ_Θ among the groups $G(\lambda)$, by allowing $\lambda = 2$. Thus, $\lambda \geq 2$ and $\lambda_n < 2$.

Assume now that q_T is a rational period function on some $G(\lambda_n)$, with multiplier system v and, for convenience, of weight $2k \geq 2$. Then there exists an entire automorphic integral F_n, of weight $2k$ and multiplier system v on $G(\lambda_n)$, with period function q_T.

To define a multiplier system, $\hat{v}$ say, on $G(\lambda)$ it suffices to specify $\hat{v}\left(\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\right)$ and $\hat{v}(T)$ since the two transformations $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ and T generate the group. Furthermore, since $T^2 = I$ is the sole relation in the group, $\hat{v}\left(\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\right)$ remains arbitrary, in contrast to the necessary condition $\hat{v}(T)^2 = (-1)^{2k}$. Thus, given the multiplier v for $G(\lambda_n)$, we derive from it a multiplier system $\hat{v}$ for $G(\lambda)$ by means of

$$(5.1) \qquad v\left(\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\right) = \exp(2\pi i\hat{\kappa}), \quad \hat{v}(T) = v(T),$$

where $0 \leq \hat{\kappa} < 1$ and $\hat{\kappa}$ is otherwise arbitrary. Then the given q_T is also a rational period function on $G(\lambda)$ of weight $2k \geq 2$ and multiplier system $\hat{v}$. As with $G(\lambda_n)$, from this follows the existence of an entire automorphic integral $\hat{F}_\lambda$, of weight $2k$ and multiplier system $\hat{v}$ on $G(\lambda)$, with period function q_T.

We now have available both Φ_n, the Mellin transform of F_n, and $\hat{\Phi}_\lambda$, the Mellin transform of $\hat{F}_\lambda$. These can be written, respectively,

$$(5.2) \qquad \Phi_n(s) = (2\pi/\lambda_n)^{-s}\Gamma(s) \sum_{m+\kappa>0} \frac{a_{m+\kappa}}{(m+\kappa)^s},$$

and

$$(5.3) \qquad \hat{\Phi}_\lambda(s) = (2\pi/\lambda)^{-s}\Gamma(s) \sum_{m+\hat{\kappa}>0} \frac{\hat{a}_{m+\hat{\kappa}}}{(m+\hat{\kappa})^s}.$$

The method of proof of Theorem 2 (§III) shows that, as F_n and $\hat{F}_\lambda$ have the same rational period function q_T, Φ_n and $\hat{\Phi}_\lambda$ have precisely the same functional equation:

$$\Phi_n(2k - s) - v(T)i^{2k}\Phi_n(s) = R_k(s), \tag{5.4}$$

$$\hat{\Phi}_\lambda(2k - s) - v(T)i^{2k}\hat{\Phi}_\lambda(s) = R_k(s). \tag{5.5}$$

(Note that the corollary to Theorem 1, §II, and its method of proof, remain valid for $G(\lambda)$.) Since there is a converse, Theorem 4 (§IV), to Theorem 2, we may infer that there is no loss of information in the passage from the rational period function q_T occurring in the transformation formula for the automorphic integral to the function $R_k(s)$ appearing in the functional equation of the Mellin transform. Thus the structure of the q_T must be inherent in the form of the $R_k(s)$, and it should be possible to derive the former from the latter.

Nevertheless, comparison of (5.4) and (5.5) argues against such a conclusion. For (5.5) reflects only the single functional equation (1.16) for q_T, while (5.4) entails both (1.16) and (1.17), a far stronger requirement upon q_T. Since (5.4) and (5.5) contain the same function $R_k(s)$, the difference—and it is an essential one—necessarily finds expression elsewhere. And, indeed, the only visible distinction between the functional equations (5.4) and (5.5) resides in the (seemingly innocuous) factor $(\rho\pi)^{-s}$ which occurs in the expressions (5.2), (5.3) for the Mellin transforms. In (5.4), $\rho > 1$; $\rho \leq 1$ in (5.5). This distinction alone may be sufficient to characterize the contrast in properties of the q_T. Or, perhaps, one needs to use the fact that the situation in which q_T satisfies (1.16) alone corresponds to a functional equation with $\rho \leq 1$ only (i.e., (5.5)), while q_T satisfying both (1.16) and (1.17), corresponds simultaneously to functional equations with $\rho \leq 1$ and with $\rho > 1$ (i.e., (5.4) and (5.5)).

We conclude with the observation that the functional equation (3.9) satisfied by $\Phi(s)$ can be cast into a more symmetrical form. With an appropriate simplification of notation in (3.25), as in Theorem 4, define

$$\Psi(s) = \Phi(s) + \sum_{j=0}^{p}\sum_{r=1}^{M} c'_{rj}\frac{(-i)^r}{\tau_j^r}B(s, r - s)\tau_j^s. \tag{5.6}$$

It then follows from (3.9) that

$$\Psi(2k - s) = v(T)i^{2k}\Psi(s), \tag{5.7}$$

the usual functional equation of Riemann and Hecke that holds for the Mellin transform of an entire automorphic form on $G(\lambda_n)$ or $G(\lambda)$. Thus, as an alternative to the viewpoint we have adopted throughout the paper—that Theorem 2 (§III) exhibits a new type of functional equation for the Mellin transform Φ of an exponential series—we can interpret Theorem 2 as giving the new solutions (5.6) of the standard Riemann-Hecke functional equation (5.7). Note that each sum,

$$\sum_{j=0}^{p} c'_{rj} \frac{(-i)^r}{\tau_j^r} B(s, r-s)\tau_j^s, \quad 1 \le r \le M,$$

added to $\Phi(s)$ in (5.6) to give $\Psi(s)$, is analogous to $\Phi(s)$ itself insofar as it is a "finite zeta-function" multiplied by the Γ-factors of $B(s, r-s)$.

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Tulane University
New Orleans, Louisiana

Temple University
Philadelphia, Pennsylvania