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THE HELICAL TRANSFORM AND THE A.E. CONVERGENCE OF FOURIER SERIES

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Introduction

Let $(X, \mathscr{F}, \mu, \varphi)$ be a dynamical system, μ being an invertible measure preserving transformation on $(X, \mathscr{F}, \mu)$. The helical transform $H_\varepsilon f(x)$ of $f \in L^1(\mu)$ is the limit a.e. of

$$H_\varepsilon^n f(x) = \sum_{k=-n}^{n}{}' \frac{f(\varphi^k x) e^{2\pi i k\theta}}{k}$$

for each ε fixed. The existence of the limit is known from the results of A. Calderón [3] and M. Cotlar [5]. (The notation Σ_j' means that we delete in the sums the term corresponding to $j = 0$.)

DEFINITION 1. A measurable function f satisfies the Wiener-Wintner property (with respect to the dynamical system $(X, \mathscr{F}, \mu, \varphi)$ if there exists a single null set $N \in X$ off which the limit $H_\varepsilon f(x)$ exists for all $\varepsilon \in \mathbf{R}$.

DEFINITION 2. A measurable function f satisfies the strong Wiener-Wintner property (with respect to $(X, \mathscr{F}, \mu, \varphi)$ if off a single null set $\varepsilon \to H_\varepsilon f(x)$ is a continuous function.

By taking an invariant function (i.e., $f \circ \varphi = f$) the discontinuity property at 0 of

$$\varepsilon \to \sum_{k=-\infty}^{\infty}{}' \frac{e^{ik\varepsilon}}{k}$$

shows easily that not all functions satisfy the strong Wiener-Wintner property (S.W.W.). This property (S.W.W.) is more likely to hold when we are outside the Kronecker factor of φ (i.e., the closure of the linear span of the

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eigenfunctions of φ). In [2] we showed that the property S.W.W. for all functions in $\overline{(I-\phi)(L^p)}$ for all $1 < p < \infty$ is equivalent to the strong (p, p) property for the maximal operator

$$\sup_n \int_{-\pi}^{\pi} \frac{e^{int} f(t)}{x-t}\, dt$$

(Carleson-Hunt theorem [4], [6]). The space $\overline{(I-\phi)(L^p)}$ is the closure in L^p of the set of functions $f - f \circ \varphi$. The dynamical system in [1] was the shift on $[0,1]^{\mathbf{Z}}$. We also proved in [1] that the Wiener-Wintner property fails in $L(\operatorname{Log}\operatorname{Log} L)^\beta$ for any $0 < \beta < 1$. One of the tools for these proofs is the double maximal helical transform

$$H^{**}f(x) = \sup_n \sup_\varepsilon \left| \sum_{k=-n}^{n}{}' \frac{f(\varphi^k(x))}{k} e^{ik\varepsilon} \right|.$$

Its discrete analog is $H^{**}a$ defined by

$$H^{**}a(j) = \sup_n \sup_\varepsilon \left| \sum_{k=-n}^{n}{}' \frac{a_{k-j} e^{ik\varepsilon}}{k} \right|.$$

One way to study $H^{**}a$ is to study first the maximal discrete helical transform H^*a

$$H^*a(j) = \sup_\varepsilon \left| \sum_{k=-\infty}^{\infty}{}' \frac{a_{k-j} e^{ik\varepsilon}}{k} \right|.$$

In [2] we proved the formal equivalence of the L^2 boundedness of the maximal operators corresponding to the partial sums of Fourier series, the range of a discrete helical walk, partial Fourier coefficients and the discrete helical transform. In the same paper we proved that the maximal operator associated to the partial Fourier coefficients I^* is not strong (p, p) for $1 < p < 2$.

We are going to prove here the formal equivalence of the strong type (p, p) estimate of $H^{**}f, H^{**}a, H^*a$, the partial sums of Fourier series of L^p functions and the maximal operators used in the proofs of the main results in [1] for $1 < p < \infty$. An estimate of the constant involved allows us to "extrapolate" using a result of [8]. We show that $H^{**}f \in L^1$ if $f \in L(\operatorname{Log} L)^4$, and give an exponential estimate of $H^{**}f$ for $f \in L_\infty$ and prove that $H^{**}f < \infty$ a.e. if $f \in L(\operatorname{Log} L)^2$ by a weak type inequality. This will allow us to extend one of our previous results in [1]. We also prove that the property S.W.W. in $L \operatorname{Log} L$ for the shift on $[0,1]^{\mathbf{Z}}$ implies the a.e. convergence of Fourier series of functions in $L \operatorname{Log} L$.

Let us say that the methods used to establish these equivalences are certainly familiar to experts in harmonic analysis and Fourier series. As it may not be the case for specialists in ergodic theory and dynamical systems we give what we believe are self-contained proofs. At the same time we will obtain an estimate of the constant $(C(p^6/(p-1)^4)$ in L^p, $1<p<\infty)$ which will allow us to "extrapolate" in the Lorentz spaces $L(\text{Log } L)^\alpha$. It did not seem to the author that these connections are direct when we are dealing with Lorentz spaces of the type $L(\text{Log } L)^\alpha(L(\text{Log } L)^\gamma)$. More precisely, the introduction of double supremum

$$\sup_n \sup_\varepsilon |H_n^\varepsilon f| = \sup_n \sup_\varepsilon \left| \sum_{k=-n}^{n}{}' \frac{f(\varphi^k x)e^{ik\varepsilon}}{k} \right|$$

which in the case of the shift on $\mathbf{Z}$ translates to

$$H^{**}(a)(j) = \sup_n \sup_\varepsilon \left| \sum_{k=-n}^{n}{}' \frac{a_{k+j}e^{ik\varepsilon}}{k} \right|$$

seems to give a more restrictive class than the single supremum

$$H^{**}a(j) = \sup_\varepsilon \left| \sum_{k=-\infty}^{\infty}{}' \frac{a_{k+j}e^{ik\varepsilon}}{k} \right|.$$

Knowing that the partial sums of the Fourier series of functions in one of the spaces $L(\text{Log } L)^\alpha(L \text{ Log } L)^\gamma$ are bounded a.e., does this imply that $H^{**}(f)$ is also bounded a.e. for f in $L(\text{Log } L)^\alpha(L \text{ Log } L)^\gamma$. We have in mind here the result of P. Sjolin [6] on the a.e. convergence of Fourier series in $L \text{ Log } L \text{Log Log } L$.

As pointed out by the referee, another interesting point about these connections is their simplicity while each maximal inequality is so far difficult to prove.

The results

THEOREM 1. *The following are equivalent for p real,* $1<p<\infty$.
(i) *Partial sums of Fourier series* (Carleson-Hunt [4], [6]).
For $f \in L^p[-\pi,\pi]$ *let*

$$S^*f(x) = \sup_n \left| \sum_{j=-n}^{n} \hat{f}(j)e^{ijx} \right|.$$

Then there is a constant C such that

$$\|S^*f\|_{L^p[-\pi,\pi]} \leq C\|f\|_{L^p[-\pi,\pi]} \quad \textit{for all } f \in L^p[-\pi,\pi].$$

(ii) *Maximal helical transform on l^p.*
For $a \in l^p(\mathbf{Z})$ and $\varepsilon \in \mathbf{R}$ define

$$H_\varepsilon a(j) = \sum_{k=-\infty}^{\infty}{}' \frac{e^{i(j-k)\varepsilon} a_{j-k}}{k}$$

and

$$H^* a(j) = \sup_{\varepsilon \in \mathbf{R}} |H_\varepsilon a(j)|.$$

There exists a constant C such that

$$\|H^* a\|_{l^p} \leq C\|a\|_{l^p} \quad \textit{for all } a \in l^p(\mathbf{Z}).$$

(iii) *Double maximal helical transform on l^p.*
For $a \in l^p(\mathbf{Z})$ and $\varepsilon \in \mathbf{R}$ define

$$H^{**} a(j) = \sup_n \sup_\varepsilon \left| \sum_{k=-n}^{n}{}' \frac{e^{i(j-k)\varepsilon} a_{j-k}}{k} \right|.$$

Then there is a constant C such that for all $a \in l^p(\mathbf{Z})$

$$|H^{**} a|_{l^p(\mathbf{Z})} \leq C|a|_{l^p(\mathbf{Z})}.$$

(iv) *Double maximal helical transform for a measure preserving transformation.*

There is a constant C such that for all dynamical systems $(X, \mathscr{F}, \mu, \varphi)$ and all $f \in L^p(\mu)$ we have

$$\left\| \sup_n \sup_\varepsilon \sum_{k=-n}^{n}{}' \frac{f(\varphi^k x)}{k} e^{ik\varepsilon} \right\|_p \leq C|f|_p.$$

(v) *Double maximal estimate for "the ergodic Fejer sums".*

There is a constant C such that for all dynamical systems $(X, \mathscr{F}, \mu, \varphi)$ and all $f \in L^p(\mu)$ we have

$$\left\| \sup_n \sup_\varepsilon \sum_{|k| \leq n}{}' \left(1 - \frac{|k|}{n+1}\right) \frac{f(\varphi^k x)}{k} e^{ik\varepsilon} \right\|_p \leq C\|f\|_p.$$

(v) *Double maximal estimate for a measure preserving flow.*

Let $\{T_s: -\infty < s < \infty\}$ *be a measuring preserving flow on a measure space* $(X, \mathscr{B}, \mu)$. *For* $f \in L^p(X, \mathscr{B}, \mu)$ *define*

$$\mathscr{H}^{**}f(x) = \sup_{N,\varepsilon}\left|\int_{N \le |s| \le 1/N} \frac{e^{is\varepsilon}f(T_s x)}{s}\,ds\right|.$$

There is a constant C such that for all measure preserving flow we have

$$\|\mathscr{H}^{**}f\|_{L^p} \le C\|f\|_p.$$

(vii) *Carleson Hunt estimate.*

For $f \in L^p_{[-\pi,\pi]}$ *define*

$$P^*f(x) = \sup_{n\in\mathbf{Z}}\left|\int_{-\pi}^{\pi} \frac{e^{int}f(t)}{x-t}\,dt\right|.$$

There is a constant C such that for all $f \in L^p[-\pi,\pi]$

$$\|P^*f\|_p \le C\|f\|_p.$$

Proof. We will prove the following implications: (i) $\Leftrightarrow$ (vii), (iii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (v), (iv) $\Rightarrow$ (vi), (vi) $\Rightarrow$ (vii), (iii) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (vii), and (vii) $\Rightarrow$ (iii).

(i) $\Leftrightarrow$ (vii) is certainly well known; (vii) $\Rightarrow$ (i) is a consequence of classical calculations involving the Dirichlet kernel and the Hilbert transform (see [7] for instance). The implication (i) $\Rightarrow$ (vii) can be proved by also using the fact that for $f \in L^p[-\pi,\pi]'$

$$f^1(x) = \sum_{J=-\infty}^{0} \hat{f}(j)e^{ijx} \quad\text{and}\quad f^2(x) = \sum_{j=0}^{0} \hat{f}(j)e^{ijx}$$

are also $L^p[-\pi,\pi]$ functions ($L^p[-\pi,\pi]$ admits projection).

(iii) $\Rightarrow$ (iv) can be obtained by a standard transference argument. For $a_j = f(\varphi^j x)$ we have positive integers N, L so that

$$\begin{aligned}
(2N+1)\int_X \sup_{n\le L}\sup_{\varepsilon}&\left|\sum_{k=-n}^{n}{}' \frac{e^{i(j-k)\varepsilon}f(\varphi^{j-k}x)}{k}\right|^p dx \\
&= \sum_{j=-N}^{N}\int_X \sup_{n\le L}\sup_{\varepsilon}\left|\sum_{k=-n}^{n}{}' \frac{e^{i(j-k)\varepsilon}f(\varphi^{j-k}x)}{k}\right|^p dx \\
&\le C\sum_{j=-N-L}^{N+L}\int |f(\varphi^j x)|^p\,dx \\
&= C(2(N+L)+1)\|f\|_p.
\end{aligned}$$

The result follows by dividing by $(2N+1)$, and letting N and then L go to infinity.

(iv) $\Rightarrow$ (v). This is a consequence of the following equality and the well-known strong (p,p) estimate for the ergodic averages. We have

$$\sum_{k=-n}^{n}{}' \frac{f(\varphi^k x)}{k} e^{ik\varepsilon} - \sum_{|k|\leq n}{}' \left(1 - \frac{|k|}{n+1}\right) \frac{f(\varphi^k x)}{k} e^{ik\varepsilon}$$
$$= \frac{1}{n+1} \sum_{k=0}^{n} \left(f(\varphi^k x) e^{ik\varepsilon} - (f(\varphi^{-k} x) e^{-ik\varepsilon}\right.$$

(iv) $\Rightarrow$ (vi). We can approximate a flow by times δ map of discrete measure preserving transformation as we did in [2].

(vi) $\Rightarrow$ (vii). It is enough to consider the particular case of the flow of translation on the real line.

(iii) $\Rightarrow$ (ii). Obvious (take a with finite support).

(ii) $\Rightarrow$ (vii). It is enough to show that there is a constant C such that

$$\int_{-\infty}^{\infty} \sup_{0\leq\varepsilon\leq 1} \left| \int_{-\infty}^{\infty} \frac{e^{i\varepsilon t} f(x-t)}{t}\, dt \right|^p \cdot dx \leq C \cdot \int_{-\infty}^{\infty} |f(t)|^p\, dt$$

for step functions of the following type:

$$f = \sum_{k=-\infty}^{\infty} a_k \cdot 1_{[k/N,(k+1)/N]}, \qquad 0 < \frac{1}{N} \leq \frac{1}{3}.$$

These functions are dense in $L^p(\mathbf{R})$:

$$\int_{-\infty}^{\infty} \sup_{\varepsilon} \left| \int_{-\infty}^{\infty} \frac{e^{i\varepsilon t} f(x-t)}{t}\, dt \right|^p dx$$
$$= \sum_{j=-\infty}^{\infty} \int_0^{1/N} \sup_{\varepsilon} \left| \sum_{k=-\infty}^{\infty} e^{ik\varepsilon/N} \int_{-1/2N}^{1/2N} \frac{e^{i\varepsilon t} f(x+(j-k)/N-t)}{t+k/N}\, dt \right|^p dx$$
$$= \sum_{j=-\infty}^{\infty} \int_0^{1/3N} \sup_{\varepsilon} \left| \sum_{k=-\infty}^{\infty} e^{ik\varepsilon/N} \int_{-1/2N}^{1/2N} \frac{e^{i\varepsilon t} f(x+(j-k)/N-t)}{t+k/N}\, dt \right|^p dx$$
$$+ \int_{1/3N}^{1/2N} \sup_{\varepsilon} \left| \sum_{k=-\infty}^{\infty} e^{ik\varepsilon/N} \int_{-1/2N}^{1/2N} \frac{e^{i\varepsilon t} f(x+(j-k)/N-t)}{t+k/N}\, dt \right|^p dx$$
$$+ \int_{1/2N}^{1/N} \sup_{\varepsilon} \left| \sum_{k=-\infty}^{\infty} e^{ik\varepsilon/N} \int_{-1/2N}^{1/2N} \frac{e^{i\varepsilon t} f(x+(j-k)/N-t)}{t+k/N}\, dt \right|^p dx$$

We will treat the first term and show how to get similar estimates for the two others.

$$\sum_{j=-\infty}^{\infty}\int_0^{1/3N}\sup_{\varepsilon}\left|\sum_{k=-\infty}^{\infty}e^{i\varepsilon k/N}\int_{-1/2N}^{1/2N}\frac{e^{i\varepsilon t}f(x+(j-k)/N-t)}{t+k/N}\,dt\right|^p dx$$

$$\leq C\left[\sum_{j=-\infty}^{\infty}\int_0^{1/3N}\sup_{\varepsilon}\left|\sum_{k=-\infty}^{\infty}e^{i\varepsilon k/N}\int_{-1/3N}^{1/3N}\frac{e^{i\varepsilon t}f(x+(j-k)/N-t)}{t+k/N}\,dt\right|^p dx\right.$$

$$+\sum_{j=-\infty}^{\infty}\int_0^{1/3N}\sup_{\varepsilon}\left|\sum_{k=-\infty}^{\infty}e^{i\varepsilon k/N}\int_{-1/2N}^{-1/3N}\frac{e^{i\varepsilon t}f(x+(j-k)/N-t)}{t+k/N}\,dt\right|^p dx$$

$$\left.+\sum_{j=-\infty}^{\infty}\int_0^{1/3N}\sup_{\varepsilon}\left|\sum_{k=-\infty}^{\infty}e^{i\varepsilon k/N}\int_{1/3N}^{1/2N}\frac{e^{i\varepsilon t}f(x+(j-k)/N-t)}{t+k/N}\,dt\right|^p dx\right].$$

Here again we will treat only the first term in detail; it is less than

$$C\left[\sum_{j=-\infty}^{\infty}\int_0^{1/3N}\sup_{\varepsilon}\left|{\sum_{k=-\infty}^{\infty}}'\frac{e^{i\varepsilon k/N}\cdot a_{j-k}}{k/N}\int_{-1/3N}^{1/3N}e^{it\varepsilon}\,dt\right|^p dx\right. \tag{1}$$

$$+\sum_{j=-\infty}^{\infty}\int_0^{1/3N}|a_j|^p\cdot\sup_{\varepsilon}\left|\int_{-1/3N}^{1/3N}\frac{e^{i\varepsilon t}}{t}\right|^p dx \tag{2}$$

$$+\sum_{j=-\infty}^{\infty}\int_0^{1/3N}\sup_{\varepsilon}\left|{\sum_{k=-\infty}^{\infty}}'\frac{e^{i\varepsilon k/N}\cdot a_{j-k}}{k/N}\int_{-1/3N}^{1/3N}\frac{e^{i\varepsilon t}\cdot t}{t+k/N}\right|^p dx. \tag{3}$$

(1) is less than

$$C\cdot\frac{1}{3N}\left(\sum_{j=-\infty}^{\infty}\sup_{\varepsilon}\left|{\sum_{k=-\infty}^{\infty}}'\frac{e^{i\varepsilon k}\cdot a_{j-k}}{k}\right|^p\right)\cdot\sup_{\varepsilon}\frac{|\sin(\varepsilon/3N)|^p_{N_p}}{|\varepsilon|^p}$$

$$\leq C\frac{1}{N}\sum_{k=-\infty}^{\infty}|a_k|^p = C\|f\|_p^p.$$

For (2), a direct computation shows that

$$\int_0^{1/3N}\sup_{0<\varepsilon\leq 1}\left|\int_{-1/3N}^{1/3N}\frac{e^{i\varepsilon t}}{t}\,dt\right|^p dx \leq \frac{C}{N}.$$

Notice that $|\int_{-1/3N}^{1/3N}e^{i\varepsilon t}/t\,dt|$ is taken in the principal value sense: it is equal to

$$\lim_{\delta \to 0} \left| \int_{\delta}^{1/3N} \frac{e^{i\varepsilon t} - e^{-i\varepsilon t}}{t} \, dt \right| = C \cdot \lim_{\delta \to 0} \left| \int_{\delta}^{1/3N} \frac{\sin \varepsilon t}{t} \, dt \right|.$$

For (3) we have

$$\sum_{j=-\infty}^{\infty} \int_{0}^{1/3N} \sup_{\varepsilon} \left| \sum_{k=-\infty}^{\infty}{}' \frac{e^{ik\varepsilon} \cdot a_{j-k}}{k/N} \int_{-1/3N}^{1/3N} \frac{e^{i\varepsilon t} t}{t + k/N} \, dt \right|^{p} dx$$
$$\leq \frac{C}{N} \sum_{j=-\infty}^{\infty} \left| \sum_{k=-\infty}^{\infty}{}' \frac{|a_{j-k}|}{k^2} \right|^{p}$$

and

$$\sum_{j=-\infty}^{\infty} \frac{C}{N} \left| \sum_{k=-\infty}^{\infty}{}' \frac{|a_{j-k}|}{k^2} \right|^{p} \leq \frac{C}{N} \sum_{k=-\infty}^{\infty} |a_k|^p = C \cdot \|f\|_p^p.$$

The other two terms can be treated similarly (by the same splitting).

It remains to show how to estimate

$$\int_{1/3N}^{1/2N} \sup_{\varepsilon} \left| \sum_{k=-\infty}^{\infty} e^{ik\varepsilon/N} \int_{-1/2N}^{1/2N} \frac{e^{i\varepsilon t} f(x + (j-k)/N - t)}{t + k/N} \, dt \right|^{p} dx$$

and

$$\int_{1/2N}^{1/N} \sup_{\varepsilon} \left| \sum_{k=-\infty}^{\infty} e^{ik\varepsilon/N} \int_{-1/2N}^{1/2N} \frac{e^{i\varepsilon t} f(x + (j-k)/N - t)}{t + k/N} \, dt \right|^{p} dx.$$

The control of these terms is similar. Again picking the first we can split

$$\int_{-1/2N}^{1/2N} \frac{e^{i\varepsilon t} f(x + (j-k)/N - t)}{t + k/N} \, dt$$

into two terms, $\int_{-1/2N}^{x}$ and $\int_{x}^{1/2N}$. Then we treat the corresponding terms as we just did. For the first integral, $x - t$ is positive and we have a similar situation; for the second integral, $x - t$ is negative and the resulting actions is a shift of the sequence (a_j) to (a_{j-1}). This proves (ii) $\Rightarrow$ (vii).

(vii) $\Rightarrow$ (iii). We need some notation. Let

$$Hf(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(t)}{\tan(x-t)}\,dt, \quad H_\eta f(x) = \frac{1}{\pi}\int_{\eta<|x-t|<\pi} \frac{f(t)}{\tan(x-t)}\,dt,$$

$$H^*f(x) = \sup_{\varepsilon\in\mathbf{Q}} \frac{1}{\pi}\left|\int_{-\pi}^{\pi} \frac{e^{i\varepsilon t}f(t)}{\tan(x-t)}\,dt\right|;$$

in fact, $H^*f(x)$ is equal to

$$\sup_{\varepsilon\in\mathbf{R}} \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{e^{i\varepsilon t}f(t)}{\tan(x-t)}\,dt,$$

see Lemma 4. For $f \in L^p[-\pi,\pi]$ and $H_* f(x) = \sup_{0<\eta<\pi} H_\eta f(x)$ we have (see [7], p. 120)

$$H_* f(x) \le C(Mf(x) + M(Hf)(x))$$

where M is the Hardy-Littlewood maximal function. So

$$\begin{aligned}(4)\qquad H_*(e^{i\varepsilon t}f)(x) &\le C\big(M(|f|)(x) + M\big(|H|(e^{i\varepsilon t}f)(x)\big)\big)\\ &\le C(M(|f|)(x) + M(|H^*f|)(x))\end{aligned}$$

and

$$\left\|\sup_{\varepsilon\in\mathbf{Q}} H_*(e^{i\varepsilon t}f)\right\|_p \le C\|f\|_p$$

because (vii) implies that H^*f is strong type (p,p)

$$\left(\frac{1}{\tan(x-t)} - \frac{1}{x-t} \in L^\infty\right).$$

We have also

$$\begin{aligned}&\left\|\sup_{\eta}\sup_{\varepsilon\in\mathbf{Q}} \frac{1}{\pi}\int_{\eta<|x-t|<\pi} \frac{e^{i\varepsilon t}f(t)}{x-t}\,dt\right\|_p\\ &\quad= \left\|\sup_{\eta}\sup_{\varepsilon\in\mathbf{R}} \frac{1}{\pi}\int_{\eta<|x-t|x<\pi} \frac{e^{i\varepsilon t}f(t)}{x-t}\,dt\right\|_p \le C\|f\|_p.\end{aligned}$$

This can now be extended to $L^p(\mathbf{R})$ (by change of variables and functions, for instance) to obtain

$$\int_{-\infty}^{\infty} \sup_{\varepsilon} \sup_{\eta} \left| \frac{1}{\pi} \int_{\eta < |x-t|} \frac{e^{i\varepsilon t} f(t)}{x-t}\, dt \right|^p dx \le C \int_{-\infty}^{\infty} |f(t)|^p\, dt.$$

As

$$\sup_{\varepsilon \in \mathbf{Q}} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\varepsilon t} f(t)}{x-t}\, dt \in L^p(\mathbf{R})$$

we deduce that

$$\int_{-\infty}^{\infty} \sup_{\eta} \sup_{\varepsilon \in \mathbf{R}} \left| \frac{1}{\pi} \int_{0<|x-t|<\eta} \frac{e^{i\varepsilon t} f(t)}{x-t}\, dt \right|^p dx \le C \int_{-\infty}^{\infty} |f(t)|^p\, dt.$$

Now we take

$$f(t) = \begin{cases} a_k & \text{if } k - 1/8 \le t \le k + 1/8, \quad k \in \mathbf{Z} \\ 0 & \text{otherwise.} \end{cases}$$

We get

$$\begin{aligned} C\|a\|^p_{l_p(\mathbf{Z})} &\ge \sum_{j=-\infty}^{\infty} \int_0^1 \sup_{\varepsilon} \sup_{\eta} \left| \int_{0<|j+x-t|<\eta} \frac{e^{i\varepsilon t} f(t)}{j+x-t}\, dt \right|^p dx \\ &= \int_{-\infty}^{\infty} \sup_{\varepsilon} \sup_{\eta} \left| \int_{0<|x-t|<\eta} \frac{e^{i\varepsilon t} f(t)}{x-t}\, dt \right|^p dx \\ &= \sum_{j=-\infty}^{\infty} \int_0^1 \sup_{\varepsilon} \sup_{\eta} \left| \int_{0<|j+x-t|<\eta} e^{i\varepsilon t} \right. \\ &\qquad\qquad \left. \times \sum_{k=-\infty}^{\infty} a_k 1_{[k-1/8,\, k+1/8]}(t)\, dt \right|^p dx \\ &\ge \sum_{j=-\infty}^{\infty} \int_0^{1/8} \sup_{-1<\varepsilon<1} \sup_{\eta} \left| \sum_{|k-j|\le[\eta]} a_k \right. \\ &\qquad\qquad \left. \cdot e^{ik\varepsilon} \int_{-1/8}^{1/8} \frac{e^{i\varepsilon t}}{j+x-k-t}\, dt \right|^p dx \\ &= \sum_{j=-\infty}^{\infty} \int_0^{1/8} \sup_{-1<\varepsilon<1} \sup_{\eta} \left| {\sum_{|k-j|\le[\eta]}}' \frac{a_k e^{ik\varepsilon}}{j-k} \right. \\ &\qquad\qquad \times \left(\int_{-1/8}^{1/8} e^{ik\varepsilon}\, dt - \int_{-1/8}^{1/8} \frac{e^{i\varepsilon t}(x-t)}{x+j-k-t}\, dt \right) \\ &\qquad \left. + e^{ij\varepsilon} a_j \int_{-1/8}^{1/8} \frac{e^{i\varepsilon t}}{x-t}\, dt \right|^p dx. \end{aligned}$$

Because

$$\int_0^{1/8} \sup_{-1<\varepsilon<1} \left| \int_{-1/8}^{1/8} \frac{e^{i\varepsilon t}}{x-t}\, dt \right|^p dx$$

is a finite absolute constant and $\int_{-1/8}^{1/8} e^{ik\varepsilon}\, dt$ is bounded below by an absolute constant the result follows from noting that for $x \in [0, 1/8]$,

$$\sum_{j=-\infty}^{\infty} \sup_{-1<\varepsilon<1} \sup_{\eta} \left| \sum_{|k-j|\le[\eta]}' \frac{e^{ik\varepsilon} a_k}{j-k} \left[\int_{-1/8}^{1/8} \frac{e^{i\varepsilon t}(x-t)}{x+j-k-t}\, dt \right] \right|^p$$

$$\le C \sum_{j=-\infty}^{\infty} \sup_{-1<\varepsilon<1} \sup_{\eta} \left| \sum_{|k-j|\le[\eta]}' \frac{|a_k|}{|j-k|} \int_{-1/8}^{1/8} \frac{1}{|x+j-k-t|}\, dt \right|^p$$

$$\le C \sum_{j=-\infty}^{\infty} \sup_{\eta} \left| \sum_{|k-j|\le[\eta]}' \frac{|a_k|}{|j-k|^2} \right|^p \le C \sum_{j=-\infty}^{\infty} \sup_{\eta} \left| \sum_{|k'|\le[\eta]}' \frac{|a'_{k+j}|}{k'^2} \right|^p$$

$$\le C \sum_{j=-\infty}^{\infty} \left| \sum_{k'=-\infty}^{\infty}{}' \frac{|a'_{k+j}|}{k'^2} \right|^p \le \|a\|_{l_p(\mathbf{Z})}^p.$$

COROLLARY 2. *For all p, $1 < p < \infty$, and all dynamical systems $(X, \mathscr{F}, \mu, \varphi)$ we have*

$$\left\| \sup_{\eta} \sup_{\varepsilon} \sum_{k=-n}^{n}{}' \frac{f(\varphi^k x)}{k} e^{ik\varepsilon} \right\|_p \le C \frac{p^6}{(p-1)^4} \|f\|_p$$

for all $f \in L^p(\mu)$.

(The constant C does not depend on p or any particular dynamical system.) The same estimate holds for invertible measure preserving flow; we have

$$\|\mathscr{H}^{**}f\|_p = \left\| \sup_{N,\varepsilon} \left| \int_{N\le|s|\le 1/N} \frac{e^{is\varepsilon} f(T_s x)}{s}\, ds \right| \right\|_p \le C \frac{p^6}{(p-1)^4} \|f\|_p.$$

for all $f \in L^p(\mu)$ and all measure preserving flows T_s on $(X, \mathscr{B}, \mu)$.

Proof. In [6] Hunt proved that the maximal operator $P^*f(x)$ in (vii) of the previous theorem satisfies a strong type (p, p) estimate with a constant

$$C_p \le \frac{p^5}{(p-1)^3} \cdot \text{constant}.$$

A look at the proof of Theorem 1 and keeping track of the constant shows that the constant in (iii) is the same as the one for $H^{**}f(x)$, the double maximal helical transform. The constant in (iii) is less than

$$\text{constant} \cdot \frac{p^6}{(p-1)^4}$$

because of the Hardy Littlewood maximal function M (inequality (4) in the proof of (vii) $\Rightarrow$ (iii)). The same conclusion also holds for measure preserving flow by again using discrete approximations of the flow by times δ map.

THEOREM 3. (i) *For all dynamical systems* $(X, \mathscr{F}, \mu, \varphi)$ $(\mu(X) = 1)$, (*resp. all measure preserving flows*), *we have, for each* $f \in L^1(\mu)$ *such that* $\int_X |f(x)| \ln^{+4}|f(x)|\, d\mu < +\infty$,

$$\int \sup_n \sup_\varepsilon \left| \sum_{k=-n}^{n}{}' \frac{f(\varphi^k x)}{k} e^{ik\varepsilon} \right| d\mu \leq C\left(1 + \int_X |f(x)| \ln^{+4}|f(x)|\, d\mu\right)$$

$$\left(resp.\ \sup_n \sup_\varepsilon \left| \int_{1/\eta \leq |t| \leq \eta} \frac{e^{i\varepsilon t} f(T_t x)}{t}\, dt \right| d\mu \right.$$
$$\left. \leq C\left(1 + \int_X |f(x)| \ln^{+4}|f(x)|\, d\mu\right)\right).$$

(ii) *There exist positive constants* λ, K, C *such that for all dynamical systems* $(X, \mathscr{F}, \mu, \varphi)$, $(\mu(X) = 1)$ *we have, for all* $f \in L^\infty$,

$$\int \exp\left(\lambda \left| \sup_n \sup_\varepsilon \left| \sum_{k=-n}^{n}{}' \frac{f(\varphi^k x)}{\|f\|_\infty \cdot k} e^{ik\varepsilon} \right| \right|^{1/2}\right) d\mu \leq K$$

and

$$\mu\left\{ x : \sup_n \sup_\varepsilon \left| \sum_{k=-n}^{n}{}' \frac{f(\varphi^k x) e^{ik\varepsilon}}{k} \right| > \lambda \right\} \leq C \exp\left(-C \frac{\sqrt{\lambda}}{\|f\|_\infty^{1/2}} \right).$$

Proof. (i) and the first part of (ii) are direct consequences of Corollary 2 and the extrapolation result found in [8, vol. II, p. 119]. (The proof in [8] works also for sublinear operators.)

For the second part of (ii) we use Corollary 2 and some ideas in [8, p. 119]. We have

$$\left\| \sup_n \sup_\varepsilon \left| \sum_{|l| \leq n}{}' \frac{e^{il\varepsilon} f(\varphi^l x)}{s} \right| \right\|_p \leq Cp^2 \|f\|_p$$

for, say, $p \geq 2$. Then for k an integer, $k \geq 2$, we have

$$\lambda^k \mu\{x : |H^{**}f(x)| > \lambda\} \leq \int |H^{**}f|^k \, d\mu \leq \|f\|_\infty^k \cdot C^k \cdot k^{2k}$$

and

$$\lambda^{+k} \|f\|_\infty^{-k} C^{-k} k^{-2k} \mu\{x : |H^{**}f(x)| > \lambda\} \leq 1.$$

As $k^{-2k} \geq e^{-2k} \cdot 2^{2k}((2k)!)^{-1}$, we have

$$\sum_{k=2}^{\infty} \frac{\lambda^{+k} C^{-k} \|f\|_\infty e^{-2k}}{(2k)!} \mu\{x : |H^{**}f(x)| > \lambda\} \leq \sum_{K=2}^{\infty} 2^{-2k}$$

and

$$\sum_{k=2}^{\infty} \frac{\sqrt{\lambda}^{2k} \sqrt{C}^{-2k} \left(\|f\|_\infty^{1/2}\right)^{-2k} e^{-2k}}{(2k)!} \mu\{x : |H^{**}f(x)| > \lambda\} \leq 1$$

The conclusion follows by noting that for $\lambda > \lambda_0$,

$$\sum_{k=2}^{\infty} \frac{\sqrt{\lambda}^{2k} \sqrt{C}^{-2k} \left(f\|_\infty^{1/2}\right)^{-2k} e^{-2k}}{(2k)!} \geq \tilde{C} \exp\left(\sqrt{\lambda} \cdot \left(\sqrt{C} \|f\|_\infty^{1/2}\right)^{-1}\right)$$

and for $0 < \lambda < \lambda_0$, the same type of inequality holds. □

The last proposition already allows us to improve one of our results [1] on the shift on $[0,1]^{\mathbf{Z}}$ for functions in $L(\operatorname{Log} L)^4$. More exactly, we can show that for this shift and $f \in L(\operatorname{Log} L)^4$ when f satisfies W.W. and $f - \int f \, d\mu^{\mathbf{Z}}$ satisfies S.W.W. We want to prove that these W.W. properties hold in fact for functions in $L(\operatorname{Log} L)^2$. To achieve this goal we need to extend Hunt's basic inequality. But before that, to avoid any measurability problem by dealing with a supremum on an uncountable set of measurable functions we will prove the following lemma:

LEMMA 4. *For all p, $1 < p < \infty$, and all $f \in L^p(\mathbf{R})$, there exists a single null set of which*

$$Hf(x) = \int_{-\infty}^{\infty} \frac{e^{i\varepsilon t} f(t)}{x - t} \, dt \text{ exists for all } \varepsilon \in \mathbf{R}.$$

(In other words, the flow of translation on the real line satisfies W.W.)

Proof. There are several ways to prove this lemma. One way is to exhibit a dense set of functions where the property W.W. holds and then use the strong (p, p) estimate of $H^{**}f$. The dense set is easily provided by the continuous differentiable functions with compact support. Now let us take g_j such that $\|f - g_i\|_p \to_i 0$, g_i continuous differentiable with compact support. We have

$$\left\{x : \sup_{\varepsilon}\left(\limsup_{y\to 0}\int_{|x-t|>y}\frac{e^{i\varepsilon t}f(t)}{x-t}\,dt - \liminf_{y\to 0}\int_{|x-t|>y}\frac{e^{i\varepsilon t}f(t)}{x-t}\,dt\right)\right\}$$

$$\subset \left\{x : \sup_{\varepsilon}\left|\limsup_{y\to 0}\int_{|x-t|>y}\frac{e^{i\varepsilon t}(f-g_j)(t)}{x-t}\,dt\right.\right.$$

$$\left.\left. - \liminf_{y\to 0}\int_{|x-t|>y}\frac{e^{i\varepsilon t}(f-g_i)(t)}{x-t}\,dt\right|\right\}$$

for each j

$$\subset \left\{x : 2\sup_{\varepsilon}\sup_{\eta}\left|\int_{|x-t|>\eta}\frac{e^{i\varepsilon t}(f-g_j)(t)}{x-t}\,dt\right|\right\}$$

and

$$\lim_j \mu\left\{x : 2\sup_{\varepsilon}\sup_{\eta}\left|\int_{|x-t|>\eta}\frac{e^{i\varepsilon t}(f-g_j)(t)}{x-t}\,dt\right|\right\} = 0$$

because of the strong type property of H^{**}. □

We can now extend Hunt's inequality

PROPOSITION 5. *For $A \subset [-\pi, \pi]$ let*

$$H^*1_A(x) = \sup_{\varepsilon\in\mathbf{R}}\left|\int_{-\infty}^{\infty}\frac{e^{i\varepsilon t}1_A(t)}{x-t}\,dt\right|.$$

Then there exists a constant C such that

$$\mu\{x \in (-\pi, \pi); H^*1_A(x) > \lambda\} \le \left(\frac{C}{\lambda}\frac{p^2}{p-1}\right)^p \mu(A)$$

for all $A \subset (-\pi, \pi)$.

Proof. Hunt's basic inequality says that

$$\mu\left\{x \in (-\pi,\pi) : \sup_{n\in\mathbf{Z}}\left|\int_{-\pi}^{\pi}\frac{e^{int}1_A(t)}{x-t}\,dt\right| > \lambda\right\} \le \left(C\frac{p^2}{p-1}\right)^p \lambda^{-p}\mu(A)$$

for all $A \in (-\pi,\pi)$ and all p, $1 < p \le 2$.

From this we deduce first that

$$(1')\quad \mu\left\{x \in \mathbf{R} : \sup_{n\in\mathbf{Z}}\left|\int_{-\pi}^{\pi}\frac{e^{int}1_A(t)}{x-t}\,dt\right| > \lambda\right\} \le \left(C\frac{p^2}{p-1}\right)^p \lambda^{-p}\mu(A)$$

This is because

$$\begin{aligned}
&\mu\{x\in\mathbf{R}: H^*1_A(x) > \lambda\}\\
&\quad = \sum_{k=-\infty}^{\infty,} \mu\{x \in (-\pi+2k\pi,\pi+2k\pi) : H^*1_A(x) > \lambda\}\\
&\quad = \sum_{k=-\infty}^{\infty} \mu\left\{x \in (-\pi,\pi) : \sup_{n\in\mathbf{Z}}\left|\int_{-\pi}^{\pi}\frac{e^{int}1_A(t)}{x+2k\pi-t}\,dt\right| > \lambda\right\}
\end{aligned}$$

and for $|k| \ge 2$ we have

$$\left|\int_{-\pi}^{\pi}\frac{e^{int}1_A(t)}{2k\pi}\,dt - \int_{-\pi}^{\pi}\frac{e^{int}1_A(t)}{x+2k\pi-t}\,dt\right| \le C\int_{-\pi}^{\pi}\frac{1_A}{k^2\pi}\,dt\Big|$$

so

$$\begin{aligned}
&\sum_{|k|\ge 2}\mu\left\{x \in (-\pi,\pi) : \sup_{n\in\mathbf{Z}}\left|\int_{-\pi}^{\pi}\frac{e^{int}1_A(t)}{x+2k\pi-t}\,dt\right| > \lambda\right\}\\
&\quad \le \sum_{|k|\ge 2}\mu\left\{x \in (-\pi,\pi) : \sup_{n\in\mathbf{Z}}\left|\int_{-\pi}^{\pi}\frac{e^{int}1_A(t)}{2k\pi}\,dt\right| > \frac{\lambda}{2}\right\}\\
&\quad\quad + \sum_{|k|\ge 2}\mu\left\{x \in (-\pi,\pi) : C\left|\int_{-\pi}^{\pi}\frac{1_A(t)}{k^2\pi}\,dt\right| > \frac{\lambda}{2}\right\}\\
&\quad \le C\sum_{k=-\infty}^{\infty}\frac{\mu(A)}{\lambda^p}\left(\frac{1}{|k\pi|}\right)^p \le \left(C\frac{p^2}{p-1}\right)^p\frac{\mu(A)}{\lambda^p}.
\end{aligned}$$

The case $k = 0$ is clear. For $k = 1$ or $k = -1$ we can use a periodic extension of 1_A. Hunt's basic estimate holds for periodic functions. Now for any set $B \subset [-k_0\pi, k_0\pi]$, k_0 positive integer, we have

$$(2') \quad \mu\left\{x \in \mathbf{R}: \sup_{n\in\mathbf{Z}}\left|\int_{-\pi}^{\pi} \frac{e^{int}1_B(t)}{x-t}\,dt\right| > \lambda\right\} \le \left(C\frac{p^2}{p-1}\right)^p \frac{\mu(B)}{\lambda^p}.$$

As $1_B(k_0t)$ is the characteristic function of a set included in $(-\pi, \pi)$, by (1′) we have

$$\mu\left\{x \in \mathbf{R}: \sup_{n\in\mathbf{Z}}\left|\int_{-\infty}^{\infty} \frac{e^{int}1_B(k_0t)}{x-t}\,dt\right| > \lambda\right\} \le \lambda^{-p}\left(C\frac{p^2}{p-1}\right)^p \int_{-\pi}^{\pi} 1_B(k_0t)\,dt$$

and

$$\mu\left\{x \in \mathbf{R}: \sup_{n\in\mathbf{Z}}\left|\int_{-\infty}^{\infty} \frac{e^{in(t/k_0)}1_B(k_0t)}{x-t/k_0}\,dt\right| > \lambda\right\}$$
$$\le \lambda^{-p}\left(C\frac{p^2}{p-1}\right)^p \int_{-\pi}^{\pi} 1_B(k_0t)\,dt.$$

This implies that

$$\mu\left\{x \in \mathbf{R}: \sup_{n'\in\mathbf{Z}}\left|\int_{-\infty}^{\infty} \frac{e^{in(t/k_0)}1_B(k_0t)}{x-t/k_0}\,dt\right| > \lambda\right\}$$
$$\le \lambda^{-p}\left(C\frac{p^2}{p-1}\right)^p \int_{-\pi}^{\pi} 1_B(k_0t)\,dt.$$

The conclusion in (2) follows by the change of variables $x' = k_0x$. Now the conclusion of Proposition 5 follows easily. It is enough to show that for each m positive integer

$$\mu\left\{x \in \mathbf{R}: \sup_{k\in\mathbf{Z}}\left|\int_{-\infty}^{\infty} \frac{e^{i(k/2^m)t}1_A(t)}{x-t}\right| > \lambda\right\} \le \frac{1}{\lambda^p}\left(C\frac{p^2}{p-1}\right)^p m(A)$$

But this is now an easy consequence of (1) and (2) (after changes of variable $u = t/2^m$, $x' = 2^mx$). We have the following corollary:

Corollary 6. *There exists a constant C such that*

$$\int_{-\pi}^{\pi} \sup_{\varepsilon\in\mathbf{R}}\left|\int_{-\pi}^{\pi} \frac{e^{i\varepsilon t}f(t)}{x-t}\,dt\right| dx \le C\left(1 + \int |f(x)|\ln^{+2}|f(x)|\,d\mu\right)$$

for all $f \in L(\mathrm{Log}\, L)^2$.

Proof. We can use the Lorentz spaces $L(p, q)$ as R. Hunt did in his remarks (p. 235–236). They are a modification of [8, p. 119].

THEOREM 7. *Let $(X, \mathscr{F}, \mu, \varphi)$ be a dynamical system on a probability measure space. There exists a constant C such that for all $f \in L(\mathrm{Log}\, L)^2$ we have*

$$\mu\{x : |H^{**}f(x)| > \lambda\} \leq \frac{C}{\lambda}\int |f|\, d\mu + \frac{C}{\lambda}\left(1 + \int |f(x)| \ln^{+2}|f(x)|\, d\mu\right).$$

Proof. Using Corollary 6 we can get the inequality

$$\tag{3'} \mu\left\{x \in (-\pi, \pi) : \sup_{0<\eta<\pi}\ \sup_{\varepsilon \in \mathbf{R}}\left|\int_{0<|x-t|<\eta} \frac{e^{i\varepsilon t}f(t)}{x-t}\, dt\right| > \lambda\right\}$$
$$\leq \frac{C}{\lambda}\int |f|\, d\mu + \frac{C}{\lambda}\left(1 + \int |f|(x)| \ln^{+2}|f|(x)|\, d\mu\right)$$

for all $\lambda > 0$. This, because of the following inequality we already used:

$$\sup_{0<\eta<\pi}\ \sup_{\varepsilon \in \mathbf{R}}\left|\int_{0<|x-t|<\eta} \frac{e^{i\varepsilon t}f(t)}{x-t}\, dt\right|$$
$$\leq C\left(M(|f|)(x) + M\left(\sup_{\varepsilon}\int_{-\pi}^{\pi} \frac{e^{i\varepsilon t}f(t)}{x-t}\, dt\right)\right)$$

where M is the Hardy-Littlewood maximal function. We want to transfer (3′) to the ergodic setting.

Let $(a_j) : |j| \leq J$ be a sequence of real numbers and

$$f(t) = \sum_{j=-J}^{J} a_j 1_{(-\pi/8J + \pi j/J,\, \pi/8J + \pi j/J)}(t).$$

Let

$$V = \left\{\left(\left|x + \frac{k\pi}{J} - \frac{j\pi}{J} - t\right| < \eta\right) \cap \left(\frac{-\pi}{8J}, \frac{\pi}{8J}\right)\right\}.$$

By (3) we have

$$\sum_{k=-J}^{J} \mu\left\{x \in \left(0, \frac{\pi}{J}\right) : \sup_{0<\eta<\pi}\ \sup_{\varepsilon \in H}\left|\sum_{j=-J}^{J}\int_V \frac{e^{i\varepsilon t}\cdot e^{i\varepsilon\pi(j/J)}a_j}{x + k\pi/J - j\pi/J - t}\, dt\right| > \lambda\right\}$$
$$\leq \frac{C}{\lambda}\sum_{j=-J}^{J}|a_j| \cdot \frac{\pi}{8J} + \frac{C}{\lambda}\left(1 + \frac{\pi}{8J}\sum_{j=-J}^{J}|a_j| \cdot \ln^{+1}|a_j|\right).$$

Now we have the following properties:

$$(4)\quad \mu\left\{x \in \left(0, \frac{\pi}{8J}\right) : \sup_{0<\eta<\pi} \sup_{|\varepsilon|<J} \left| \sum_{j=-J}^{J} \int_V \frac{e^{i\varepsilon t} \cdot e^{i\varepsilon\pi(j/J)} a_j}{x + k\pi/J - j\pi/J - t}\, dt \right| > \lambda \right\}$$

$$\leq \mu\left\{x \in (0, \pi/J) : \sup_{0<\eta<\pi} \sup_{\varepsilon \in \mathbf{R}} \left| \int_{0<|x-t|<\eta} \frac{e^{i\varepsilon t} f(t)\, dt}{x - t} \right| > \lambda \right\},$$

$$(5)\quad \int_{-\pi/8J}^{\pi/8J} \frac{e^{i\varepsilon t} \cdot e^{i\varepsilon\pi(j/J)} a_j}{x + k\pi/J - j\pi/J - t}\, dt$$

$$= \int_{-\pi/8J}^{\pi/8J} \frac{e^{i\varepsilon t}}{k\pi/J - j\pi/J}\, dt$$

$$- \int_{-\pi/8J}^{\pi/8J} \frac{e^{i\varepsilon t}(x - t)}{\left(\frac{k\pi - j\pi}{J} + x - t\right)\left(\frac{k\pi - j\pi}{J}\right)}\, dt$$

and

$$(6)\quad \inf_{|\varepsilon|<J} \left| \int_{-\pi/8J}^{\pi/8J} e^{i\varepsilon t}\, dt \right| \geq \gamma \left| \int_{-\pi/8J}^{\pi/8J} \frac{e^{i\varepsilon t}(x - t)}{\left(\frac{k\pi - j\pi}{J}\right)\left(\frac{k\pi - j\pi}{J} + x - t\right)}\, dt \right|$$

$$\leq C \int_{-\pi/8J}^{\pi/8J} \frac{|x - t|}{\left(\frac{k\pi - j\pi}{J}\right)^2}\, dt \leq \frac{C}{J^2} \cdot \frac{J^2}{(k - j)^2}$$

for $k \neq j$ and $x \in (0, \pi/8J)$. Finally

$$(\mathrm{A})\quad \sum_{j=-J}^{J} \mu\left\{x \in \left(0, \frac{\pi}{8J}\right) : \sup_{n \leq J} \sup_{|\varepsilon'| \leq 1} \left| \sum_{|k-j| \leq n}' \frac{e^{i\varepsilon' j} \alpha_i}{k - j} \right| > \alpha\lambda \right\}$$

$$\leq \frac{C}{\lambda}\left(\sum_{j=-J}^{J} |a_j| \cdot \frac{\pi}{8J} \right) + \frac{C}{\lambda}\left(1 + \frac{\pi}{8J} \sum_{j=-J}^{J} |a_j| \ln^{-2}|a_j| \right)$$

$$(\mathrm{B})\quad + \sum_{k=-J}^{J} \mu\left\{x \in \left(0, \frac{\pi}{8J}\right) : \sup_{n \leq J} \left| \sum_{|k-j| \leq n}' \frac{|e^{i\varepsilon' j}|\, |\alpha_j|}{(k - j)^2} \right| > \beta\lambda \right\}$$

$$(\mathrm{D})\quad + \sum_{k=-J}^{J} \mu\left\{x \in \left(0, \frac{\pi}{8J}\right) : \sup_{0<\eta<\pi} \sup_{|\varepsilon'| \leq 1} \left| \int_{\mathbf{Z}} \frac{e^{i\varepsilon' t} a_j}{x - t}\, dt \right| > \beta\lambda \right\}$$

where $\mathbf{Z} = \{(|x - t| < \eta) \cap (-\pi/8J, \pi/8J)\}$ and α and β are constants independent of J, k, j and λ. We have

$$\text{(B)} \qquad = \frac{\pi}{8J}\operatorname{card}\left\{k \in (-J, J) : \left| \sum_{|k-j| \le J}' \frac{|a_j|}{(k-j)^2} \right| > \beta\lambda \right\}$$

and

$$\text{(D)} \qquad \le \frac{C}{\lambda}\left(\sum_{j=-J}^{J} |a_j| \cdot 2\frac{\pi}{8J} \right) + \frac{C}{\lambda}\left(1 + \frac{\pi}{8J} \sum_{j=-J}^{J} |a_j| \ln^{+2}|a_j| \right).$$

Then by (3) and the equality

$$(D) = \sum_{k=-J}^{J} \mu\left\{ x \in \left(\frac{j\pi}{J}, \frac{(j+1)\pi}{J} \right) : \sup_{0<\eta<\pi} \sup_{|\varepsilon'| \le 1} \left| \int_W \frac{e^{i\varepsilon' t} a_j}{x-t}\, dt \right| > \beta\lambda \right\}$$

where

$$W = \left\{ (|x-t| < \eta) \cap \left(\frac{-\pi}{8J} + \frac{\pi j}{J}, \frac{\pi}{8J} + \frac{(j+1)\pi}{J} \right) \right\}$$

we have

$$\sum_{k=-J}^{J} \mu\left\{ x \in \left(\frac{j\pi}{J}, \frac{(j+1)\pi}{J} \right) : \sup_{0<\eta<\pi} \sup_{|\varepsilon'| \le 1} \left| \int_{|x-t|<\eta} \frac{e^{i\varepsilon' t} f(t)}{x-t}\, dt \right| > \beta\lambda \right\}$$
$$= \mu\left\{ x \in (-\pi, \pi) : \sup_{0<\eta<\pi} \sup_{|\varepsilon'| \le 1} \left| \int_{|x-t|<\eta} \frac{e^{i\varepsilon' t} f(t)}{x-t}\, dt \right| > \beta\lambda \right\}$$

where

$$f(t) = \sum_{k=-J}^{J} a_j 1\left(\frac{-\pi}{8J} + \frac{\pi j}{J}, \frac{\pi}{8J} + \frac{\pi j}{J} \right)(t).$$

So

$$\text{(E)} \quad \frac{\pi}{8J}\operatorname{card}\left\{ k \in (-J, J) : \sup_{0<\eta<\pi} \sup_{|\varepsilon'| \le 1} \left| \sum_{|k-j| \le J}' \frac{e^{i\varepsilon' j} a_i}{k-j} \right| > \alpha\lambda \right\}$$
$$\le 2(\text{A}) + \frac{\pi}{8J}\operatorname{card}\left\{ k \in (-J, J) : \left| \sum_{|k-j| \le J}' \frac{|a_j|}{(k-j)^2} \right| > \beta\lambda \right\}.$$

Now suppose $(X, \mathscr{F}, \mu, \varphi)$ is a dynamical system and $f \in L(\text{Log } L)^2$. If

$$\begin{aligned} a_j &= f(\varphi^j x) \quad \text{for } |j| \le J, \\ &= 0 \qquad\quad \text{otherwise,} \end{aligned}$$

then

$$\begin{aligned} &\frac{\pi}{8J}\operatorname{card}\left\{k \in (-J, J) : \sup_{n \le N} \sup_{|\varepsilon'| \le 1} \left| \sum_{|l| \le n} \frac{e^{i\varepsilon' l} f(\varphi^{l+k} x)}{l} \right| > \alpha\lambda \right\} \\ &\quad \le \frac{C}{\lambda}\left(\sum_{j=-J}^{J} |f(\varphi^j(x))| \right) + \frac{C}{\lambda}\left(1 + \frac{\pi}{8J} \sum_{j=-J}^{J} |f(\varphi^j(x) \ln^{+2}|f(\varphi^j x)| \right) \\ &\qquad + \frac{\pi}{8J}\operatorname{card}\left\{k \in (-J, J) : \left| \sum_{|l| \le J} \frac{e^{i\varepsilon' l} f(\varphi^{l+k} x)}{l^2} \right| > \beta\lambda \right\} \end{aligned}$$

for each fixed integer $N \le J$.

By integrating both sides of the inequality with respect to μ we get

$$\begin{aligned} &\frac{\pi}{8J}\mu x \operatorname{card}\{(x, k) : H_k^N f(x) > \alpha\lambda\} \\ &\quad \le \frac{C}{\lambda}\int |f| \, d\mu + \frac{C}{\lambda}\left(1 + \frac{\pi}{4}\int |f(x)| \ln^{+2}|f(x)| \, d\mu \right) \\ &\qquad + \frac{C}{\lambda}\int \sum_{l=-\infty}^{\infty,} \frac{|f(\varphi^{l+k} x)|}{l^2} \, d\mu(x); \end{aligned}$$

because φ is measure preserving (card is the counting measure on $\mathbf{Z}$),

$$H_k^N f(x) = \sup_{n \le N} \sup_{|\varepsilon'| \le 1} \left| \sum_{|l| \le n} \frac{e^{i\varepsilon' l} f(\varphi^{l+k} x)}{l} \right|.$$

As

$$\begin{aligned} \mu \times \operatorname{card}\{(x, k) : H_k^N f(x) \ge \alpha\lambda\} &\ge \sum_{k=-J}^{J} \mu\{x \, H_k^N f(x) > \lambda\} \\ &= (2J + 1)\mu\{x : H_0^N f(x) > \lambda\} \end{aligned}$$

the conclusion of Theorem 6 now follows easily.

As an application of Theorem 6 we can improve one of our results in [1] on the shift on $[0, 1]^{\mathbf{Z}}$.

COROLLARY 8. *Let φ be the shift on $([0,1]^{\mathbf{Z}}, B[0,1]^{\mathbf{Z}}), \mu^{\mathbf{Z}})$. For any $f \in L^1(\mu^{\mathbf{Z}})$ such that*

$$\int |f(x)| \ln^{+2} |f(x)| \, d\mu < \infty$$

we have the following:

(i) *f satisfies W.W.*;

(ii) *$f - \int f \, d\mu^{\mathbf{Z}}$ satisfies S.W.W.*

Proof. (i) We know by [1, Theorem 7] that any $f \in L^p(\mu^{\mathbf{Z}})$ satisfies W.W. and $f - \int f \, d\mu^{\mathbf{Z}}$ then satisfies S.W.W. Let $f_n \in L^p(\mu^{\mathbf{Z}})$ be such that $f - f_n \to_n 0$ a.e. For each positive integer k we have

$$\sup_{\varepsilon} \left(\overline{\lim_n} \sum_{k=-n}^{n}{}' \frac{f(\varphi^k x)}{k} e^{ik\varepsilon} - \underline{\lim_n} \sum_{k=-n}^{n}{}' \frac{f(\varphi^k x)}{k} e^{ik\varepsilon} \right)$$
$$\leq 2 \sup_n \sup_{\varepsilon} \left| \sum_{l=-n}^{n}{}' \frac{(f - f_k)(\varphi^l x)}{l} e^{ik\varepsilon} \right|.$$

So

$$\left\{ x : \sup_{\varepsilon} \left(\overline{\lim_n} \sum_{k=-n}^{n}{}' \frac{f(\varphi^k x)}{k} e^{ik\varepsilon} - \underline{\lim_n} \sum_{k=-n}^{n}{}' \frac{f(\varphi^k x)}{k} e^{ik\varepsilon} \right) > 0 \right\}$$
$$\subset \left\{ x : 2 \sup_n \sup_{\varepsilon} \left| \sum_{l=-n}^{n}{}' \frac{(f - f_k)(\varphi^l x)}{l} e^{ik\varepsilon} \right| > 0 \right\} \text{ for each } k.$$

But

$$\left\{ x : 2 \sup_n \sup_{\varepsilon} \left| \sum_{l=-n}^{n}{}' \alpha \frac{(f - f_k)(\varphi^l x)}{l} e^{il\varepsilon} \right| > \lambda \right\}$$
$$2 \frac{C}{\lambda} \alpha \int |f - f_k| \, d\mu + \frac{C}{\lambda} \left(1 + \int \alpha |f - f_k| \ln^{+2} \alpha |f - f_k| \, d\mu \right)$$

for each $\alpha > 0$ fixed.

Letting k go to infinity, for $\rho = \lambda/\alpha$ we get

$$\lim_k \mu \left\{ x : 2 \sup_n \sup_{\varepsilon} \left| \sum_{l=-n}^{n}{}' \frac{(f - f_k)(\varphi^l x)}{l} e^{il\varepsilon} \right| > \rho \right\} \leq \frac{2C}{\rho\alpha}.$$

Now letting α go to infinity we obtain

$$\lim_k \mu\left\{x : 2\sup_n \sup_\varepsilon \left|\sum_{l=-n}^{n}{}' \frac{(f-f_k)(\varphi^l x)}{l} e^{il\varepsilon}\right| > \rho\right\} = 0.$$

This proves that

$$\left\{x : \sup_\varepsilon \left[\overline{\lim} \sum_{k=-n}^{n}{}' \frac{f(\varphi^k x)}{k} e^{ik\varepsilon} \underline{\lim} \sum_{k=-n}^{n}{}' \frac{f(\varphi^k x)}{k} e^{ik\varepsilon}\right]\right\}$$

is a null set and f satisfies W.W.

(ii) The function $g : f - \int f \, d\mu^{\mathbf{Z}}$ belongs to

$$\begin{aligned} &\overline{(I-\phi)(L^1(\mu^{\mathbf{Z}})} \text{ (closure in } L^1 \text{ of the set } (I-\phi)(L^1(\mu^{\mathbf{Z}}))) \\ &\quad = \{g - g \circ \varphi ; g \in L^1(\mu^{\mathbf{Z}})\}. \end{aligned}$$

As $\overline{(I-\phi)(L^p(\mu^{\mathbf{Z}}))}$ is dense in L^1 norm in $\overline{(I-\phi)(L^1(\mu^{\mathbf{Z}}))}$ we can find functions $g_n \in (I-\phi)(L^p(\mu^{\mathbf{Z}}))$ such that $g - g_n \to$ a.e. As g satisfies W.W., for each n and $\delta > 0$ we can make sense of the following inequality:

$$\begin{aligned} &\sup_{|\varepsilon - \varepsilon'| \le \delta} |H_\varepsilon g(x) - H_{\varepsilon'} g(x)| \\ &\quad \le \sup_{|\varepsilon - \varepsilon'| \le \delta} (|H_\varepsilon(g - g_n(x)| + |H_{\varepsilon'}(g - g_n)(x)| + |(H_\varepsilon - H_{\varepsilon'}) g_n(x)|). \end{aligned}$$

We have

$$|H_\varepsilon(g - g_n(x)| + |H_{\varepsilon'}(g - g_n)(x)| \le 2\sup_m \sup_\varepsilon \left|\sum_{k=-m}^{m}{}' \frac{(g - g_n)(\varphi^k x)}{k} e^{ik\varepsilon}\right|.$$

Using analogous arguments as in (i), we get that off a single null set

$$\lim_n \sup_{j\varepsilon - \varepsilon'| \le \delta} [|H_\varepsilon(g - g_n(x)| + |H_{\varepsilon'}(g - g_n)(x)|] = 0.$$

For the last we use the continuity of $\varepsilon \to H_\varepsilon(g_n)(x)$. Finally we have

$$\lim_{\delta \to 0} \sup_{|\varepsilon - \varepsilon'| \le \delta} |H_\varepsilon g(x) - H_{\varepsilon'} g(x)| \le O(1) + 0$$

which proves (ii).

Remark 9. We do not know if the partial sums of the Fourier series of L Log L functions converge a.e.

PROPOSITION 10. *Let $\mathscr{L}$ be the space L Log L and $([0,1]^{\mathbf{Z}}, B[0,1]^{\mathbf{Z}}), \mu^{\mathbf{Z}}, \varphi)$ the dynamical system where φ is the shift on $[0,1]^{\mathbf{Z}}$. If for $f \in \mathscr{L}$, $g = f - \int f\,d\mu^{\mathbf{Z}}$ satisfies S.W.W. then the partial sums of the Fourier series of functions in $\mathscr{L}$ converges a.e.*

Proof. As in [1], S.W.W. holds for functions $g = f - \int f\,d\mu^{\mathbf{Z}}$ where $f \in \mathscr{L}$ implies that

$$\sup_n \sup_\varepsilon \left| \sum_{k=-n}^{n}{}' \frac{h(\varphi^k x)}{k} e^{ik\varepsilon} \right| < \infty \quad \text{a.e. for all } h \in \mathscr{L}.$$

This follows from the uniform convergence of the "ergodic Féjer sums" and the Wiener-Wintner property for the Cesaro averages. They are related to the helical transform by the formula

$$\sum_{k=-n}^{n}{}' \frac{h(\varphi^k x)}{k} e^{ik\varepsilon} - \sum_{|k| \le n}{}' \left(1 - \frac{|k|}{n+1}\right) \frac{f(\varphi^k x)}{k} e^{ik\varepsilon}$$
$$= \frac{1}{n+1} \sum_{k=-n}^{n}{}' f(\varphi^k x) e^{ik\varepsilon}.$$

On $\mathscr{L} = L$ Log L we can define a norm $\|\ \ \|_*$ where

$$\|f\|_* = \int_0^1 f^*(t) \log\left(\frac{1}{t}\right) dt = \int_0^1 \left(\frac{1}{t} \int_0^t f^*(s)\,ds\right) dt$$

where $f^*(t)$ is the decreasing rearrangement of f defined on $[0, \infty]$. We have

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \le t\},\ t \ge 0 \text{ with } \mu_f(\lambda) = \mu\{|f(x)| > \lambda\}.$$

By Banach's principle there exists a decreasing function $C(\lambda)$ defined on $[0, \infty]$, $C(\lambda) \to 0$, $\lambda \to \infty$, such that

$$\mu^{\mathbf{Z}}\left\{x : \sup_n \sup_\varepsilon \left| \sum_{k=-n}^{n}{}' \frac{f(\varphi^k x)}{k} e^{ik\varepsilon} \right| > \lambda \|f\|_* \right\} \le C(\lambda).$$

The shift γ being aperiodic by the well-known conjugacy lemma, the conjugate $S_q \varphi S_q^{-1}$ of φ are dense in the weak topology. Since $\|f S_q^{-1}\|_* = \|f S_q\|_* = \|f\|_*$, for all dynamical systems on a probability $(\Omega, \alpha, \psi, \nu)$ measure space isomorphic to $([0,1]^{\mathbf{Z}}, B[0,1]^{\mathbf{Z}}), \mu^{\mathbf{Z}})$ this gives

$$\nu\left\{w : \sup_n \sup_\varepsilon \left| \sum_{k=-n}^{n}{}' \frac{f(\varphi^k x)}{k} e^{ik\varepsilon} \right| > \lambda \|f\|_* \right\} \leq C(\lambda).$$

Approximating again a flow by times δ map gives the same type of inequality for any measure preserving flow. In the particular case of the flow of translation on the real line this implies that

$$\nu\left\{x \in [-\pi, \pi] : \sup_n \sup_\varepsilon \left| \int_{1/\eta \leq |t| < \eta} \frac{e^{i\varepsilon t} f(x-t)}{t} \, dt \right| > \lambda \|f\|_* \right\} \leq \tilde{C}'(\lambda)$$

and then

$$\nu\left\{x \in [-\pi, \pi] : \sup_\eta \left| \int \frac{e^{int} f(t)}{x - t} \, dt \right| > \lambda \|f\|_* \right\} \leq \tilde{C}(\lambda).$$

This gives the a.e. convergence of the partial sums $S_n f$.

Remarks 11. (a) Another consequence of Proposition 5 is an improvement of (i) in Theorem 3. We can prove that $H^{**}f \in L^1$ if $f \in L(\mathrm{Log}\, L)^3$.

(b) At the present time we do not know if $H^{**}f < \infty$ if $f \in L \,\mathrm{Log}\, L \,\mathrm{Log}\,\mathrm{Log}\, L$.

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