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STANDARD HOMOMORPHISMS AND CONVERGENT SEQUENCES IN WEIGHTED CONVOLUTION ALGEBRAS

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1. Introduction

In this paper we continue the study, from our joint paper [11] with J.P. McClure, of the relation between homomorphisms, semigroups, types of convergence, and closed ideals in weighted convolution algebras on the half-line $\mathbf{R}^+ = [0, \infty)$. In particular, we are interested in the question of which continuous homomorphisms preserve dense principal ideals, which we showed to be equivalent [13], [14] to the question of which convolution semigroups are strongly continuous.

We will call a positive Borel function $\omega(x)$ on $\mathbf{R}^+$ a *weight* if both ω and $1/\omega$ are bounded on all finite intervals $[0, a]$. The weight $\omega(x)$ is an *algebra weight* if, in addition, $\omega(x)$ is right continuous, $\omega(0) = 1$, and $\omega(x + y) \leq \omega(x)\omega(y)$ for all x and y in $\mathbf{R}^+$. For a weight ω, we let $L^1(\omega)$ be the Banach space of those (equivalence classes of) locally integrable functions f on $\mathbf{R}^+$ for which $f\omega$ belongs to $L^1(\mathbf{R}^+)$, with the inherited norm $\|f\|_\omega = \|f\| = \int_0^\infty |f(t)|\omega(t)\,dt$. The other weighted spaces we consider are defined analogously. Thus $M(\omega)$ is the space of locally finite complex Borel measures on $\mathbf{R}^+$ for which the norm $\|\mu\| = \int_{\mathbf{R}^+}\omega(t)d|\mu|(t)$ is finite; $L^\infty(1/\omega)$ is the space of f for which f/ω is in $L^\infty(\mathbf{R}^+)$, with the inherited norm $\|f\| = \operatorname{ess\,sup} |f(t)|/\omega(t)$; and $C_0(1/\omega)$ is the closed subspace of $L^\infty(1/\omega)$ comprised of continuous functions with $\lim_{t\to\infty} f(t)/\omega(t) = 0$. Occasionally we will consider $L^1(\omega)$ and $M(\omega)$ when $\omega(x)$ is just a bounded non-negative Borel function.

When $\omega(x)$ is an algebra weight, $L^1(\omega)$ is a Banach algebra under the convolution product $f * g(x) = \int_0^x f(x - t)g(t)\,dt$. Under the analogous convolution of measures on $\mathbf{R}^+$, the space $M(\omega)$ is also a Banach algebra which, under the usual identification of f with $f(t)\,dt$, contains $L^1(\omega)$ as a closed ideal. Moreover [9, Th. 1.4], [13, Th. 2.2, p. 592], for our algebra weights we

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can identify $M(\omega)$ as the multiplier algebra of $L^1(\omega)$ and also as the dual space of $C_0(1/\omega)$ under the natural duality

$$\langle \mu, h \rangle = \int_{\mathbf{R}^+} h(t)\, d\mu(t).$$

With these identifications $M(\omega)$ has not only its usual norm topology but also a strong operator topology and a weak* topology. The subalgebra $L^1(\omega)$ has in addition the weak topology given by $L^1(\omega)^* = L^\infty(1/\omega)$. Just as in our earlier paper [11], many of our arguments involve the comparison of convergence in these and other topologies. There is actually no loss of generality in assuming that the weight ω is an algebra weight, for whenever $L^1(\omega)$ is an algebra under convolution we can always normalize so that ω is an algebra weight [13, Th. 2.1, p. 591]. Therefore, except in those few cases in which we explicitly allow otherwise, we will assume implicitly that $L^1(\omega)$ is an algebra with ω an algebra weight.

In the next section, we will give a precise description of the basic questions which we will study about ideals, homomorphisms, semigroups, and convergence in the $L^1(\omega)$ algebras. We will also discuss the relation between the questions and prove some preliminary results which answer some of the questions in special cases. In Sections 3 and 4 we will compare various types of convergence of sequences $\lambda_n * f$ where $\{\lambda_n\}$ is a bounded sequence in $M(\omega)$. Our main interest is the special case of determining when a semigroup $\{\mu_t\}$ in $M(\omega)$ is *strongly continuous* that is when $\lim_{t \to 0^+} \mu_t * f = f$ for all f in $L^1(\omega)$, but presumably these more general results will have independent interest. In Section 5, we consider the classical case of $L^1(\mathbf{R}^+)$ that is $\omega(x) \equiv 1$ for which more can be said than for general weights. In the final section, we study the structure of the set of continuous homomorphisms between convolution algebras.

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2. Basic questions and preliminary results

Probably the most important question about ideal theory in the $L^1(\omega)$ algebra is determining when the principal ideal $L^1(\omega) * f$ is dense. Clearly a necessary condition is that $\alpha(f)$ (the infimum of the support of f) must be 0. Also, unless $L^1(\omega)$ is radical, that is unless $\lim_{t \to \infty} \omega(t)^{1/t} = 0$, the maximal ideals will contain some $L^1(\omega) * f$ with $\alpha(f) = 0$, so the key question seems to be [14, p. 357]:

Question 1. If f belongs to the radical algebra $L^1(\omega)$ and if $\alpha(f) = 0$, must the ideal $L^1(\omega) * f$ be dense?

In [11] we consider the following related question:

Question 2. If ϕ: $L^1(\omega_1) \to L^1(\omega_2)$ is a continuous non-zero homomorphism, does $L^1(\omega_1) * f$ dense in $L^1(\omega_1)$ imply that $L^1(\omega_2) * \phi(f)$ is dense in $L^1(\omega_2)$?

Since $L^1(\omega_1) * f$ dense always implies that $\alpha(\phi(f)) = 0$ [13, Lemma 4.5, p. 605], a negative answer to Question 2 for some radical ω_2 will give a negative answer to Question 1, and we hope that positive answers to Question 2 will help obtain positive answers to Question 1. We say that a homomorphism which satisfies the conditions of Question 2 is a *standard homomorphism*. In [11, Th. (2.2)] we give a number of natural conditions on ϕ each equivalent to ϕ being standard. These characterizations make it possible to give sufficient conditions on ϕ (for instance if ϕ has dense range [11, Th. (2.2)(c)]) for standardness and sufficient conditions on ω_2 for all ϕ to be standard [11, Th. (3.4)].

Most of the proofs showing that some ϕ is standard involve showing that a related semigroup is strongly continuous. Specifically, one can extend ϕ in a unique way [13, Th. 3.4, p. 596] to a homomorphism, which we also designate by ϕ, from $M(\omega_1)$ to $M(\omega_2)$. Let $\{\delta_t\}$ be the semigroup of point masses in $M(\omega_1)$ (recall

$$\delta_t * f(x) = \begin{cases} 0, & x < t, \\ f(x-t), & x \geq t, \end{cases}$$

so (δ_t) can be identified with the right translation semigroup on $L^1(\omega_1)$). Now if $\mu_t = \phi(\delta_t)$, then it turns out [11, Th. (2.2)] that ϕ is standard if and only if $\{\mu_t\}$ is strongly continuous on $L^1(\omega_2)$. Many of the arguments about $\mu_t * f$ in [11], [14] do not use the fact that $\{\mu_t\}$ is a semigroup. If we let $g = \phi(f)$, where f is standard and $\alpha(f) = 0$, then $\alpha(g) = 0$ [13, Lemma 4.5, p. 605] and $\lim_{t \to 0^+} \mu_t * g = \lim_{t \to 0^+} \phi(\delta_t * f) = \phi(f) = g$, so the natural translation of the question of determining whether a semigroup strongly converges for sequences seems to be:

Question 3. If $\{\lambda_n\}$ is a bounded sequence in $M(\omega)$ and $\lim_{n \to \infty} \lambda_n * g = \lambda * g$ for some g in $L^1(\omega)$ with $\alpha(g) = 0$, does $\lim_{n \to \infty} \lambda_n * f = \lambda * f$ for all f in $L^1(\omega)$?

From the above discussion it is clear that if the answer to Question 3 is yes for all $\{\lambda_n\}$ in $M(\omega)$, then the analogous question for $\mu_t = \phi(\delta_t)$ is also yes. It is not clear that a negative answer to Question 3 implies a negative answer to Question 2, but it does imply a negative answer to Question 1. For if $\{\lambda_n\}$ is a bounded sequence in $L^1(\omega)$ and we define the *convergence ideal* of $\{\lambda_n\}$ by

$$I(\{\lambda_n\}) = I = \{f \in L^1(\omega) : \lambda_n * f \text{ converges in norm in } L^1(\omega)\},$$

then I is clearly an ideal. Also since the collection of operators $f \mapsto \lambda_n * f$ $(n = 1, 2, \dots)$ is a uniformly bounded set, and since $f \in I$ if and only if $\{\lambda_n * f\}$ is Cauchy, the convergence ideal is a closed subspace; so we have:

LEMMA (2.1). *The convergence ideal of a bounded sequence in $L^1(\omega)$ is a closed ideal in $L^1(\omega)$.*

In our definition of the convergence ideal, we do not specify what the limit of $\lambda_n * f$ is, as we do in Question 3. But it follows easily from Lemma (2.2) below that if $\lambda_n * g \to \lambda * g$ for some $g \neq 0$, then $\lambda_n * f \to \lambda * f$ for all f in the convergence ideal of $\{\lambda_n\}$, for the same measure λ. Thus Question 3 asks if the convergence ideal of $\{\lambda_n\}$, which is a closed ideal containing $L^1(\omega) * g$, is all of $L^1(\omega)$. Hence a negative answer to Question 3 does provide a negative answer to Question 1.

Although we do not have definitive answers for all $L^1(\omega)$ about norm convergence of $\lambda_n * f$, the situation for weak* convergence is much simpler for all weights and will allow us to answer Question 3 (and hence Question 2) for norm convergence for some weights. The basic facts on weak* convergence are in the following lemma which slightly extends [13, Lemma (3.2), p. 595].

LEMMA (2.2). *If $\{\lambda_n\}$ is a bounded sequence in $M(\omega)$, then the following are equivalent:*

(a) *There is a $\nu \neq 0$ in $M(\omega)$ for which $\lambda_n * \nu$ converges weak* in $M(\omega) = C_0(1/\omega)^*$.*

(b) *There is a measure λ for which $\lambda_n * \mu$ converges weak* to $\lambda * \mu$ for all μ is $M(\omega)$. In particular $\lambda_n = \lambda_n * \delta_0$ converges weak* to λ.*

Proof. We need only prove that (a) implies (b), and, by [13, Lemma 3.2, p. 595], it will be enough to show that $\{\lambda_n\}$ converges in the weak* topology. Since closed balls in $M(\omega)$ are weak*-compact every subsequence of $\{\lambda_n\}$ has a weak*-convergent subsequence. Therefore we just need to show that weak* convergent subsequences all have the same limit. So let $\{\lambda'_n\}$ and $\{\lambda''_n\}$ be two subsequences of $\{\lambda_n\}$ with weak* limits λ' and λ'', respectively. Then by the weak* continuity of convolution [9, Lemma 2.1], we have

$$\lambda' * \nu = \lim_{n\to\infty} \lambda'_n * \nu = \lim_{n\to\infty} \lambda''_n * \nu = \lambda'' * \nu.$$

Since $M(\omega)$ is an integral domain (by the Titchmarsh convolution theorem), we have $\lambda' = \lambda''$ as required. This completes the proof.

It follows from Lemma (2.2) that if the convergence ideal of a bounded sequence $\{\lambda_n\}$ is not $\{0\}$, then $\{\lambda_n\}$ converges weak*. For semigroups the converse is also true. Suppose $\{\mu_t\}_{t\geq 0}$ is a weak*-continuous semigroup (equivalently $\lim_{t\to 0^+} \mu_t * \nu = \nu$ for some $\nu \neq 0$), then [13, Th. (2.1) and Cor. (2.11), pp. 160 and 165] there is a $g \neq 0$ in $L^1(\omega)$ with $\alpha(g) = 0$ for which $\lim_{t\to 0^+} \mu_t * g = g$. Thus Question 3 for semigroups is equivalent to whether every weak* convergent semigroup is strongly continuous. It is also equivalent to Question 2 [13, Th. (2.9), p. 164]; that is every weak* continuous semigroup in $M(\omega)$ is strongly continuous on $L^1(\omega)$ if and only if every continuous homomorphism from some $L^1(\omega)$ to $L^1(\omega)$ is standard. It is thus natural to ask the following weak* version of Question 3.

Question 4. Suppose that ω is an algebra weight and g belongs to $L^1(\omega)$. If $\{\lambda_n\}$ converges weak* to λ in $M(\omega)$; does $\lambda_n * g$ converge to $\lambda * g$ in norm?

Following Bade and Dales [3, Def. 1.3, p. 81], we say that the weight ω is *regulated at* $a \geq 0$ provided $\lim_{x\to\infty} \omega(x+t)/\omega(x) = 0$ for all $t > a$. With this terminology we can now give a complete answer to Question 4.

THEOREM (2.3). *Suppose that ω is an algebra weight and that $a \geq 0$.*

(a) *If ω is regulated at a, then whenever $\{\lambda_n\}$ is a sequence in $M(\omega)$ and the function g in $L^1(\omega)$ has $\alpha(g) \geq a$ we have that $\lambda_n \to \lambda$ (weak*) implies that $\lambda_n * g \to \lambda * g$ in norm.*

(b) *If ω is not regulated at a, then there is a sequence $\{\lambda_n\}$ in $M(\omega)$ with $\lambda_n \to 0$ weak*, but $\lambda_n * g$ diverges in norm for all g in $L^1(\omega)$ with $\alpha(g) \leq a$.*

Proof. Since a weak*-convergent sequence is bounded, part (a) is just [11, Th. (3.2)].

Suppose therefore that ω is not regulated at a. Then there is a sequence $s_n \to \infty$ and a $t_0 > a$, for which $\omega(s_n + t_0)/\omega(s_n)$ is bounded away from 0. Consider the sequence of measures $\lambda_n = \delta_{s_n}/\omega(s_n)$. For each h in $C_0(1/\omega)$,

$$\langle \lambda_n, h\rangle = \int_{\mathbf{R}^+} h(t)\, d\lambda_n(t) = h(s_n)/\omega(s_n) \to 0 \qquad \text{as } n \to \infty,$$

since h belongs to $C_0(1/\omega)$. Thus $\lambda_n \to 0$ weak*.

Suppose g belongs to $L^1(\omega)$ and $\alpha(g) \leq a$. We complete the proof by assuming that $\|\lambda_n * g\| \to 0$ and contradicting $\omega(s_n + t_0)/\omega(s_n)$ being bounded below (this part of the proof adapts arguments from [3, pp. 82 and 86]). We have

$$\int_0^\infty |g(t)| \frac{\omega(s_n + t)}{\omega(s_n)}\, dt = \|\lambda_n * g\| \to 0.$$

Hence there is a subsequence $\{s_n'\}$ of $\{s_n\}$ for which

$$\lim_{n \to \infty} g(t) \frac{\omega(s_n' + t)}{\omega(s_n')} = 0$$

for almost every $t > a \geq \alpha(g)$. Hence there is a t_1 with $a < t_1 < t_0$, $g(t_1) \neq 0$, and

$$\lim_{n \to \infty} g(t_1) \frac{\omega(s_n' + t_1)}{\omega(s_n')} = 0.$$

Then

$$\frac{\omega(s_n' + t_0)}{\omega(s_n')} \leq \omega(t_0 - t_1) \frac{\omega(s_n' + t_1)}{\omega(s_n')} \to 0 \qquad \text{as } n \to \infty.$$

This contradicts the fact that $\omega(s_n + t_0)/\omega(s_n)$ is bounded below and hence completes the proof.

Another answer to Question 4 is given by the following corollary.

COROLLARY (2.4). *Suppose that ω is an algebra weight.*

(a) *If ω is regulated at 0, then if the sequence $\{\lambda_n\}$ in $M(\omega)$ converges to λ in the weak*-topology on $M(\omega) = C_0(1/\omega)^*$, we also have $\lambda_n \to \lambda$ in the strong operator topology of $M(\omega)$ acting on $L^1(\omega)$.*

(b) *If ω is not regulated at any $a \geq 0$, then there is a sequence $\{\lambda_n\}$ in $M(\omega)$ for which $\lambda_n * g \to 0$ weak* for all $g \in L^1(\omega)$, but $\{\lambda_n * g\}$ diverges in norm for all $g \neq 0$ in $L^1(\omega)$.*

We can also use Theorem (2.3) to give a positive answer to Question 3 for regulated weights. We omit the proof which is essentially the same as in the special case when $\{\lambda_n\}$ is a semigroup [11, Th. (3.4)], [14, Th. (2.8), p. 164].

THEOREM (2.5). *Suppose that $\{\lambda_n\}$ is a bounded sequence in $M(\omega)$ and that* $\lim_{n\to\infty} \lambda_n * g = \lambda * g$ *for some g in $L^1(\omega)$ with $\alpha(g) = 0$. If ω is regulated at any $a \geq 0$, then* $\lim_{n\to\infty} \lambda_n * f = \lambda * f$ *for all f in $L^1(\omega)$.*

Suppose that $\{\lambda_n\}$ is a bounded sequence in $M(\omega)$. In the next section, Section 3, we examine the consequences of weak* convergence of $\{\lambda_n\}$. Then in Section 4 we determine sufficient conditions for $\lambda_n * f$ to converge to $\lambda * f$ in norm for some, or all, f in $L^1(\omega)$. Though the gap between our necessary consequences of weak* convergence and our sufficient conditions for norm convergences may appear small, we know from Corollary 2.4(b) that the gap is unbridgable for arbitrary sequences when ω is not regulated. The hope is that in the specific case of semigroups, where we have so much more structure [14], these conditions will help us determine when we can prove strong continuity. We will not normally bother to restate our general convergence results for the special case of semigroup convergence, but we will occasionally mention the specific applications to determining which homomorphisms are standard. Technically, since the index set for a semigroup is $\mathbf{R}^+$ and not the positive integers, we should be talking not about sequences, but about nets $\{\lambda_i\}$ where index sets have a cofinite sequence. However the extension of our results from sequences to such nets always follows from a routine passage to a subsequence, so the awkwardness is statements and proofs of convergence theorems for nets compared to sequences is not justified. After the two sections on convergence of sequences, we will return to the question of which homomorphisms are standard. In Section 5 we examine the special case $L^1(\mathbf{R}^+)$, and in Section 6 we look at the structure of the set of all standard homomorphisms.

3. Consequences of weak* convergence

The following two theorems give the basic consequences of weak* convergence for pointwise convergence and for convergence in a weaker norm, respectively. The rest of the section will then be devoted to consequences of these first two theorems.

THEOREM (3.1). *Suppose that the sequence $\{\lambda_n\}$ converges weak* to λ in the algebra $M(\omega) = C_0(1/\omega)^*$.*

(a) *If f is a continuous function on $\mathbf{R}^+$ with $f(0) = 0$, then*

$$\lim_{n\to\infty} \lambda_n * f(x) = \lambda * f(x) \quad \textit{for all } x \geq 0.$$

(b) *If g is a locally integrable function on $\mathbf{R}^+$, then there is a subsequence $\{\lambda'_n\}$ for which $\lambda'_n * g$ converges to $\lambda * g$ almost everywhere on $\mathbf{R}^+$.*

THEOREM (3.2). *Suppose that* $\{\lambda_n\}$ *converges weak* to* λ *in* $M(\omega)$ *and that* $\eta \geq 0$ *belongs to* $L^1(\mathbf{R}^+) \cap L^\infty(\mathbf{R}^+)$. *Then for all* g *in* $L^1(\omega)$ *we have*

$$\lim_{n\to\infty} \int_0^\infty |\lambda_n * g(t) - \lambda * g(t)|\omega(t)\eta(t)\, dt = 0.$$

When g *is in* $L^1(\omega) \cap L^\infty(\omega)$, *we only need* η *to be in* $L^1(\mathbf{R}^+)$.

The convergence of the integral in Theorem (3.2) just says that $\lambda_n * g$ converges to $\lambda * g$ in norm in $L^1(\omega\eta)$. Notice that we do not require that $L^1(\omega\eta)$ be an algebra, or even that $\omega\eta$ and $1/\omega\eta$ be locally bounded. We prove Theorems (3.1) and (3.2) together, since we will use (3.1)(a) to prove (3.2), and need (3.2) to prove (3.1)(b).

Proofs of Theorems (3.1) *and* (3.2). We start with (3.1)(a). We need to show that for $b \geq 0$, $\lim_{n\to\infty} \lambda_n * f(b) = \lambda * f(b)$. For each b, we define the function $\phi_b(t)$ on $\mathbf{R}^+$ by $\phi_b(t) = f(b - t)$ for $0 \leq t \leq b$ and $\phi_b(t) = 0$ for $t > b$. Since $f(0) = 0$, we have $\phi_b(b) = 0$, so that ϕ_b is a continuous function with compact support. Thus ϕ_b belongs to $C_0(1/\omega)$. Since $\lambda_n \to \lambda$ weak* in $C_0(1/\omega)^*$,

$$\begin{aligned}\lambda_n * f(b) &= \int_{[0,b)} f(b-t)\, d\lambda_n(t) = \int_{\mathbf{R}^+} \phi_b(t)\, d\lambda_n(t) = \langle \lambda_n, \phi_b\rangle \to \langle \lambda, \phi_b\rangle \\ &= \lambda * f(b)\end{aligned}$$

as $n \to \infty$. This proves (3.1)(a).

We now prove (3.2) for η in $L^1(\mathbf{R}^+) \cap L^\infty(\mathbf{R}^+)$. First suppose that f is a continuous function in $L^\infty(\omega)$ with $f(0) = 0$. Since ω is an algebra weight, the analog for measures of formula 2.5 of [12] shows that

$$\|\lambda_n * f(t)\omega(t)\|_\infty \leq \|\lambda_n\|_\omega \|f\|_{\infty,\omega}$$

which is bounded since $\{\lambda_n\}$ is a weak*-convergent, and therefore a norm-bounded, sequence in $M(\omega)$. By (3.1)(a) we also have

$$\lim_{n\to\infty} \lambda_n * f(t)\omega(t) = \lambda * f(t)\omega(t)$$

pointwise on $\mathbf{R}^+$. Since η belongs to $L^1(\mathbf{R}^+)$, it follows from the dominated convergence theorem that $\lambda_n * f(t)\omega(t)\eta(t)$ approaches $\lambda * f(t)\omega(t)\eta(t)$ in $L^1(\mathbf{R}^+)$ for those f in $L^\infty(\omega)$ which are continuous and have $f(0) = 0$.

Since η belongs to $L^\infty(\mathbf{R}^+)$, we have $L^1(\omega) \subseteq L^1(\omega\eta)$ with the imbedding continuous. Thus if we define T_n: $L^1(\omega) \to L^1(\omega\eta)$ by $T_n f = \lambda_n * t$, then the sequence $\{T_n\}$ of continuous linear operators is bounded. We already know that $T_n f = \lambda_n * f \to \lambda * f$ in $L^1(\omega\eta)$ on a dense subspace of $L^1(\omega)$. Hence $T_n g \to \lambda * g$ in $L^1(\omega\eta)$ for all g in $L^1(\omega)$. This proves (3.2) for η in $L^1(\mathbf{R}^+) \cap L^\infty(\mathbf{R}^+)$.

We now prove (3.1)(b). Fix some $\eta \geq 0$ in $L^1(\mathbf{R}^+) \cap L^\infty(\mathbf{R}^+)$ with η and $1/\eta$ both locally bounded; for instance, $\eta(t) = e^{-t}$. Then Lebesgue measure and the measure $\omega(t)\eta(t)\,dt$ are mutually absolutely continuous. If g belongs to $L^1(\omega)$, we have from Theorem (3.2) that $\lambda_n * g$ converges to $\lambda * g$ in norm in $L^1(\omega\eta)$. Hence there is a subsequence $\{\lambda'_n\}$ for which $\lambda'_n * g$ converges to $\lambda * g$ almost everywhere with respect to the measure $\omega(t)\eta(t)\,dt$, and hence with respect to Lebesgue measure. This completes the proof of Theorem (3.1).

Finally suppose that g belongs to $L^1(\omega) \cap L^\infty(\omega)$ and that $\eta \geq 0$ belongs to $L^1(\mathbf{R}^+)$. It will be enough to show that whenever $\lambda_n \to \lambda$ weak*, then some subsequence $\{\lambda'_n\}$ has $\lambda'_n * g \to \lambda * g$ in norm in $L^1(\omega\eta)$. By (3.1)(b), we can choose a subsequence so that $\lambda'_n * g \to \lambda * g$ almost everywhere in $\mathbf{R}^+$. Just as with our proof for continuous functions in $L^\infty(\omega)$, we have $\{\|\lambda'_n * g(t)\omega(t)\|_\infty\}$ bounded. It then follows from the dominated convergence theorem that

$$\lim_{n\to\infty} \int_0^\infty |\lambda'_n * g(t) - \lambda * g(t)|\omega(t)\eta(t)\,dt = 0.$$

This completes the proof of Theorems (3.1) and (3.2).

The following is an easy consequence of Theorem (3.2).

COROLLARY (3.3). *Suppose that $\{\lambda_n\}$ converges to λ weak* in the algebra $M(\omega)$ and that f is a locally integrable function on $\mathbf{R}^+$. Then for all $b \geq 0$, we have*

$$\lim_{n\to\infty} \int_0^b |\lambda_n * f(t) - \lambda * f(t)|\,dt = 0.$$

Proof. Fix b and define g in $L^1(\omega)$ by $g(t) = f(t)$ for $t \leq b$ and $g(t) = 0$ for $t > b$. Also choose $\eta \geq 0$ in $L^1(\mathbf{R}^+) \cap L^\infty(\mathbf{R}^+)$ with $\eta(t) = 1/\omega(t)$ on $[0, b]$. Then we have

$$\begin{aligned}\int_0^b |\lambda_n * f(t) - \lambda * f(t)|\,dt &= \int_0^b |\lambda_n * g(t) - \lambda * g(t)|\eta(t)\omega(t)\,dt \\ &\leq \int_0^\infty |\lambda_n * g(t) - \lambda * g(t)|\omega(t)\eta(t)\,dt.\end{aligned}$$

The last integral converges to 0 by Theorem (3.2). This completes the proof.

The following result generalizes Corollary 3.16 in [13, p. 602] where we showed that there was some $L^1(\omega_3) \supseteq L^1(\omega_2)$ with ϕ: $L^1(\omega_1) \to L^1(\omega_3)$ standard.

COROLLARY (3.4). *Suppose that ϕ: $L^1(\omega_1) \to L^1(\omega_2)$ is a continuous non-zero homomorphism. If the algebra $L^1(\omega_3) \supseteq L^1(\omega_2)$ has ω_3/ω_2 integrable, then as a map from $L^1(\omega_1)$ to $L^1(\omega_3)$ the homomorphism ϕ is standard.*

Proof. We extend ϕ in the usual way to the corresponding measure algebras. The statement $L^1(\omega_3) \supseteq L^1(\omega_2)$ says precisely that ω_3/ω_2 belongs to $L^\infty(\mathbf{R}^+)$. Thus we can apply Theorem (3.2) with $\eta = \omega_2/\omega_1$. Let $\{\delta_t\}$ be the semigroup of point masses in $M(\omega_1)$ and let $\mu_t = \phi(\delta_t)$ in $M(\omega_2)$. We know [13, Th. (3.6) (A), p. 599] that $\mu_t \to \delta_0$ weak* in $M(\omega_2)$. Hence it follows from Theorem (3.2) that for all g in $L^1(\omega_2)$ we have $\lim_{t\to 0^+} \|\mu_t * g - g\|_{\omega_3} = 0$. Thus by [11, Th. (2.2)(b)], ϕ: $L^1(\omega_1) \to L^1(\omega_3)$ is a standard homomorphism, as required.

The next result is essentially a re-statement of Theorem (3.2).

THEOREM (3.5). *Suppose that g belongs to the algebra $L^1(\omega)$. If $\eta \geq 0$ belongs to $L^1(\mathbf{R}^+) \cap L^\infty(\mathbf{R}^+)$, then convolution by g is a compact operator from $M(\omega)$ to $L^1(\omega\eta)$.*

Proof. Suppose that $\{\lambda_n\}$ is a bounded sequence in $M(\omega)$. We need to show that $\{\lambda_n * g\}$ has a norm convergent subsequence in $L^1(\omega\eta)$. Since $\{\lambda_n\}$ is a bounded sequence in a dual space, it has a subsequence $\{\lambda'_n\}$ which converges weak* to some λ in $M(\omega)$. It then follows from Theorem (3.2) that $\lambda'_n * g$ converges to $\lambda * g$ in the norm of $L^1(\omega\eta)$. This completes the proof.

For a bounded sequence $\{\lambda_n\}$ of finite positive Borel measures on $\mathbf{R}^+$, the Helley-Bray theorem [7, Prop. (7.19), p. 217], [5, Th. 11.1.2, p. 304] says $\lambda_n \to \lambda$ weak* in $M(\mathbf{R}^+)$ if and only if $\lambda_n[0, x] \to \lambda[0, x]$ at each point of continuity of $\lambda[0, x]$. The following result is a rough analogue for complex measures in $M(\omega)$.

THEOREM (3.6). *For a bounded sequence $\{\lambda_n\}$ in the algebra $M(\omega)$, the following are equivalent.*

(a) $\lambda_n \to \lambda$ *weak* in* $M(\omega)$.

(b) *Every subsequence $\{\lambda'_n\}$ has a subsequence $\{\lambda''_n\}$ with $\lambda''_n[0, x] \to \lambda[0, x]$ almost everywhere.*

Proof. Let $\{\lambda''_n\}$ be the indicated subsequence in (b). Folland's proof [7, Prop. (7.19) (a), p. 217] shows that $\lambda''_n \to \lambda$ weak * in $M(\omega)$. Thus every subsequence of $\{\lambda_n\}$ has a subsequence converging weak* to λ and hence $\{\lambda_n\}$ converges weak* to λ.

Conversely suppose that $\{\lambda_n\}$ converges weak* to λ and let $\{\lambda'_n\}$ be an arbitrary subsequence. Applying Theorem (3.1)(b) to the function $u(x) \equiv 1$ on $\mathbf{R}^+$ shows that $\{\lambda'_n\}$ has a subsequence $\{\lambda''_n\}$ with $\lambda''_n * u(x) = \lambda''_n[0, x]$ converging almost everywhere to $\lambda * u(x) = \lambda[0, x]$. This completes the proof of the theorem.

Remark 3.7. Suppose that $\{\lambda_n\}$ converges weak* to λ in some $M(\omega)$. Since a continuous function is a sum of a continuous function vanishing at 0 with a multiple of $u(x) \equiv 1$, it follows from Theorem (3.1)(a) and (3.6) that there is a single subsequence $\{\lambda'_n\}$ and a single set E of measure 0 for which $\lim_{n\to\infty} \lambda'_n * g(x) = \lambda * g(x)$ for all continuous $g(x)$ and all $x \notin E$.

4. Conditions for norm convergence

In this section we find conditions on a bounded sequence $\{\lambda_n\}$ in $M(\omega)$ sufficient for $\{\lambda_n * f\}$ to converge in norm for some or all f in $L^1(\omega)$, and we use these conditions to prove that certain homomorphisms are standard. All of our conditions will be strong enough to imply that $\{\lambda_n\}$ converges weak*, so we will be able to apply the results of previous sections. We start with two basic results that show that in our context weak convergence and a certain type of convergence, which is intermediate between weak and weak* convergence, each imply norm convergence.

THEOREM (4.1). *Suppose that $\{\lambda_n\}$ is a bounded sequence in the algebra $M(\omega)$ and that f belongs to $L^1(\omega)$. If $\lambda_n * f$ converges to $\lambda * f$ weakly in $L^1(\omega)$, then $\lim_{n\to\infty} \lambda_n * f = \lambda * f$ in norm in $L^1(\omega)$.*

Proof. We choose $\varepsilon > 0$ and show that

$$\limsup \|\lambda_n * f - \lambda * f\| \le \varepsilon.$$

By hypothesis, the set $\{\lambda_n * f - \lambda * f : n = 1, 2, \ldots\} \cup \{0\}$ is weakly sequentially compact and hence weakly compact in $L^1(\omega)$. Hence consequences of the Dunford-Pettis theorem (see [6, Cor. IV, 8.10 p. 292] and [6, Cor. IV.8.11, p. 294]) show that there is a $b > 0$ for which

$$\int_b^\infty |\lambda_n * f(t) - \lambda * f(t)|\omega(t)\, dt < \varepsilon \quad \text{for all } n. \tag{1}$$

Also $\lambda_n * f$ converges to $\lambda * f$ weak* in $M(\omega)$, so $\lambda_n \to \lambda$ weak* (by Lemma

(2.2)). It thus follows from Corollary (3.3) that

$$\lim_{n\to\infty}\int_0^b |\lambda_n * f(t) - f * f(t)|\omega(t)\,dt = 0.$$

Combining this with formula (1) yields

$$\limsup \|\lambda_n * f - \lambda * f\|_\omega \le \varepsilon,$$

as required.

From the above theorem together with Theorem (2.3), we see that weak convergence of $\lambda_n * f$ in $L^1(\omega)$ is enough to guarantee norm convergence in $L^1(\omega)$, but weak* convergence in $M(\omega)$ is not enough.

Perhaps the most natural space between the space $C_0(1/\omega)$, which determines weak* convergence, and the space $L^\infty(1/\omega)$, which determines weak convergence, is the space $UC(1/\omega)$ of *uniformly continuous* functions in $L^\infty(1/\omega)$ defined by

$$UC(1/\omega) = \{f \in L^\infty(1/\omega) : f \text{ is continuous and } \|r_x f - f\|_{\infty, 1/\omega} \to 0 \text{ as } x \to 0^+\},$$

where $r_x f(y) = f(x+y)$. It is easy to show that $UC(1/\omega)$ is a closed subspace of $L^\infty(1/\omega)$.

To study types of convergence intermediate in strength between weak and weak* convergence we will need to consider a product $\hat{*}$ which makes $L^\infty(1/\omega)$ an $L^1(\omega)$ Banach module; for f in $L^1(\omega)$ and g in $L^\infty(1/\omega)$ let

$$f \,\hat{*}\, g(x) = \int_0^\infty g(x+y) f(y)\,dy.$$

Thus, $UC(1/\omega)$ is also an $L^1(\omega)$ Banach module, since it is closed. To give an alternative description of $UC(1/\omega)$ useful for our purposes, we first need the following lemmas.

LEMMA 4.2. *The sequence* $e_n = n\chi_{[0,1/n]}$ $(n = 1, 2, \ldots)$ *is a bounded approximate identity for the Banach module* $UC(1/\omega)$.

Proof. For f in $UC(1/\omega)$ we have

$$\begin{aligned}
\|e_n \,\hat{*}\, f - f\| &= \sup_x \frac{1}{\omega(x)}\left|\int_0^\infty f(x+y)e_n(y)\,dy - f(x)\right| \\
&= \sup_x \frac{1}{\omega(x)}\left|n\int_0^{1/n} f(x+y)\,dy - n\int_0^{1/n} f(x)\,dy\right| \\
&\le \sup_x n\int_0^{1/n} \frac{|f(x+y) - f(x)|}{\omega(x)}\,dy \\
&= \sup_x n\int_0^{1/n} \frac{|r_y f(x) - f(x)|}{\omega(x)}\,dy. \qquad (2)
\end{aligned}$$

Now given $\varepsilon > 0$, choose $\delta > 0$ such that $\|r_y f - f\|_{\infty, 1/\omega} < \varepsilon$, for every $0 < y < \delta$. Then for every n with $n > 1/\delta$ from (2) we have $\|e_n \hat{*} f - f\| < \varepsilon$, and the lemma is proved.

LEMMA 4.3. *The space $UC(1/\omega)$ factors as $L^1(\omega) \hat{*} L^\infty(1/\omega)$.*

Proof. For f in $L^\infty(1/\omega)$ and g in $L^1(\omega)$ we have

$$\begin{aligned}
\|r_x(f \hat{*} g) - f \hat{*} g\| &= \sup_y \frac{1}{\omega(y)} \left| \int_0^\infty \left[f(x+y+t) g(t)\, dt \right.\right. \\
&\qquad \left.\left. - \int_0^\infty f(y+t) g(t) \right] dt \right| \\
&= \sup_y \frac{1}{\omega(y)} \left| \int_0^\infty f(y+t)\, d(\delta_x * g)(t) \right. \\
&\qquad \left. - \int_0^\infty f(y+t) g(t)\, dt \right| \\
&\le \sup_y \int_0^\infty \frac{|f(y+t)|}{\omega(y+t)} \cdot \omega(t)\, d|((\delta_x * g) - g)(t)| \\
&\le \|f\|_{\infty, 1/\omega} \|\delta_x * g - g\|_\omega \\
&\to 0 \quad \text{as } x \to 0^+.
\end{aligned} \tag{3}$$

Hence $L^1(\omega) \hat{*} L^\infty(1/\omega) \subset UC(1/\omega)$. On the other hand since $L^1(\omega)$ has a bounded approximate identity for $L^\infty(1/\omega)$ (Lemma 4.2) we have $L^1(\omega) \hat{*} L^\infty(1/\omega)$ closed [15, Thm. 32.22, p. 268]. Also from Lemma 4.2 and by the module version of Cohen's factorization theorem every element of $UC(1/\omega)$ factors as a product of an element in $L^1(\omega)$ and another element of $L^\infty(1/\omega)$. Therefore

$$L^1(\omega) \hat{*} L^\infty(1/\omega) = UC(1/\omega).$$

THEOREM (4.4). *Suppose that $\{\lambda_n\}$ is a bounded sequence in the algebra $M(\omega)$.*

(a) *If $\langle \lambda_n, h \rangle \to \langle \lambda, h \rangle$ for all h in $UC(1/\omega)$, then $\{\lambda_n\}$ converges to λ in the strong operator topology on $L^1(\omega)$.*

(b) *Let f belong to $L^1(\omega)$. If $\langle \lambda_n * f, h \rangle \to \langle \lambda * f, h \rangle$ for all h in $UC(1/\omega)$, then $\lim_{n \to \infty} \lambda_n * f = \lambda * f$ in the norm of $L^1(\omega)$.*

Proof. By Theorem (4.1) and Lemma 4.3 to prove (a) it suffices to show that for g in $L^1(\omega)$ and ψ in $L^\infty(1/\omega)$ we have

$$\lim \langle \lambda_n * g, \psi \rangle = \langle \lambda * g, \psi \rangle.$$

But a Fubini type argument shows that $\langle \lambda_n * g, \psi \rangle = \langle \lambda_n, g \mathbin{\hat{*}} \psi \rangle$, and $\langle \lambda_n, g \mathbin{\hat{*}} \psi \rangle$ converges to $\langle \lambda, g \mathbin{\hat{*}} \psi \rangle = \langle \lambda * g, \psi \rangle$, by our hypothesis.

To prove (b) we apply (a) to the sequence $\lambda_n * f$. We thus have $\lambda_n * f * g \to \lambda * f * g$ in norm for all g in $L^1(\omega)$. In other words, the convergence ideal of $\{\lambda_n\}$ contains $L^1(\omega) * f$, and hence $\mathrm{cl}(L^1(\omega) * f)$ (see Lemma (2.1)). But $L^1(\omega)$ has a bounded approximate identity, so f belongs to the closure of $L^1(\omega) * f$. Thus $\lambda_n * f \to \lambda * f$ in norm, so the proof is complete.

Bade and Dales [3, Th. 1.5, p. 71] show that the weight $\omega(x)$ is regulated at 0 if and only if $L^1(\omega) \mathbin{\hat{*}} L^\infty(1/\omega) \subseteq C_0(1/\omega)$. Since $C_0(1/\omega) \subseteq UC(1/\omega)$, from Lemma 4.3 it follows that ω is regulated at 0 if and only if $L^1(\omega) \mathbin{\hat{*}} L^\infty(1/\omega) = C_0(1/\omega)$. In fact Bade and Dales prove that $\omega(x)$ is regulated at a if and only if $f \mathbin{\hat{*}} L^\infty(1/\omega) \subseteq C_0(1/\omega)$ for all f in $L^1(\omega)$ with $\alpha(f) \geq a$. This fact can be used to give a different proof of Theorem (2.3)(a).

We now develop some more specialized sufficient conditions for strong-operator-topology convergence. While we continue to consider arbitrary λ as a limit, the most important case is $\lambda_n \to \delta_0$. This is the case we apply to prove that a homomorphism is standard, and this is also the classical "summability kernel" case.

THEOREM (4.5). *Suppose that $\{\lambda_n\}$ converges weak* to λ in $M(\omega)$. If $\limsup \|\lambda_n\| \leq \|\lambda\|$, then $\{\lambda_n\}$ converges to λ in the strong operator topology of $M(\omega)$ acting on $L^1(\omega)$.*

Notice that the fact that $\lambda_n \to \lambda$ weak* implies that

$$\liminf \|\lambda_n\| \geq \|\lambda\|;$$

so the hypothesis in the theorem just says $\lim_{n\to\infty} \|\lambda_n\| = \|\lambda\|$. Before proving the theorem we give an application to homomorphisms.

COROLLARY (4.6). *If the homomorphism ϕ: $L^1(\omega_1) \to L^1(\omega_2)$ has norm* 1, *then ϕ is a standard homomorphism.*

Proof. The extension of ϕ to the corresponding measure algebras also has norm 1 [13, Th. 3.4, p. 596]. So if $\mu_t = \phi(\delta_t)$, then $\|\mu_t\| \leq \|\delta_t\| = \omega_1(t)$. But $\{\mu_t\}$ is a weak*-continuous semigroup [11, Th. 3.4 (8), p. 596], and

$$\limsup_{t\to 0^+} \|\mu_t\| = \lim_{t\to 0^+} \omega_1(t) = 1.$$

Thus it follows from Theorem (4.5) that $\{\mu_t\}$ is a strongly continuous semigroup, and hence ϕ is a standard homomorphism [11, Th. (2.2)].

Proof of Theorem (4.5). By Theorem (4.4), it will be enough to show that $\lim_{n\to\infty}\langle\lambda_n, h\rangle = \langle\lambda, h\rangle$ for all h in $UC(1/\omega)$. This is a standard result for probability measures and is a known result [7, Prob. 7.26, p. 219] for weak* convergent complex measures in unweighted L^1-spaces. Since it is not clear how to obtain the weighted result directly from the unweighted result, we sketch a proof that $\lim \int_{\mathbf{R}^+} h\,d\lambda_n = \int_{\mathbf{R}^+} h\,d\lambda$. For convenience we take $\|h\| \le 1$ in $UC(1/\omega)$. Given $\varepsilon > 0$, we will show that

$$\left|\int_{\mathbf{R}^+} h\,d\lambda_n - \int_{\mathbf{R}^+} h\,d\lambda\right| < 2\varepsilon, \tag{4}$$

for n sufficiently large.

Choose a continuous function ψ with compact support K and with norm $\|\psi\| = 1$ in $C_0(1/\omega)$ (so that $|\psi(t)| \le \omega(t)$) which satisfies

$$\langle\lambda, \psi\rangle = \int_{\mathbf{R}^+} \psi\,d\lambda > \|\lambda\| - \varepsilon/2.$$

Now choose n large enough so that

$$\left|\int \psi\,d\lambda_n\right| > \|\lambda\| - \varepsilon/2 \quad\text{and}\quad \|\lambda_n\| < \|\lambda\| + \varepsilon/2.$$

Then we have

$$\|\lambda\| - \varepsilon/2 \le \left|\int_{\mathbf{R}^+} \psi\,d\lambda_n\right| = \left|\int_K \psi\,d\lambda_n\right| \le \int_K \omega d|\lambda_n| \le \|\lambda_n\| < \|\lambda\| + \varepsilon/2.$$

So if we let $U = \mathbf{R}^+\setminus K$, we then have

$$\int_U \omega d|\lambda_n| = \|\lambda_n\| - \int_K \omega d|\lambda_n| < \varepsilon,$$

for sufficiently large n. In a similar way we obtain $\int_U \omega d|\lambda| < \varepsilon$.

Now choose g: $\mathbf{R}^+ \to [0, 1]$ continuous with compact support, with $g \equiv 1$ on K. Since $\lambda_n \to \lambda$ weak* in $M(\omega) = C_0(1/\omega)^*$ we have

$$\lim_{n\to\infty} \int_{\mathbf{R}^+} gh\,d\lambda_n = \int_{\mathbf{R}^+} gh\,d\lambda.$$

But

$$\left|\int_{\mathbf{R}^+} gh\,d\lambda_n - \int_{\mathbf{R}^+} h\,d\lambda_n\right| = \left|\int_U (1-g)h\,d\lambda_n\right| \le \int_U \omega d|\lambda_n| \le \varepsilon$$

for sufficiently large n. Similarly $|\int_{\mathbf{R}^+} gh\, d\lambda - \int_{\mathbf{R}^+} h\, d\lambda| \leq \varepsilon$. Combining these last two estimates gives formula (4) and proves the theorem.

5. Standard homomorphisms and convergence in $L^1(\mathbf{R}^+)$

In this section we study standardness of endomorphisms and strong-operator-topology convergence of semigroups and sequences in $L^1(\mathbf{R}^+)$. In $L^1(\mathbf{R}^+)$ we can combine our earlier results, particularly from [11], with properties of the Laplace and Fourier transforms. We let Π be the open half-plane,

$$\Pi = \{z = x + iy \colon x > 0\},$$

and $\overline{\Pi}$ the corresponding closed half-plane. Then for μ in $M(\mathbf{R}^+)$, the *Laplace transform* is defined by

$$\hat{\mu}(z) = \int_{\mathbf{R}^+} e^{-zt}\, d\mu(t) \quad \text{for } z \text{ in } \overline{\Pi}.$$

When f is in $L^1(\mathbf{R}^+)$, we extend its Laplace transform continuously to $\overline{\Pi} \cup \{\infty\}$ by letting $\hat{f}(\infty) = 0$. As is well known (see for instance [4, Th. 4.4, p. 189]), the Laplace transform identifies $\overline{\Pi}$ with the character space of $L^1(\mathbf{R}^+)$ and $\{\infty\}$ with the zero homomorphism. To some extent our restriction to $L^1(\mathbf{R}^+)$ is a normalization. All our results have obvious translations to the spaces $L^1(e^{kt})$ for k real. Some of our results hold for arbitrary semisimple $L^1(\omega)$, but we will occasionally use Nyman's theorem or the Wiener Tauberian theorem, which hold only for special weights.

For the form of Nyman's theorem we need, recall that a function f in $L^1(\omega)$ with $\alpha(f) = d$ is *standard* in $L^1(\omega)$ if

$$\mathrm{cl}(L^1(\omega) * f) = \{g \in L^1(\omega) \colon \alpha(g) \geq d\} = L^1(\omega)_d.$$

Thus if $\alpha(f) = 0$, then f is standard if and only if $L^1(\omega) * f$ is dense in $L^1(\omega)$, as in Question 1 in Section 2. For semi-simple $L^1(\omega)$, it is clear that f cannot be standard if $\hat{f}(z)$ is ever 0 on the maximal ideal space of $L^1(\omega)$. The version of Nyman's theorem we need is the following converse for $L^1(\mathbf{R}^+)$.

LEMMA (5.1) (Nyman's theorem). *If f is a non-zero function in $L^1(\mathbf{R}^+)$, then f is standard in $L^1(\mathbf{R}^+)$ if and only if $\hat{f}(z)$ is never 0 on $\overline{\Pi}$.*

Proof. Let $a = \alpha(f)$ and $f = \delta_a * g$. Then $\hat{g}(z) = e^{az}\hat{f}(z)$ is also never 0 on $\overline{\Pi}$. Since $\alpha(g) = 0$, it follows from the usual form of Nyman's theorem [4, Cor. 6.4, p. 201] that $L^1(\mathbf{R}^+) * g$ is dense in $L^1(\mathbf{R}^+)$. The lemma now follows from the fact that convolution with δ_a is an isometry from $L^1(\mathbf{R}^+)$ onto its standard ideal $L^1(\mathbf{R}^+)_a$.

The next two results give sufficient conditions for weak* and strong-operator-topology convergence in $M(\mathbf{R}^+)$ in terms of the Laplace transforms. Notice that it follows from the definition of the Laplace transform that if $\lambda_n \to \lambda$ weak* in $M(\mathbf{R}^+)$, then $\hat{\lambda}_n(z) \to \hat{\lambda}(z)$ pointwise for z in Π. Similarly if $f_n \to f$ weakly in $L^1(\mathbf{R}^+)$, then $\hat{f}_n(z) \to \hat{f}(z)$ pointwise on $\overline{\Pi}$.

LEMMA (5.2). *Suppose $\{\lambda_n\}$ is a bounded sequence in $L^1(\mathbf{R}^+)$. If $\{\hat{\lambda}_n(z)\}$ converges pointwise on some set of uniqueness $A \subseteq \Pi$ of the Laplace transform, then $\{\lambda_n\}$ converges weak* in $M(\mathbf{R}^+)$.*

Proof. It follows from weak*-compactness that every subsequence of $\{\lambda_n\}$ has a weak*-convergent subsequence. Therefore we need only show that if two subsequences $\{\lambda'_n\}$ and $\{\lambda''_n\}$ have weak* limits λ' and λ'' respectively, then $\lambda' = \lambda''$. But $\hat{\lambda}'(z) = \lim_{n\to\infty} \hat{\lambda}_n(z) = \hat{\lambda}''(z)$ for all z in A. Since A is a uniqueness set this proves the lemma.

THEOREM (5.3). *Suppose that $\{\lambda_n\}$ is a bounded sequence in $M(\mathbf{R}^+)$. If there is an f in $L^1(\mathbf{R}^+)$ with $f(z)$ never 0 on the imaginary axis and for which $\lim_{n\to\infty} \lambda_n * f = \lambda * f$ in norm in $L^1(\mathbf{R}^+)$, then $\lambda_n * g \to \lambda * g$ for all g in $L^1(\mathbf{R}^+)$.*

Proof. We consider $L^1(\mathbf{R}^+)$ and $M(\mathbf{R}^+)$ as closed subalgebras of $L^1(\mathbf{R})$ and $M(\mathbf{R})$ respectively, by extending the functions and measures to be zero on the negative real axis. Let

$$J = \left\{h \in L^1(\mathbf{R}) \colon \lim_{n\to\infty} \lambda_n * h = \lambda * h \text{ in } L^1(\mathbf{R})\right\}.$$

Just as in Lemma (2.1), J is a closed ideal in $L^1(\mathbf{R})$ and f belongs to J. Translated to $L^1(\mathbf{R})$, the hypothesis on f says that the Fourier transform of f never vanishes. It thus follows from the Wiener Tauberian theorem [16, Th. 6.4, p. 228], that $J = L^1(\mathbf{R})$. In particular, $J \supseteq L^1(\mathbf{R}^+)$, so the theorem is proved.

The rest of the results in this section will use the added structure given by semigroups and homomorphisms. We first look at the zeroes of the Laplace transform of a semigroup.

THEOREM (5.4). *Suppose that $\{\mu_t\}$ is a weak*-continuous semigroup in $M(\mathbf{R}^+)$ with $\lim_{t\to 0^+} \mu_t = \delta_0$ in weak*-topology. Then:*

(a) *each $\hat{\mu}_t(z)$, for $t > 0$, has the same zeroes in $\overline{\Pi}$;*

(b) *no $\hat{\mu}_t(z)$ can have a zero in Π or at any point on the boundary of Π at which $\hat{\mu}_t(z)$ has an analytic continuation;*

(c) *$\{\mu_t\}$ is a strongly continuous semigroup if and only if some $\hat{\mu}_t(z)$ has no zero on the imaginary axis, and hence on $\overline{\Pi}$.*

Proof. That the $\hat{\mu}_t(z)$ have the same zeroes follows from the semigroup property $\hat{\mu}_{s+t}(z) = \hat{\mu}_s(z)\hat{\mu}_t(z)$. Now suppose that $\hat{\mu}_t(z_0) = 0$ and $\hat{\mu}_t$ is analytic or has an analytic continuation at z_0. Then for all positive integers n, we have $\hat{\mu}_t(z_0) = (\hat{\mu}_{t/n}(z_0))^n$, so the zero at z_0 is of infinite order, but this forces $\hat{\mu}_t$, and hence μ_t, to be 0, contradicting the assumption $\lim_{t\to 0^+} \mu_t = \delta_0$ in weak*-topology.

Finally, if $\hat{\mu}_t$ has a zero at some point z_0 in $\overline{\Pi}$ and f in L^1 has $\hat{f}(z_0) \neq 0$, then $\lim_{t\to 0^+} \mu_t * f(z_0) = \lim_{t\to 0^+} \hat{\mu}_t(z_0)\hat{f}(z_0) = 0 \neq \hat{f}(z_0)$; so $\mu_t * f$ cannot converge to f as $t \to 0^+$. Conversely, suppose $\hat{\mu}_t(z)$ is never 0. Choose some f in $L^1(\mathbf{R}^+)$ with $\hat{f}(z)$ never 0 on $\overline{\Pi}$ and let $g = \mu_1 * f$. Then $\hat{g}(z)$ is not 0 on the boundary of Π and

$$\lim_{t\to 0^+} \mu_t * g = \lim_{t\to 0^+} \mu_{1+t} * f = g,$$

since $\{\mu_t\}$ is strongly continuous for $t > 0$ [13, Th. 3.6 (B), p. 594], [14, Th. (2.1), p. 160]. Hence $\{\mu_t\}$ is strongly continuous by Theorem (5.3), and the proof is complete.

We now derive and apply representation formulas for endomorphisms and semigroups in $L^1(\mathbf{R}^+)$. Suppose $w(z)$ is analytic on some open subset U of the complex plane and that z_0 belongs to the boundary of U in the extended complex plane. We say that $w(z_0) = \infty$ if and only if $\lim_{z\to z_0} e^{-tw(z)} = 0$ for some (equivalently all) $t > 0$. That is

$$w(z_0) = \infty \text{ if and only if } \lim_{\substack{z\to z_0 \\ z\in U}} \operatorname{Re}(w(z)) = \infty.$$

Thus we can unambiguosuly say $e^{-\infty} = 0$.

With this convention, we can now state our representation theorem for semigroups. We normalize to the case of bounded semigroups. If $\{\mu_t\}$ is any weak* continuous semigroup with $\lim_{t\to\infty} \|\mu_t\|^{1/t} < e^c$, then $\lambda_t = e^{-ct}\mu_t$ is a bounded weak* continuous semigroup. Moreover, $\{\lambda_t\}$ is strongly continuous if and only if $\{\mu_t\}$ is.

THEOREM (5.5). *If $\{\mu_t\}$ is a bounded weak*-continuous semigroup not of the form $\mu_t = e^{-ct}\delta_0$, then there is an analytic function $w\colon \Pi \to \Pi$ with a continuous extension to a map from $\overline{\Pi}$ to $\overline{\Pi} \cup \{\infty\}$ for which $\hat{\mu}_t(z) = e^{-tw(z)}$ for all $t > 0$ and all z in $\overline{\Pi}$. Moreover $\{\mu_t\}$ is a strongly continuous semigroup if and only if $w(z)$ is finite on $\overline{\Pi}$.*

Proof. First we fix some $t > 0$. By Theorem (5.4)(b), $\hat{\mu}_t(z)$ is never 0 on the simply connected set Π. Thus there is an analytic function $w(z)$ on Π with $\hat{\mu}_t(z) = e^{-tw(z)}$. Since $\{\mu_t\}$ is a semigroup and, for each fixed z, $\hat{\mu}_t(z)$ is a continuous function of t, the function $w(z)$ is the same for all t. Since $|\hat{\mu}_t(z)| \leq \|\mu_t\|$ and $\{\mu_t\}$ is bounded, we must have $t\,\mathrm{Re}(w(z))$ bounded below; so that $w(\Pi) \subseteq \overline{\Pi}$. But $w(z)$ is not constant since $e^{-tw(z)}$ is not a multiple of $\hat{\delta}_0(z) = 1$. Hence w is an open map and $w(\Pi) \subseteq \Pi$. Since $\hat{\mu}_t(z) = e^{-tw(z)}$ has a continuous extension to $\overline{\Pi}$, $w(z)$ also has a continuous extension to a map from $\overline{\Pi}$ to $\overline{\Pi} \cup \{\infty\}$. By our convention $\hat{\mu}_t(z_0) = 0$ if and only if $w(z_0) = \infty$, so it follows from Theorem (5.4)(c) that $\{\mu_t\}$ is strongly continuous if and only if $w(z)$ is finite on $\overline{\Pi}$. This completes the proof.

If $\lambda_t = e^{-ct}\mu_t$ is a bounded semigroup, then $\hat{\lambda}_t(z) = e^{-ct}\hat{\mu}_t(z)$, so Theorem (5.5) still holds except that $w(z)$ maps Π to the open half plane $\{z = x + iy\colon\ x > c\}$. We apply the representation theorem to describe the limit of $\hat{\mu}_t(z)$ as $t \to 0^+$.

THEOREM (5.6). *Suppose that* $\{\mu_t\}$ *is a weak*-continuous semigroup in* $M(\mathbf{R}^+)$.

(a) $\lim_{t \to 0^+} \hat{\mu}_t(z) = 1$ *uniformly on compact subsets of* Π.

(b) *The semigroup* $\{\mu_t\}$ *is strongly continuous if and only if* $\lim_{t \to 0^+} \hat{\mu}_t(z) = 1$ *for all* z *on the imaginary axis.*

Proof. Let $\hat{\mu}_t(z) = e^{-tw(z)}$. Part (a) follows from the finiteness of $w(z)$ on Π. When $\{\mu_t\}$ is strongly continuous, then $(\mu_t * f)\hat{\ }(z) = \hat{\mu}_t(z)\hat{f}(z)$ approaches $\hat{f}(z)$ for all f in $L^1(\mathbf{R}^+)$ and all z in $\overline{\Pi}$. Hence $\hat{\mu}_t(z) \to 1$ pointwise on $\overline{\Pi}$. Conversely, if $\hat{\mu}_t(z) = e^{-tw(z)}$ approaches 1, then $w(z) \neq \infty$. This completes the proof.

We now give the representation theorem for endomorphisms. Virtually all of the following theorem is in [10], where it is used to prove that all endomorphisms are one–one.

THEOREM (5.7). *Suppose that* ϕ *is a continuous nonzero endomorphism of* $L^1(\mathbf{R}^+)$. *Then there exists a continuous function* $w(z)$ *from* $\overline{\Pi}$ *to* $\overline{\Pi} \cup \{\infty\}$ *for which* $\phi(f)(z) = \hat{f}(w(z))$ *for all* z *in* $\overline{\Pi}$. *Moreover*:

(a) $w(z)$ *is a non-constant analytic function on* Π *which maps* Π *to itself*;

(b) *the extension of* ϕ *to* $M(\mathbf{R}^+)$ *also satisfies* $(\phi(\mu))\hat{\ }(z) = \hat{\mu}(w(z))$ *for all* z *in* $\overline{\Pi}$;

(c) $w(\infty) = \infty$.

Proof. The character space of $L^1(\mathbf{R}^+)$ is $\overline{\Pi}$ with the character given by z in $\overline{\Pi}$ just the map $f \to \hat{f}(z)$. Hence there is a continuous $w\colon \overline{\Pi} \to \overline{\Pi} \cup \{\infty\}$ (essentially the restriction of the adjoint ϕ^* to the characters) for which $\phi(f)\hat{\ }(z) = \hat{f}(w(z)))$. The first author [10, p. 310] shows that $w(z)$ is a nonconstant analytic function on Π and therefore $w(\Pi) \subseteq \Pi$. Part (b) follows from the fact that for all μ in $M(\mathbf{R}^+)$ and f in $L^1(\mathbf{R}^+)$, $\phi(\mu * f)\hat{\ }(z)$ is equal to $\hat{\mu}(w(z))\hat{f}(w(z))$ and also to $(\phi(\mu)^\wedge(z))(\hat{f}(w(z))$. For part (c) we choose an f with $\hat{f}(z)$ nonzero on $\overline{\Pi}$. Then $(\phi f)^\wedge(\infty) = 0 = \hat{f}(w(\infty))$. Hence $w(\infty) = \infty$, and the proof is complete.

Notice that part (b) says that if μ_t is the semigroup $\phi(\delta_t)$, then $\hat{\mu}_t(z) = e^{-tw(z)}$ just as in Theorem (5.6), the representation theorem for semigroups. In this case, however, it follows from (5.7)(c), that $\lim_{z\to\infty}\hat{\mu}_t(z) = e^{-\infty} = 0$. Actually, the representation theorems for semigroups and endomorphisms can be gotten from each other using the relations between semigroups and homomorphisms given in [13] and [14], but the direct proofs we gave above are simpler.

We now use the representation theorem together with our general result [11, Th. (2.2)] for characterizing standard homomorphisms to characterize standard endomorphisms of $L^1(\mathbf{R}^+)$. We do not repeat the various conditions in [11, Th. (2.2)] which are equivalent to standardness for arbitrary $L^1(\omega)$.

THEOREM (5.8). *Suppose that ϕ is a continuous non-zero endomorphism of $L^1(\mathbf{R}^+)$ with representation $(\phi f)^\wedge(z) = \hat{f}(w(z))$ for z in $\overline{\Pi}$. Then the following are equivalent*:

(a) *ϕ is a standard homomorphism.*
(b) *Whenever $f(z)$ is never 0 on $\overline{\Pi}$, then $(\phi f)^\wedge(z)$ is never 0 on $\overline{\Pi}$.*
(c) *There is an f in $L^1(\mathbf{R}^+)$ with $(\phi f)^\wedge(z)$ never 0 on the imaginary axis.*
(d) *$w(z)$ is finite on the imaginary axis.*
(e) *$w(z)$ is finite on $\overline{\Pi}$.*

Proof. It follows from the definition of standard homomorphism, given after Question 2 in Section 2, that if ϕ is standard then there are f with $\phi(f)$ standard and hence $(\phi(f))^\wedge(z)$ never zero on $\overline{\Pi}$. Hence (a) $\Rightarrow$ (c). On the other hand if (b) holds, then there is an f in $L^1(\mathbf{R}^+)$ for which $\phi(f)^\wedge(z)$ is nver 0 on $\overline{\Pi}$. By the form of Nyman's theorem given in Lemma (5.1), this implies that $\phi(f)$ is standard on $L^1(\mathbf{R}^+)$. Hence (b) $\Rightarrow$ (a) by [11, Th. (2.2)(c)].

We complete the proof by showing the equivalence of (b) through (e). It is clear that (b) $\Rightarrow$ (c). Suppose (c) holds. Then

$$(\phi(f))^\wedge(z) = \hat{f}(w(z)) \neq 0$$

whenever z belongs to the imaginary axis. But $\hat{f}(\infty) = 0$ for all f in $L^1(\mathbf{R}^+)$, so (c) $\Rightarrow$ (d). We know from Theorem (5.7) that $w(z)$ is always finite on Π, so (d) $\Rightarrow$ (e). Finally suppose $w(z)$ is finite on $\overline{\Pi}$ and that $\hat{f}(z)$ never vanishes on $\overline{\Pi}$. Then, for all z in $\overline{\Pi}$, we have $(\phi f)^\wedge(z) = \hat{f}(w(z)) \neq 0$. Thus (e) $\Rightarrow$ (b), and the proof is complete.

By combining part (c) of the above theorem with our version of Nyman's theorem we obtain the following Corollary. For general $L^1(\omega)$ we only know the following result under the additional assumption that $\alpha(f) = 0$.

COROLLARY (5.9). *Suppose that ϕ is a standard endomorphism of $L^1(\mathbf{R}^+)$. Whenever f is standard in $L^1(\mathbf{R}^+)$, then $\phi(f)$ is also standard in $L^1(\mathbf{R}^+)$.*

It is possible to prove the implication (e) $\Rightarrow$ (a) in Theorem (5.8) by using our characterization of strongly continuous semigroups in Theorem (5.5). If we let $\mu_t = \phi(\delta_t)$, then $\hat{\mu}_t(z) = e^{-tw(z)}$. When $w(z)$ is finite on $\overline{\Pi}$, it follows from Theorem (5.5) that $\{\mu_t\}$ is strongly continuous. But this implies that ϕ is standard [11, Th. (2.2)].

With a bit more effort it is possible to weaken condition (c) in Theorem (5.8) to only requiring some g in $\mathrm{cl}(\phi(L^1(\mathbf{R}^+)))$ or even, in the smallest closed ideal containing $\phi(L^1(\mathbf{R}^+))$, satisfy $\hat{g}(z)$ is never 0 on the imaginary axis. Such a g would belong to the convergence ideal of the semigroup $\{\mu_t\}$ (see [11, Th. (2.4)] or [14, p. 161]). But, by Theorem (5.3), this would imply that $\{\mu_t\}$ is strongly continuous.

6. The space of standard homomorphisms

In this section we prove a few closure properties of the collections of standard homomorphisms. Results of this type have been very useful in investigating other properties of homomorphisms between convolution algebras.

Our basic result is the following:

THEOREM (6.1). *Suppose that ω' and ω are algebra weights. Then the collection of standard homomorphisms from $L^1(\omega')$ to $L^1(\omega)$ is closed in the uniform operator topology and also is dense in the strong operator topology in the set of all nonzero continuous homomorphisms.*

Proof. Suppose that ϕ is the uniform limit of a sequence $\{\phi_m\}$ of standard homomorphisms and that $\{\lambda_n\}$ is a bounded sequence in $M(\omega')$. The usual "$\varepsilon/3$-argument" shows that if for each m we have $\lim_{n\to\infty} \phi_m(\lambda_n) = \phi_m(\lambda)$, then we also have $\lim_{n\to\infty} \phi(\lambda_n) = \phi(\lambda)$. Applying this result to the semigroup $\{\delta_t\}$ (or to an approximate identity in $L^1(\omega')$) shows that ϕ is also a standard homomorphism [11, Th. (2.2)].

Now suppose that $\phi\colon L^1(\omega') \to L^1(\omega)$ is a nonzero homomorphism. For each $\lambda > 0$, we define the homomorphism ϕ_λ by

$$\phi_\lambda(f)(t) = e^{-\lambda t}(\phi(f)(t))$$

and the weight $\omega_\lambda(t) = e^{\lambda t}\omega(t)$. Then ϕ_λ is a homomorphism from $L^1(\omega')$ to $L^1(\omega_\lambda) \subseteq L^1(\omega)$. Since $\omega/\omega_\lambda = e^{-\lambda t}$ belongs to $L^1(\mathbf{R}^+) \cap L^\infty(\mathbf{R}^+)$, it follows from Corollary (3.4) that each ϕ_λ is standard. On the other hand for each g in $L^1(\omega)$, so in particular for each $\phi(f)$ in the range of ϕ, we have, by the dominated convergence theorem, that

$$\left\| e^{-\lambda t}g(t) - g(t) \right\|_\omega = \int_0^\infty |1 - e^{-\lambda t}||g(t)|\omega(t)\, dt$$

converges to 0 as $\lambda \to 0^+$. Thus ϕ is the strong-operator-topology limit at $\{\phi_\lambda\}$; so the proof is complete.

We also have the following simple algebraic closure property.

THEOREM (6.2). *Any composition of standard homomorphisms is a standard homomorphism.*

The theorem is an easy consequence of most of the characterizations of standard homomorphisms [11, Th. (2.2)]. It is immediate from the fact that a homomorphism is standard if and only if it preserves dense principal ideals, or if and only if it is continuous in the strong operator topology.

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