ILLINOIS JOURNAL OF MATHEMATICS
Volume 38, Number 1, Spring 1994

FEJER THEOREMS ON COMPACT SOLVMANIFOLDS

CAROLYN PFEFFER

Section 1. Introduction

In the theory of Fourier series, it is well known that if $f \in L^2(T)$ is continuous, where T is the unit circle), then the Fourier series

$$g(x) = \sum_{n \in \mathbf{Z}} \hat{f}(n) e^{2\pi i n x}$$

need not converge uniformly or even pointwise to f. However, the Fejer theorem asserts that there exists a set of constants $\{a_{n,k}\}_{n,k=1}^{\infty}$, such that for each fixed n only finitely many k differ from 0, and so that if we define

$$\sigma_n(x) = \sum_{k \in \mathbf{Z}} a_{n,k} \hat{f}(k) e^{2\pi i k x},$$

then $\sigma_n \to f$ uniformly on T.

We note that the map $f \mapsto \hat{f}(n) e^{2\pi i n x}$ is an orthogonal projection onto a subspace of $L^2(T)$ which is translation-invariant; if Λ denotes the quasi-regular representation of $\mathbf{R}$ in $L^2(T)$, then Λ restricted to the subspace $\{\mathbf{C} e^{2\pi i n x}\}$ is equivalent to an irreducible representation of $\mathbf{R}$.

Similarly, if S is a solvable Lie group with cocompact discrete subgroup Γ, the right quasiregular representation decomposes $L^2(S/\Gamma)$ into a countable direct sum of orthogonal irreducible subspaces. Those irreducible representations of S which appear in the decomposition may appear with multiplicity, always finite. Although the decomposition of $L^2(S/\Gamma)$ isn't unique, the direct sum of all irreducible π-spaces is independent of the decomposition; we call it the primary summand of π. We order the primary summands $\{H_n\}$ and let P_n denote orthogonal projection onto the nth primary summand.

In this paper we address the question of whether Fejer theorems exist for the three-dimensional compact solvmanifolds which are quotients of the

Received February 5, 1992.
1991 Mathematics Subject Classification. Primary 43A85; Secondary 43A46.

following solvable Lie groups:

1. S_r is the semidirect product of **R** with $\mathbf{R}^2$. **R** acts on $\mathbf{R}^2$ via the one-parameter subgroup

$$\sigma_h(t) = \begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix}.$$

2. S_h is the semidirect product of **R** with $\mathbf{R}^2$, where **R** acts on $\mathbf{R}^2$ via the one-parameter subgroup

$$\sigma_h(t) = \begin{pmatrix} \lambda^t & 0 \\ 0 & \lambda^{-t} \end{pmatrix},$$

where λ is a positive real number satisfying $\lambda + \lambda^{-1} = 3$.

It is known that these are the only three-dimensional, solvable, non-nilpotent Lie groups with cocompact discrete subgroups (see [AGH], Section 2.2). In addition,

$$N_h = \{(0, x, y) \in S_h\} \quad \text{and} \quad N_r = \{(0, x, y) \in S_r\}$$

form abelian, normal subgroups of S_h and S_r, respectively; we will use the symbol N without the subscript when the context is clear.

The existence of Fejer theorems on compact quotient spaces of nilpotent Lie groups with flat Kirillov orbits has been demonstrated by L. Richardson in [Ri1]; this work builds heavily upon the results of L. Richardson, J. Brezin, and W. Rudin. Richardson (in [Ri2], examples 5.3 and 5.4) used results of Rudin on irreducible idempotent measures (see [Ru], Theorem 3.1.3) to demonstrate that orthogonal projections on π-primary summands of $S_r \setminus \Gamma$ preserve continuity of functions in $C(S_r \setminus \Gamma)$, the space of continuous functions on $S_r \setminus \Gamma$, and that those on primary summands of $S_h \setminus \Gamma$ do not; he also gave a characterization of those compact nilmanifolds for which primary summand projections preserve the continuity of functions [Ri2, Theorem 3.10]. J. Brezin expanded this result to show that primary summand projections in L^2 of a nilmanifold preserve continuity of functions if and only if the associated nilpotent Lie group has flat Kirillov (coadjoint) orbits ([Bre], Theorem 2.5).

In Section 2, we demonstrate the existence of a Fejer theorem on quotients of the form $S_r \setminus \Gamma$, using approximate identities on a fundamental domain of $N_r \setminus N_r \cap \Gamma = T^2$.

Our goal in Section 3 is to prove:

THEOREM 3.1. *Let $S = \sum_{n=1}^{r} \lambda_n P_n$ be an operator, $S : L^2(S_h \setminus \Gamma) \to L^2(S_h \setminus \Gamma)$, which is a linear combination of primary summand operators. Then S maps at least one element of $C(S_h \setminus \Gamma)$ to an essentially unbounded function.*

This theorem shows that the standard type of Fejer theorem does not hold on this solvmanifold, and thus we are led to the following question: is there a sequence of operators of this type such that for each $f \in C(S_h \setminus \Gamma)$, there exists an $n \in N$ such that if $k \geq n$, then we have $S_k f \in L^\infty(S_h \setminus \Gamma)$ (and hence in $C(S_h \setminus \Gamma)$), and $S_k f \to f$ uniformly on $S_h \setminus \Gamma$?

Section 2

In this section, let Γ be a fixed cocompact discrete subgroup of S_r; we assume that S_r is coordinatized so that $\mathbf{R}$ acts on N via the map

$$\sigma(t) = \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix}.$$

Then the points of $N_r \cap \Gamma$ form a nonstandard lattice subgroup of N_r, and the nondegenerate coadjoint orbits of S_r are cylinders centered about the t-axis in $\mathbf{s}_\mathbf{r}^*$, the vector space dual of the Lie algebra of S_r. The infinite-dimensional representations of S_r correspond to these nondegenerate orbits, and those appearing in the spectrum of the quasiregular representation of S_r correspond to orbits which contain "integral points", elements $\lambda \in \mathbf{s}_\mathbf{r}^*$ which satisfy

$$\chi_\lambda(N \cap \Gamma) = e^{2\pi i \lambda(\log(N \cap \Gamma))} = 1.$$

Such representations appear with finite multiplicity in the decomposition of $L^2(S_r \setminus \Gamma)$ into irreducible subspaces. If n is the multiplicity of π, then the sum of any n irreducible independent π-spaces (i.e., invariant subspaces of $L^2(S_h \setminus \Gamma)$ upon which the quasiregular representation is equivalent to π) is called the primary summand of π, and is canonically determined by π. The decomposition of a primary summand into irreducible subspaces is, however, highly nonunique.

Since there are only countably many primary summands, we may order them as $\{H_n\}$.

Orthogonal projection onto the primary summand H_n is given as follows. If the irreducible representation π_n in the spectrum of the right quasiregular representation of S_r corresponds to the coadjoint orbit O_n, and if k is the multiplicity of π_n in the decomposition of the right quasiregular representation of S_r, then O_n contains k Γ-orbits of integral points. Let Ω_n be the intersection of the set of integral points in $\mathbf{R}X^* + \mathbf{R}Y^*$ with O_n. Then the elements of Ω_n can be represented as pairs (N_1, N_2) satisfying $N_1^2 + N_2^2 = \lambda_n^2$, where λ is the radius of O_n. Note that due to our choice of coordinatization for S_r, the (N_1, N_2) may not be pairs of integers.

If $f \in L^2(S_r \setminus \Gamma)$, then for a.e. fixed $t \in [0, 1]$, the function $f_t(x, y) = f(\Gamma(t, x, y))$ is in L^2 of the 2-torus $N/N \cap \Gamma$. We let $\hat{f}_t(N_1, N_2)$ denote the partial Fourier transform of f with respect to x and y, evaluated at the lattice point (N_1, N_2).

We then have

$$P_n(f)(\Gamma(t, x, y)) = \sum_{(N_1, N_2) \in \Omega_n} \hat{f}_t(N_1, N_2) \chi_{(N_1, N_2)}(x, y)$$

for a.e. (t, x, y).

THEOREM 2.1. *Let $P_n : L^2(S_r \setminus \Gamma) \mapsto H_n$ be orthogonal projection onto H_n. Then there exists a sequence $\{S_k\}$ of operators, $S_k = \sum_{n=1}^{r} \alpha_{n,k} P_n$, such that:*

(i) *For each S_k, the number of nonzero $\alpha_{n,k}$ is finite.*
(ii) *S_k maps $C(S_r \setminus \Gamma)$ to $C(S_r \setminus \Gamma)$ for each k.*
(iii) *For all $f \in C(S_r \setminus \Gamma)$, $S_n(f) \to f$ uniformly as $n \to \infty$.*

Proof. Let $\{h_k\}_{k=1}^{\infty}$ be a compactly supported, C^∞, rotation-invariant approximate identity on N. Let D be a fundamental domain of $N/N \cap \Gamma$ containing the identity of N as an interior point; we choose the h_k so that their supports are contained inside D, and so that for each $\varepsilon > 0$ there exists a $k \in \mathbf{N}$ such that if $n > k$, then the support of h_n is contained in an ε-ball around the origin in N.

Since each h_k is C^∞, we have

$$\phi_{r,k}(x_1, x_2) = \sum_{|(N_1, N_2)| < r} \hat{h}_k(N_1, N_2) \exp(2\pi i(N_1 x_1 + N_2 X_2))$$

converging uniformly to h_k as $r \to \infty$. Note also that since h_k is rotation-invariant,

$$\hat{h}_k(N_1, N_2) = \hat{h}_k(M_1, M_2) \quad \text{if} \quad N_1^2 + N_2^2 = M_1^2 + M_2^2.$$

Note also that the sum may be over a nonstandard lattice (i.e., the coordinates of the lattice may not be integral).

Define $\phi_k = \phi_{r_k, k}$ for each k so that

$$\|\phi_k - h_k\| < 1/k$$

is satisfied on $N/N \cap \Gamma$. Now let $f : S_r \setminus \Gamma \mapsto \mathbf{C}$ be a continuous function on $S_r \setminus \Gamma$, so that in particular $f_t(x, y) = f(\Gamma(t, x, y))$ is continuous on the 2-torus $(N/N \cap \Gamma)$.

We define the function

$$f_t * \phi_k(x, y) = \int_{N/N\cap\Gamma} f_t(x - x_0, y - y_0)\phi_k(x_0, y_0)\, dx_0 dy_0$$

where the x_0, y_0 range over a fundamental domain for $N/(N \cap \Gamma)$.

We wish to show that $f_t * \phi_k$ converges uniformly to f_t for each t and, in fact, that

$$f_t * \phi_k \mapsto f_t(x, y) = f(t, x, y)$$

in the sup norm on $S_r \setminus \Gamma$.

First consider $\sup_{\Gamma(t,x,y)\in S_r\setminus\Gamma}|f_t - f_t * h_k|$. This is given by

$$\sup_{\Gamma(t,x,y)\in S_r\setminus\Gamma} \left|\int_{N/N\cap\Gamma} [f_t(x, y) - f_t(x - x_0, y - y_0)]h_k(x_0, y_0)\, dx_0\, dy_0\right|.$$

Since $S_r \setminus \Gamma$ is compact, $f(t, x, y)$ is uniformly continuous on $S_r \setminus \Gamma$. Let k be large enough that

$$|f_t(x, y) - f_t(x - x_0, y - y_0)| < \varepsilon/2$$

for all $(x, y) \in S_r \setminus \Gamma$ and $(x_0, y_0) \in$ support h_k. Then we have that

$$\sup_{\Gamma(t,x,y)} \left|\int_{N/N\cap\Gamma} [f_t(x, y) - f_t(x - x_0, y - y_0)]h_k(x_0, y_0)\, dx_0\, dy_0\right|$$
$$< (\varepsilon/2) \cdot \left|\int_{N/N\cap\Gamma} h_k(x_0, y_0)\, dx_0\, dy_0\right| = \varepsilon/2.$$

On the other hand, we have

$$\sup_{\Gamma(t,x,y)} |f_t * h_k - f_t * \phi_k| \leq \sup_{t\in[0,1]} \|f_t\|_\infty \|h_k - \phi_k\|_1 \leq \sup_{t\in[0,1]} \|f_t\|_\infty \cdot \frac{1}{k},$$

where the second norm is on $N/N \cap \Gamma$.

Thus if k is large enough, this term can also be made less than $\varepsilon/2$, so that $f_t * \phi_k \to f_t$ uniformly in the sup norm on $S_r \setminus \Gamma$, as desired.

Define $\alpha_{\lambda,k} = \hat{h}_k(N_1, N_2)$ if $N_1^2 + N_2^2 = \lambda^2$. This is well defined, since h_k was initially rotation-invariant, and therefore $\hat{h}_k(N_1, N_2) = \hat{h}_k(M_1, M_2)$ if $N_1^2 + N_2^2 = M_1^2 + M_2^2$. Then if the projection P_n corresponds to the coadjoint orbit having radius $\lambda > 0$, the operator S_n as defined in the statement of Theorem 2.1 satisfies

$$S_n f(\Gamma(t,x,y)) = f_t * \phi_k(x,y),$$

which as we have seen converges uniformly on $S_r \setminus \Gamma$ to f. This completes the proof of Theorem 2.1.

Section 3

Let Γ be a fixed cocompact, discrete subgroup of S_h. Our goal in this section is to prove:

THEOREM 3.1. *Let $S : L^2(S_h \setminus \Gamma) \to L^2(S_h \setminus \Gamma)$, $S = \Sigma_{n=1}^{r} \lambda_n P_n$, be the finite sum of projections of $L^2(S_h \setminus \Gamma)$ onto primary summands of $L^2(S_h \setminus \Gamma)$. Then S maps at least one element of $C(S_h \setminus \Gamma)$ to an essentially unbounded function.*

Suppose $S_h = \mathbf{R} \propto N_h$ is coordinatized so that $\mathbf{R}$ acts on N_h via the 1-parameter subgroup

$$\sigma_h(t) = \begin{pmatrix} \lambda^t & 0 \\ 0 & \lambda^{-t} \end{pmatrix}.$$

Then the nondegenerate coadjoint orbits will be "hyperbolic cylinders", saturated in the t-direction, given by the equations $xy = k$, $k \in \mathbf{R}$. Let π be an infinite-dimensional irreducible representation of S_h in the spectrum of the right quasiregular representation of S_h on $L^2(S_h \setminus \Gamma)$, and let

$$P_\pi : L^2(S_h \setminus \Gamma) \mapsto L^2(S_h \setminus \Gamma)$$

be orthogonal projection of L^2 onto the π-primary summand of L^2; P_π does not preserve continuity of functions on $S_h \setminus \Gamma$ [Ri2, Example 5.3]. Let (a,b) be a fixed lattice point in the coadjoint orbit O_π, lying in the plane $\mathbf{R}X^* + \mathbf{R}Y^*$; we note that with the chosen coordinatization of S, the torus $N_h \setminus N_h \cap \Gamma$ will be a nonstandard torus, and so the elements $\lambda = (a,b) \in O_\pi$ satisfying $\chi_\lambda(N_h \cap \Gamma) = 1$ will not have integer coordinates. The set of such elements forms a lattice in the plane which we call L^*. $\Omega_\pi = L^* \cap O_\pi$ consists of finitely many Γ-orbits of integral points, and so the union of these sets for finitely many orbits O_π consists of finitely many Γ-orbits as well.

As described in Section 2, the projection P_n onto the nth primary summand H_n of $S_h \setminus \Gamma$ is given by

$$P_n(f)(\Gamma(t,x,y)) = \sum_{(a,b)\in\Omega_n} \hat{f}_t(a,b)\chi_{(a,b)}(x,y).$$

LEMMA 3.2. *The union of finitely many* Γ-*orbits is a Sidon set* (*see* [Ru], *p.* 127).

LEMMA 3.3. *Let* $S = \sum_{n=1}^{r}\lambda_n P_n$ *be as in the statement of Theorem* 3.1. *Define* U *to be the union of the finitely many sets* Ω_π *which correspond to projections appearing in the sum for* S. *Suppose* $f \in S(L^2(S_h \setminus \Gamma))$, $f \in L^\infty(S_h \setminus \Gamma)$. *Then for almost all fixed* $t = t_0$, *we have*

$$f(\Gamma(t,x,y)) = \sum_{(a,b)\in U} \hat{f}_t(a,b)\chi_{(a,b)}(x,y)$$

absolutely and uniformly convergent to f.

Proof. Follows immediately from Lemma 3.2, together with the definition of a Sidon set.

COROLLARY 3.4. *Suppose* $f \in C(S_h \setminus \Gamma)$. *If* $S(f) \in L^\infty(S_h \setminus \Gamma)$, *then* $S(f)$ *is continuous* (*see* [Pf], *Corollary* 1.3).

Suppose that S is as described in Theorem 3.1, and that S maps $C(S_h \setminus \Gamma)$ into $L^\infty(S_h \setminus \Gamma)$. Then Corollary 3.4 shows that S must map $C(S_h \setminus \Gamma)$ to $C(S_h \setminus \Gamma)$; the Riesz representation theorem, together with the fact that S commutes with the right quasiregular representation of S_h in $L^2(S_h \setminus \Gamma)$, then shows that S is given by convolution with a product measure of δ_0 with a measure τ on the torus T^2 (see [Ri2], Theorem 3.7). The Fourier-Stieltjes transform $\hat{\tau}$ of τ is a function of finite range, supported in U; therefore we have that τ is the linear combination of finitely many measures on T^2 which are irreducible convolution idempotents [Ru, Theorem 3.4.3].

A function ϕ on Z^2 can be the Fourier-Stieltjes transform of a convolution idempotent only if it is the characteristic function of an element of the coset ring of Z^2 [Ru, Theorem 3.1.3]. We use the following lemma, which can be proved easily using induction on the number of functions in the sum.

LEMMA 3.5. *Suppose that the functions* ϕ_n, $n = 1, \ldots, k$, *are characteristic functions of sets in the coset ring of* Z^2. *Then* $\sum_{n=1}^{r}\alpha_n\phi_n$, $\alpha_n \in \mathbf{C}$ *is supported in the coset ring of* Z^2.

Thus if we can show that $\hat{\tau}$ is not supported in the coset ring of Z^2, we have a contradiction, and the proof of Theorem 3.1 is complete. U is the union of sets of elements in Z^2 satisfying a polynomial relation which transforms to $xy = k$, for some $k \in R$, under a linear isomorphism of R^2. However, we have:

LEMMA 3.6. *Let Q be in the coset ring of Z^k. Then the Zariski closure of Q in R^k is a finite union of linear varieties* [*Bre*, *Theorem* 2.3].

Since the Zariski closure of the support of $\hat{\tau}$ is contained in the zero sets of polynomials which transform to the hyperbolae $xy = k$, the conclusion of Lemma 3.6 cannot be satisfied, and we have a contradiction. This completes the proof of Theorem 3.1.

REFERENCES

[Aus-Bre] L. AUSLANDER and J. BREZIN, *Uniform distribution in solvmanifolds*, Adv. in Math. **7** (1971), 111–144.

[AGH] L. AUSLANDER ET. AL., *Flows on homogeneous spaces*, Ann. of Math. Stud., no. 53, Princeton, 1953.

[Bre] J. BREZIN, *Geometry and the method of Kirillov*, Lecture Notes in Mathematics **466**, 1973, p. 13–25.

[Pf] C. PFEFFER, *Primary summand functions on compact solvmanifolds*, Proc. Amer. Math. Soc., **116** (1992), 213–217.

[Ri1] L. RICHARDSON, *N-step nilpotent Lie groups with flat Kirillov orbits*, Colloq. Math., no. 52.2, p. 285–287.

[Ri2] ______, A class of idempotent measures on compact manifolds, **135** (1975), 129–154.

[Ru] W. RUDIN, *Fourier analysis on groups*. Interscience, New York, 1962.

FLORIDA ALTANTIC UNIVERSITY
BOCA RATON, FLORIDA