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RESTRICTIONS OF FOURIER TRANSFORMS TO FLAT CURVES IN $\mathbf{R}^2$

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1. Introduction

Given a smooth (lower-dimensional) submanifold S of $\mathbf{R}^n$ and a smooth compactly supported measure σ on S, one may ask for what values of p and q an *a priori* estimate of the form

$$\|\hat{f}|_S\|_{L^q(\sigma)} \leq C_{p,q}\|f\|_{L^p(\mathbf{R}^n)} \quad \forall f \in \mathscr{S}(\mathbf{R}^n) \tag{1.1}$$

holds, where $\hat{f}|_S$ denotes the restriction of the Fourier transform of f to S, and $\mathscr{S}(\mathbf{R}^n)$ is the Schwartz class of functions. Estimates of this type are known as *restriction* theorems. Note that if $p = 1$ the estimate holds trivially (for any q). On the other hand, if $p = 2$ such an estimate cannot hold, since S has Lebesgue measure zero in $\mathbf{R}^n$. E. M. Stein was the first to observe that a restriction theorem holds for $q = 2$ and some $p > 1$ when S is the n-sphere, or more generally, when an estimate of the form

$$|\hat{\sigma}(\xi)| \leq C(1 + |\xi|)^{-\varepsilon} \ \forall \xi \in \mathbf{R}^n \tag{1.2}$$

holds with some $\varepsilon > 0$ for the Fourier transform of the measure σ on S (see [F], [S]). The estimate (1.2) holds, for instance, if S is of *finite type*, namely each point of S has at most a finite order contact with any hyperplane. Hence it follows that (1.1) holds for all finite type S with $q = 2$ and a nontrivial p, that is, some $p \in (1,2)$. See [S] for more details. Also see [F],[T], [Z], [C], [DM], [So] and further references cited in those works.

On the other hand, it is well known that a nontrivial restriction estimate need not hold if the curvature vanishes to *infinite* order at some point of S in such a way that (1.2) should fail—we will call such S (infinitely) *flat*. (So the surface of a circular cylinder, say, is not flat, since (1.2) holds for it.) For example, if S is the flat curve in $\mathbf{R}^2$ given as the graph of the function $\gamma(t) = e^{-1/t^2}$ near the origin, then a homogeneity argument shows that (1.1) fails for *every* $p > 1$. However, in this paper we show that an analog of (1.1) does hold for a class of strictly convex curves whose curvature vanishes, to infinite (or finite) order, at the origin, where the L^p space on the right side of

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(1.1) is replaced by a suitable Orlicz space L^{Φ}. See Theorem 3.2 for precise statements. For the example $\gamma(t) = e^{-1/t^2}$ mentioned above, our estimate may be written as

(1.3)
$$\left(\int_0^{\delta} |\hat{f}(t, e^{-1/t^2})|^q \, dt\right)^{1/q} \leq C_q + C_q \int_{\mathbf{R}^2} \frac{|f|}{[1 + \log^+(1/|f|)]^{1/2q}} dx,$$

for $q \geq 2$. This estimate is equivalent to the *norm* estimate

$$\left(\int_0^{\delta} |\hat{f}(t, \gamma(t))|^q \, dt\right)^{1/q} \leq C_q \|f\|_{\Phi},$$

where Φ is a Young's function equivalent to the function $t/[1 + \log^+(1/t)]^{1/2q}$ (see Remark 3.4). The estimate (1.3) is optimal for the given values of q (see Proposition 3.13 and Corollary 3.14). Our method of proof is closely related to those in [F] and [Z]. See [W] for an excellent discussion of this problem and other problems in harmonic analysis associated to flat curves and surfaces. See also [BMO] for a recent study of a related problem.

The organization of the paper is as follows: In section 2 we collect some facts and definitions in the Orlicz space theory that will be useful in the paper. In section 3 we state the results and give a few examples. Section 4 contains some lemmas and proofs of the main results.

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2. Orlicz spaces

In this section we recall a few definitions and facts from the Orlicz space theory which will be needed in later sections. For more details see [JT], [R]. We say that the function Φ: $[0, \infty) \to [0, \infty]$ is a Young's function provided that Φ is convex, increasing (= non-decreasing), nontrivial ($0 \not\equiv \Phi(t) \not\equiv \infty$ for $t > 0$), and $\Phi(0) = 0$. Given a Young's function Φ, the Orlicz space $L^{\Phi}(\mathbf{R}^n)$ is the Banach space of (equivalence classes of) measurable functions f such that

$$\int \Phi(|f|/l) \, dx < \infty \text{ for some constant } l > 0.$$

$L^\Phi(\mathbf{R}^n)$ is equipped with the norm

$$(2.1)\quad \|f\|_\Phi = \inf\left\{l > 0 : \int \Phi(|f|/l)\,dx \le 1\right\} \qquad \text{(Luxemburg norm)}.$$

For example if $\Phi(t) = t^p$, when $1 \le p < \infty$, or $\Phi(t) = \lim_{r\to\infty} t^r$, when $p = \infty$, we have $\|f\|_\Phi = \|f\|_p$. The Young's complement of Φ is the Young's function Ψ defined by

$$(2.2)\qquad \Psi(t) = \sup_{s\ge 0}\,[st - \Phi(s)].$$

If Φ is given by $\Phi(t) = \int_0^t \phi(s)\,ds$, where ϕ is a strictly increasing function, then it may be shown that Ψ is given by $\Psi(t) = \int_0^t \phi^{-1}(s)\,ds$.

From (2.2) it follows that

$$st \le \Phi(s) + \Psi(t), \text{ for } s, t \ge 0 \quad \text{(Young's inequality)}.$$

Young's inequality implies Hölder's inequality

$$(2.3)\qquad \int |fg|\,dx \le 2\|f\|_\Phi \cdot \|g\|_\Psi.$$

There is another norm on L^Φ given by

$$(2.4)\qquad \|f\|_\Phi^* \doteq \sup\left|\int fg\,dx\right| \quad \text{(Orlicz norm)},$$

where the supremum is taken over all g such that $\int \Psi(|g|)\,dx \le 1$. The two norms are equivalent:

$$\|f\|_\Phi \le \|f\|_\Phi^* \le 2\|f\|_\Phi.$$

This together with (2.4) and Young's inequality gives

$$(2.5)\qquad \|f\|_\Phi \le 1 + \int \Phi(|f|)\,dx.$$

The following extension of the Hausdorff-Young inequality to Orlicz spaces is in [JT] (see Theorem 3.2 there).

THEOREM 2.6. *Assume that ν is a positive, continuous and strictly increasing function on $(0,\infty)$ such that $\nu(s)/s$ is increasing. Let*

$$m(t) = 1/[\nu^{-1}(1/t)],\; N(t) = \int_0^t \nu(s)\,ds,$$

and

$$M(t) = \int_0^t m(s)\,ds.$$

Then for all $f \in L^M(\mathbf{R}^n)$,

$$\|\hat{f}\|_N \le 2\|f\|_M.$$

3. Statement of results

In this paper we consider the curve $\{(t, \gamma(t)) \colon t \ge 0\}$ in $\mathbf{R}^2$, where γ satisfies the following basic hypotheses.

(3.1) *Basic hypotheses on* γ. We assume that $\gamma\colon [0,\infty) \to [0,\infty)$ is a twice-differentiable, strictly convex function such that $\gamma(0) = \gamma'(0) = 0$ and $\gamma(t)/t^3$ is increasing for $t > 0$.

Notation. Given γ and $1 \le q < \infty$ we define the function γ_q by $\gamma_q(t) = t^{q-1} \cdot \gamma(t^q)$. Note that the convexity of γ implies that of γ_q, hence the inverse function γ_q^{-1} of γ_q is concave. Given a nice function h on $\mathbf{R}^2$ we often abbreviate the norm

$$\left(\int_0^\delta |h(t, \gamma(t))|^q\,dt\right)^{1/q}$$

of the restriction of h to Γ by $\|h\|_{L^q(\Gamma)}$ or $\|h\|_q$, where

$$\Gamma = \Gamma_q \doteq \{(t, \gamma(t)) : 0 \le t \le \delta\}.$$

Also, $\|\cdot\|_{L^{a,b}(\Gamma)} = \|\cdot\|_{a,b}$ will denote the Lorentz norm taken with respect to the measure dt on Γ (see [SW]). If $q \in [1,\infty]$, q' denotes the conjugate exponent of q, that is $1/q + 1/q' = 1$. We let C denote a finite positive conjugate exponent of q, that is $1/q + 1/q' = 1$. We let C denote a finite positive constant which may not be the same at each occurrence. $f \lesssim g$ will mean there exists a constant C such that $f(x) \le Cg(x)$ for all relevant x, and we write $f \approx g$ if $f \lesssim g$ and $g \lesssim f$.

We now state our results in the following three theorems.

THEOREM 3.2. *Suppose that* γ *satisfies* (3.1) *and that* $\gamma''(t)/t$ *is increasing on* $(0,\delta)$ *for some* $\delta > 0$. *Then for* $1 \le q < \infty$ *and* $0 < d < 1$ *there exists a* *constant* $C = C_{d,q}$ *such that for all* $f \in \mathcal{S}(\mathbf{R}^2)$,

$$\left(\int_0^\delta |\hat{f}(t,\gamma(t))|^q \, dt\right)^{1/q} \le C + C\int_{\mathbf{R}^2} |f| \cdot \left[\gamma_q^{-1}(|f|)\right]^d dx. \tag{3.3}$$

Moreover, the estimate (3.3) *fails for every* q *if* $d > 1$.

Remark 3.4. We remark that (3.3) is equivalent to the norm estimate

$$\|\hat{f}\|_{L^q(\Gamma)} \le C\|f\|_\Phi, \tag{3.5}$$

where Φ is the Young's function defined by

$$\Phi(t) = \int_0^t \left[\gamma_q^{-1}(s)\right]^d ds.$$

To see this first observe that the fact that γ_q^{-1} is increasing and concave implies

$$\Phi(t) \approx t\left[\gamma_q^{-1}(t)\right]^d, \tag{3.6}$$

since

$$\Phi(t) \ge \int_{t/2}^t \left[\gamma_q^{-1}(s)\right]^d ds \ge \frac{t}{2}\left[\gamma_q^{-1}(t/2)\right]^d \gtrsim t\left[\gamma_q^{-1}(t)\right]^d.$$

Now from (2.1) and (2.5) it follows that (3.5) is equivalent to

$$\|\hat{f}\|_{L^q(\Gamma)} \le C + C\int_{\mathbf{R}^2} \Phi(|f|)\, dx,$$

which is equivalent to (3.3) because of (3.6). Also note that (3.3) is equivalent to

$$\left(\int_0^\delta |\hat{f}(t,\gamma(t))|^q \, dt\right)^{1/q} \le C + C\int_{\mathbf{R}^2} |f| \cdot \left[\gamma_q^{-1}(|f| \wedge 1)\right]^d dx,$$

which may be obtained by combining (3.3) with the trivial $L^1 - L^\infty$ estimate (see Step 3 of the proof of Theorem 3.2). (Thus, only the values of γ near the origin are relevant for our (*local*) problem.) Here $|f| \wedge 1$ stands for $\min\{|f|, 1\}$. Similar remarks can be made about (3.8) and (3.12) below.

THEOREM 3.7. *Suppose that* γ *satisfies* (3.1) *and that* $\gamma''(t)/e^{-t^{-a}}$ *is increasing on* $(0, \delta)$ *for some constants* $a > 0$ *and* $0 < \delta < 1$. *Then for* $1 < q < \infty$,

$$\|\hat{f}\|_{L^{q,2}(\Gamma)} \leq C + C\int_{\mathbf{R}^2} |f| \cdot \gamma_q^{-1}(|f|)\, dx. \tag{3.8}$$

Note that when $q > 2$ the estimate (3.8) is stronger than (3.3) with $d = 1$, since $L^{p,r} \subset L^{p,s}$ if $r \leq s$ (see p. 192 in [SW]). In particular, (3.3) holds for $q \geq 2$ and $d = 1$ under the hypotheses of Theorem 3.7. See Corollary 3.14 below.

THEOREM 3.9. *Suppose that* γ *satisfies* (3.1). *In addition assume that for some real number* $k \geq 3$,

$$\gamma''(t)/t^{k-2} \text{ is increasing on } (0, \delta) \text{ for some } \delta > 0, \tag{3.10}$$

and for $-1 < a < 1$,

$$\gamma(t)t^a \lesssim \gamma'(t^{(k+a)/(k-1)}) \quad \text{if } 0 \leq t \leq 1. \tag{3.11}$$

Then for $1 < q < \infty$ *and*

$$b = 2 - \frac{4}{(k+1)q + 2},$$

we have

$$\|\hat{f}\|_{L^{q,b}(\Gamma)} \leq C + C\int_{\mathbf{R}^2} |f| \cdot \gamma_q^{-1}(|f|)\, dx. \tag{3.12}$$

In particular, (3.3) *holds when* $q \geq 2k/(k+1)$ *and* $d = 1$.

Theorem 3.2 applies to both flat and nonflat curves, whereas Theorems 3.7 and 3.9 concern the (endpoint) case $d = 1$ for some sufficiently flat curves and nonflat curves, respectively. The fact that the estimates in these theorems are sharp or nearly sharp is expressed in the following proposition (a necessary condition).

PROPOSITION 3.13. *If*

$$\|\hat{f}\|_{L^q(\Gamma)} \leq C\|f\|_\Phi$$

(*or equivalently* $\|\hat{f}\|_{L^b(\Gamma)} \leq C + C\int_{\mathbf{R}^2}\Phi(|f|)\, dx$) *holds for all* $f \in \mathscr{S}(\mathbf{R}^2)$ *for* *some* $1 \le q < \infty$ *and a Young's function* Φ, *then*

$$\Phi(t) \gtrsim t \cdot \gamma_q^{-1}(t) \qquad \text{for } 0 \le t \le 1.$$

If $1 < q < \infty$ *we may replace the norm* $\|\hat{f}\|_{L^q(\Gamma)}$ *above by* $\|\hat{f}\|_{L^{q,\infty}(\Gamma)}$ *and obtain the same conclusion.*

Proof. We use a variant of the infinitesimal homogeneity argument due to A. W. Knapp (see e.g. [T]). Let

$$f_\varepsilon(s,t) = \varepsilon e^{-\pi(\varepsilon s)^2} \cdot \gamma(\varepsilon) e^{-\pi(\gamma(\varepsilon)t)^2} \quad \text{for } \varepsilon > 0 \text{ and } s, t \in \mathbf{R}.$$

Then

$$\hat{f}_\varepsilon(u,v) = e^{-\pi(u/\varepsilon)^2} \cdot e^{-\pi(v/\gamma(\varepsilon))^2}.$$

If $\varepsilon \le \delta$,

$$\begin{aligned} \|\hat{f}_\varepsilon\|_{L^q(\Gamma)} &\gtrsim \|\chi_{[-\varepsilon,\varepsilon]\times[-\gamma(\varepsilon),\gamma(\varepsilon)]}\|_{L^q(\Gamma)} \\ &= \left[\int_0^\delta \chi_{[0,\varepsilon]}\, dt\right]^{1/q} = \varepsilon^{1/q}. \end{aligned}$$

Next we show that $\|f_\varepsilon\|_\Phi \le \varepsilon\gamma(\varepsilon)/[\Phi^{-1}(\varepsilon\gamma(\varepsilon))] \doteq l_\varepsilon$ for ε sufficiently small. (We may assume Φ is strictly increasing near 0, so that Φ^{-1} is defined there.) We have

$$\begin{aligned} \int \Phi\left(\frac{|f_\varepsilon(s,t)|}{l_\varepsilon}\right) ds\, dt &= \int \Phi\left(e^{-\pi(\varepsilon s)^2 - \pi(\gamma(\varepsilon)t)^2} \Phi^{-1}(\varepsilon\gamma(\varepsilon))\right) ds\, dt \\ &\le \int e^{-\pi(\varepsilon s)^2 - \pi(\gamma(\varepsilon)t)^2} \cdot \Phi\left(\Phi^{-1}(\varepsilon\gamma(\varepsilon))\right) ds\, dt \\ &= \int e^{-\pi(\varepsilon s)^2 - \pi(\gamma(\varepsilon)t)^2} \cdot \varepsilon\gamma(\varepsilon)\, ds\, dt \\ &= \int_{\mathbf{R}^2} e^{-\pi(s^2+t^2)}\, ds\, dt = 1. \end{aligned}$$

The last inequality is a consequence of the facts that Φ is convex, $\Phi(0) = 0$, and $e^{-\pi(\varepsilon s)^2 - \pi(\gamma(\varepsilon)t)^2} \le 1$. Therefore, by the definition (2.1),

$$\|f_\varepsilon\|_\Phi \le l_\varepsilon = \frac{\varepsilon\gamma(\varepsilon)}{\Phi^{-1}(\varepsilon\gamma(\varepsilon))}.$$

Thus, the hypothesis $\|\hat{f}\|_{L^q(\Gamma)} \le C\|f\|_\Phi$ $\forall f \in \mathscr{S}(\mathbf{R}^2)$ implies

$$\varepsilon^{1/q} \le C\frac{\varepsilon\gamma(\varepsilon)}{\Phi^{-1}(\varepsilon\gamma(\varepsilon))} \text{ for small } \varepsilon.$$

Putting $\varepsilon = s^q$ and rewriting the inequality gives

$$s\gamma_q(s) \le \Phi\big(C\gamma_q(s)\big) \text{ for small } s.$$

Finally, letting $u = C\gamma_q(s)$ we get

$$\frac{u}{C} \cdot \gamma_q^{-1}\left(\frac{u}{C}\right) \le \Phi(u) \text{ for small } u.$$

This gives

$$\Phi(u) \gtrsim u \cdot \gamma_q^{-1}(u) \text{ for } 0 \le u \le 1,$$

since the concavity of γ_q^{-1} implies that $\gamma_q^{-1}(u/C) \approx \gamma_q^{-1}(u)$. □

It is easy to see that Theorems 3.7, 3.9 and Proposition 3.13 imply the following result.

COROLLARY 3.14. *Suppose that Φ is a Young's function and assume one of the following*:

(1) *The hypotheses of Theorem* 3.7 *hold and* $2 \le q < \infty$;

(2) *The hypotheses of Theorem* 3.9 *hold and* $2k/(k+1) \le q < \infty$.

Then the a priori inequality $\|\hat{f}\|_{L^q(\Gamma)} \le C\|f\|_\Phi$ *holds if and only if* $\Phi(t) \gtrsim t \cdot \gamma_q^{-1}(t)$ *for* $0 \le t \le 1$.

3.15 *Examples*. In the first three of the following examples it is to be understood that $\gamma(0) = 0$ and γ is defined by the given formula on a sufficiently small interval $(0, \delta)$, and extended suitably for $t \ge \delta$.

(1) $\gamma(t) = e^{-t^{-a}}$ with $a > 0$. Then $\gamma_q^{-1}(t) \approx [\gamma^{-1}(t)]^{1/q} = [\log(1/t)]^{-1/qa}$ (for small t). Theorem 3.7 applies here and in this case (3.8) may be rewritten as

$$\|\hat{f}\|_{L^{q,2}(\Gamma)} \le C + C\int_{\mathbf{R}^2} \frac{|f|}{[1 + \log^+(1/|f|)]^{1/qa}}\, dx. \tag{3.16}$$

More generally we may take

$$\gamma(t) = \frac{1}{(\exp)^k(t^{-a})} \text{ with } a > 0 \text{ and } k = 1, 2, \ldots,$$

where $(\exp)^k$ denotes the composition of k copies of $\exp(t) = e^t$. Then a similar conclusion holds:

$$\|\hat{f}\|_{L^{q,2}(\Gamma)} \leq C + C\int_{\mathbf{R}^2} \frac{|f|}{\left[1 + (\log^+)^k(1/|f|)\right]^{1/qa}}\, dx.$$

Here $(\log^+)^k$ is the k-fold iteration of $\log^+$.

(2) $\gamma(t) = e^{-[\log(1/t)]^a}$, $a > 1$. Theorem 3.2 applies. Note that this curve is flat at the origin, but not as flat as the ones in (a).

(3) $\gamma(t) = t^k \cdot [\log(1/t)]^\alpha$ with $k > 3$, $\alpha \in \mathbf{R}$ or $k = 3$, $\alpha \leq 0$. Theorem 3.2 applies here. If $k \geq 3$ and $\alpha \leq 0$, then Theorem 3.9 applies and we get (3.12).

(4) Let $\gamma(t) \approx t^k$ ($k \geq 3$) satisfy, say, the hypotheses of Theorem 3.9. In this case (3.12) reduces to

$$\|\hat{f}\|_{L^{q,b}(\Gamma)} \leq C\|f\|_{L^p(\mathbf{R}^2)} \tag{3.17}$$

with $1 < q < \infty$,

$$b = 2(k+1)q/[(k+1)q+2],$$

and

$$p = (k+1)q/[(k+1)q-1],$$

since $\Phi(t) \approx t \cdot \gamma_q^{-1}(t) \approx t^{(k+1)q/[(k+1)q-1]}$. An application of the Marcinkiewicz interpolation theorem for Lorentz spaces (see [H], [SW]) to (3.17) yields the stronger estimate (for the same p, q)

$$\|\hat{f}\|_{L^{q,p}(\Gamma)} \leq C\|f\|_{L^p(\mathbf{R}^2)}. \tag{3.18}$$

The estimate (3.18) is proved in [So] under a different hypothesis, where it is also proved to be sharp in the sense that the $L^{q,p}$ norm on the left side cannot be replaced by an $L^{q,s}$ norm for any $s < p$. This also shows that at least in some cases (3.3) fails for $d = 1$, although a Lorentz norm version like (3.12) may still be true in all cases.

4. Proofs of Theorems 3.2, 3.7 and 3.9

First we prove some lemmas which will be used in proving the theorems.

LEMMA 4.1. *Suppose that α and β are strictly increasing functions, continuous on $[0, \delta)$ for some $\delta > 0$, differentiable on $(0, \delta)$ and $\alpha(0) = \beta(0) = 0$. If* *the function $\alpha'(t)/\beta'(t)$ is increasing on* $(0, \delta)$, *then*

$$\alpha^{-1}\left(\frac{\alpha(t)-\alpha(s)}{D}\right) \geq \beta^{-1}\left(\frac{\beta(t)-\beta(s)}{D}\right) \tag{4.2}$$

for every $D \geq 1$ *and* $0 < s \leq t < \delta$.

Proof. Fix a number $D \geq 1$ and define

$$F(s,t) = \frac{\alpha(t)-\alpha(s)}{D} - \alpha\left(\beta^{-1}\left(\frac{\beta(t)-\beta(s)}{D}\right)\right)$$

for $0 < s \leq t < \delta$. Then (4.2) is equivalent to $F \geq 0$. Since $F(s,s) = 0$, the conclusion will follow from

$$\frac{\partial F}{\partial t}(s,t) \geq 0 \text{ for } 0 < s \leq t < \delta,$$

that is,

$$\frac{\alpha'(t)}{D} - \alpha'\left(\beta^{-1}\left(\frac{\beta(t)-\beta(s)}{D}\right)\right) \cdot \frac{1}{\beta'\left(\beta^{-1}\left(\frac{\beta(t)-\beta(s)}{D}\right)\right)} \cdot \frac{\beta'(t)}{D} \geq 0,$$

which may be rewritten as

$$\frac{\alpha'(t)}{\beta'(t)} \geq \frac{\alpha'(x)}{\beta'(x)},$$

with

$$x = \beta^{-1}\left(\frac{\beta(t)-\beta(s)}{D}\right).$$

But the last inequality is true, because

$$x = \beta^{-1}\left(\frac{\beta(t)-\beta(s)}{D}\right) \leq \beta^{-1}(\beta(t)-\beta(s)) \leq \beta^{-1}(\beta(t)) = t$$

and $\alpha'(s)/\beta'(s)$ is increasing on $(0, \delta)$. □

LEMMA 4.3. *Assume γ satisfies* (3.1). *For* $1 \leq q < \infty$ *and* $0 < d \leq 1$, *let*

$$\xi(t) = t \cdot (\gamma^{-1}(t))^d \text{ and } \eta(t) = t^{1-1/q}(\xi^{-1}(t))^{1/q}.$$

Then there exists a constant $b > 0$ *such that*

$$t \cdot \left[\gamma_q^{-1}(t)\right]^d \geq \eta^{-1}(t) \text{ for } 0 \leq t \leq b. \tag{4.4}$$

Proof. Recall the notation $\gamma_q(t) = t^{q-1} \cdot \gamma(t^q)$. Clearly the functions γ, ξ, η and their inverses are increasing. Hence, (4.4) is equivalent to

$$\eta\left(t \cdot \left(\gamma_q^{-1}(t)\right)^d\right) \geq t.$$

Letting $u = \gamma_q^{-1}(t)$ and rewriting the inequality we get

$$\xi^{-1}(u^{q-1+d} \cdot \gamma(u^q)) \geq u^{(q-1)(1-d)} \cdot \gamma(u^q).$$

A straightforward calculation shows that the last inequality is equivalent to

$$\gamma(u^q) \geq u^{(q-1)(1-d)} \cdot \gamma(u^q),$$

which holds for $u \leq 1$, since $q \geq 1$ and $d \leq 1$. Hence (4.4) holds with $b = \gamma_q(1)$. □

LEMMA 4.5. *Let* $a, b \in (0, \infty)$ *and* $0 < \delta < 1$. *Then*

$$J \doteq \int_0^\delta \int_s^\delta |\log(1 - e^{(t^{-a} - s^{-a})})|^b \, dt \, ds < \infty.$$

Proof. By making the change of variables $u = t^a - s^a$ in the inner integral,

$$\begin{aligned} J &= \int_0^\delta \int_0^{\delta^a - s^a} |\log(1 - e^{-u(u+s^a)^{-1} \cdot s^{-a}})|^b \cdot (u + s^a)^{1/a - 1} \, du \, ds \\ &= \int_0^\delta \int_0^{s^{2a}} + \int_0^\delta \int_{s^{2a}}^{\delta^a - s^a} \doteq K_1 + K_2. \end{aligned}$$

If $u \leq s^{2a}$, then

$$u(u + s^a)^{-1} \cdot s^{-a} = \frac{u}{s^{2a}} \cdot \frac{1}{1 + u/s^a} \geq \frac{u}{s^{2a}}\left(1 - \frac{u}{s^a}\right);$$

hence

$$\begin{aligned} e^{-u(u+s^a)^{-1} \cdot s^{-a}} &\leq e^{-u \cdot s^{-2a}(1 - u/s^a)} \\ &\leq 1 - \frac{u}{s^{2a}}\left(1 - \frac{u}{s^a}\right) + \frac{1}{2}\left(\frac{u}{s^{2a}}\left(1 - \frac{u}{s^a}\right)\right)^2 \\ &\leq 1 - \frac{1}{2} \cdot \frac{u}{s^{2a}}\left(1 - \frac{u}{s^a}\right) = 1 - \frac{u(s^a - u)}{2s^{3a}}. \end{aligned}$$

Thus,

$$K_1 \le \int_0^\delta \int_0^{s^{2a}} \left[\log\left(\frac{2s^{3a}}{u(s^a - u)}\right)\right]^b \cdot (u + s^a)^{1/a-1}\, du\, ds$$
$$\lesssim \int_0^\delta \int_0^{s^{2a}} \left[\frac{s^{3a}}{u \cdot s^a}\right]^{b\varepsilon} \cdot (u + s^a)^{1/a-1}\, du\, ds.$$

which is easily seen to be finite (for ε small enough).

Next, if $u \in [s^{2a}, \delta^a - s^a]$, then

$$u(u + s^a)^{-1} s^{-a} \ge \frac{s^{2a}}{(s^{2a} + s^a) \cdot s^a} = \frac{1}{1 + s^a} \ge 1 - s^a,$$

so that

$$e^{-u(u+s^a)^{-1} \cdot s^{-a}} \le e^{-(1-s^a)}$$
$$\le 1 - (1 - s^a) + \frac{1}{2}(1 - s^a)^2 = \frac{1}{2} + \frac{1}{2}s^{2a}.$$

Hence,

$$K_2 \le \int_0^\delta \int_{s^{2a}}^{\delta^a - s^a} \left[\log\left(\frac{2}{1 - s^{2a}}\right)\right]^b \cdot (u + s^a)^{1/a-1}\, du\, ds$$
$$\lesssim \int_0^\delta \int_0^{\delta^a} (u + s^a)^{1/a-1}\, du\, ds < \infty. \qquad \square$$

We are now ready to prove the theorems. The proof of Theorem 3.2 is carried out in three steps. (As we remarked already, the fact that (3.3) fails when $d > 1$ follows from Proposition 3.13.) In Step 1 we reduce the problem to that of estimating an integral. In Step 2 we establish the case $q = 1$; and in Step 3 we use (Orlicz space) interpolation to deduce the cases $1 < q < \infty$. The proofs of Theorems 3.7 and 3.9 are based on a similar reduction and slightly more refined estimates which depend on the additional hypotheses.

Proof of Theorem 3.2. *Step* 1 (*reduction of the problem*). By Remark (3.4), to prove (3.3) we may show

$$\|\hat{f}\|_{L^q(\Gamma)} \le C\|f\|_\Phi, \tag{4.6}$$

where $\Phi(t) = \int_0^t \phi(s)\, ds$ and $\phi(s) = [\gamma_q^{-1}(s)]^d$. Let

$$Tg(\xi) = \int_0^\delta e^{2\pi i[\xi_1 t + \xi_2 \gamma(t)]} g(t)\, dt,$$

for $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$ and $g \in C^\infty([0, \delta))$. ($T$ is the adjoint of the restriction operator $Rf \doteq \hat{f}|_\Gamma$.) By dualization using (2.4) and Hölder's inequality (2.3), (4.6) will follow from

$$\|Tg\|_\Psi \le C\|g\|_{q'}, \tag{4.7}$$

where $1/q + 1/q' = 1$ and Ψ is the Young's complement of Φ given by

$$\Psi(t) = \int_0^t \psi(s)\,ds, \psi(s) = \phi^{-1}(s) = \gamma_q(s^{1/d}).$$

Clearly we may also assume that $g \ge 0$. If we put $N(t) = \Psi(\sqrt{t})$, it is immediate from (2.1) that

$$\|Tg\|_\Psi^2 = \|(Tg)^2\|_N.$$

Note that N is a Young's function, since

$$\begin{aligned}\nu(t) \doteq N'(t) = \psi(\sqrt{t})\frac{1}{2\sqrt{t}} &= \frac{\gamma_q(t^{1/2d})}{2\sqrt{t}} \\ &= \frac{1}{2}\left(t^{q/2d}\right)^{1-(1+d)/q} \cdot \gamma(t^{q/2d}),\end{aligned}$$

and so ν is strictly increasing by the assumption that $\gamma(s)/s^3$ is increasing (see (3.1)). Now

$$\begin{aligned}(Tg(\xi))^2 &= \int_0^\delta \int_0^\delta e^{2\pi i[(s+t)\xi_1 + (\gamma(s)+\gamma(t))\xi_2]} g(t)g(s)\,dt\,ds \\ &= 2\int_\Delta e^{2\pi i[(s+t)\xi_1 + (\gamma(s)+\gamma(t))\xi_2]} g(t)g(s)\,dt\,ds.\end{aligned}$$

where $\Delta \doteq \{(s, t): 0 < s < t < \delta\}$. Since the strict convexity of γ implies that the transformation $(s, t) \mapsto (x, y) = (s + t, \gamma(s) + \gamma(t))$ is one-to-one on Δ, we get

$$\int_\Delta = \int_{\mathbf{R}^2} e^{2\pi i(x\xi_1 + y\xi_2)}\tilde{g}(x, y)|J|^{-1}\,dx\,dy = \left(\tilde{g} \cdot |J|^{-1}\right)^\wedge(\xi_1, \xi_2),$$

where $\tilde{g}(x, y) = g(t)g(s)$ and

$$|J| = \left|\frac{\partial(x, y)}{\partial(s, t)}\right| = \gamma'(t) - \gamma'(s).$$

Observe that the function

$$\frac{\nu(t)}{t} = \frac{1}{2}\left(t^{q/2d}\right)^{1-(1+3d)/q} \cdot \gamma\left(t^{q/2d}\right)$$

is increasing, since $\gamma(t)/t^3$ is increasing and $1 - (1 + 3d)/q \geq -3$ if $q \geq 1$ and $d \geq 1$. Therefore, we may apply Theorem 2.6 to obtain

$$\|(Tg)^2\|_N = 2\|\left(\tilde{g} \cdot |J|^{-1}\right)^{\wedge}\|_N \leq 4\|\tilde{g} \cdot |J|^{-1}\|_M,$$

where $M(t) = \int_0^t m(s)\,ds$ and $m(s) = 1/[\nu^{-1}(1/s)]$. Hence, to prove (4.7) it suffices to prove that there exists a constant C such that

$$\int M\left(\tilde{g} \cdot |J|^{-1}\right) dx\,dy \leq C.$$

for all $g \geq 0$ with $\|g\|_{q'} = 1$, because of (2.5) and the linearity of the operator T and the fact that $\|\cdot\|_\Psi$ is a norm. Since ν is increasing, so is m, and hence

$$M(t) \leq t \cdot m(t).$$

So

$$\begin{aligned} \int M\left(\tilde{g} \cdot |J|^{-1}\right) dx\,dy &\leq \int \tilde{g} \cdot |J|^{-1} \cdot m\left(\tilde{g} \cdot |J|^{-1}\right) dx\,dy \\ &= \int_0^\delta \int_s^\delta g(t)g(s) \cdot m\left(g(t)g(s)|J|^{-1}\right) dt\,ds \\ &\doteq K, \end{aligned}$$

by reversing the change of variables made above. It remains to estimate K, assuming $\|g\|_{q'} = 1$.

Step 2 (*proof of the case* $q = 1$ *of* (3.3)). In this case we get

$$\begin{aligned} \nu(t) &= \frac{1}{2}t^{-1/2} \cdot \gamma\left(t^{1/2d}\right) \\ &\leq \frac{1}{2}t^{-1/2} \cdot t^{1/2d} \cdot \gamma'\left(t^{1/2d}\right) \text{ (by the convexity of } \gamma) \\ &\leq \gamma'\left(t^{1/2d}\right) \text{ for } 0 \leq t \leq 1, \text{ if } d < 1. \end{aligned}$$

So

$$\nu^{-1}(t) \geq \left[(\gamma')^{-1}(t)\right]^{2d} \text{ for } t \leq t_0.$$

Hence,

$$m(u) = \frac{1}{\nu^{-1}(1/u)} \leq \left[(\gamma')^{-1}(1/u)\right]^{-2d} \text{ for } u \geq u_0 > 0. \tag{4.8}$$

Since $\|g\|_\infty = \|g\|_{q'} = 1$,

$$K \lesssim \int_0^\delta \int_s^\delta m(|J|^{-1})\, dt\, ds$$
$$\lesssim \int_0^\delta \int_s^\delta \left[(\gamma')^{-1}(\gamma'(t) - \gamma'(s))\right]^{-2d} dt\, ds \text{ by (4.8).}$$

Now we use Lemma 4.1 with $\alpha(t) = \gamma'(t)$ and $\beta(t) = t^2$ and $D = 1$. Note that $\alpha'(t)/\beta'(t) = \gamma''(t)/2t$ in increasing on $(0, \delta)$ by hypothesis. Hence,

$$K \lesssim \int_0^\delta \int_s^\delta \left(\sqrt{t^2 - s^2}\right)^{-2d} dt\, ds < \infty$$

since $d < 1$. This finishes the proof of the case $q = 1$.

Step 3 (*proof of the case* $1 < q < \infty$ *of* (3.3)). Here we use an extension of the Riesz-Thorin interpolation theorem due to M. M. Rao [R]. The case $q = 1$ of (3.3) can be rewritten as

$$\|\hat{f}\|_{L^1(\Gamma)} \leq C\|f\|_{\Phi_1},$$

where Φ_1 is a Young's function with $\Phi_1(t) \leq t \cdot (\gamma^{-1}(t))^d$ (for any fixed $0 < d < 1$). Interpolating with the trivial estimate $\|\hat{f}\|_{L^\infty(\Gamma)} \leq \|f\|_{L^1(\mathbf{R}^2)}$, using Theorem 1 in [R], gives

$$\|\hat{f}\|_{L^q(\Gamma)} \leq C\|f\|_{\Phi_{1/q}}, \tag{4.9}$$

where $\Phi_{1/q}^{-1}(t) = t^{1-1/q}(\Phi_1^{-1}(t))^{1/q}$. Since $\Phi_1(t) \leq t(\gamma^{-1}(t))^d = \xi(t)$, we have

$$\Phi_{1/q}^{-1}(t) = t^{1-1/q}\left(\Phi_1^{-1}(t)\right)^{1/q} \geq t^{1-1/q}\left(\xi_1^{-1}(t)\right)^{1/q} = \eta(t).$$

Hence,

$$\Phi_{1/q}(t) \leq \eta^{-1}(t) \lesssim t\left(\gamma_q^{-1}(t)\right)^d, \qquad 0 \leq t \leq 1, \tag{4.10}$$

because $\eta^{-1}(t) \le t(\gamma_q^{-1}(t))^d$, for $1 \le t \le b$, by Lemma 4.3. Writing $f = f_1 + f_2$, where $f_1 = f \cdot \chi_{\{|f| \le 1\}}$, we obtain

$$\begin{aligned}
\|\hat{f}\|_{L^q(\Gamma)} &\le \|\hat{f}_1\|_{L^q(\Gamma)} + \|\hat{f}_2\|_{L^q(\Gamma)} \\
&\le C\|f_1\|_{\Phi_{1/q}} + C\|f_2\|_1 \qquad \text{by (4.9)} \\
&\le C + C\int \Phi_{1/q}(|f_1|)\,dx + C\|f_2\|_1 \qquad \text{by (2.5)} \\
&\le C + C\int_{\{|f| \le 1\}} |f|\left(\gamma_q^{-1}(|f|)\right)^d dx + C\int_{\{|f|>1\}} |f|\,dx \qquad \text{by (4.10)} \\
&= C + C\int_{\mathbf{R}^2} |f|\left(\gamma_q^{-1}(|f| \wedge 1)\right)^d dx \\
&\le C + C\int_{\mathbf{R}^2} |f|\left(\gamma_q^{-1}(|f|)\right)^d dx.
\end{aligned}$$

This completes the proof of Theorem 3.2. □

Proof of Theorem 3.7. By repeating the arguments in Step 1 of the proof of the previous theorem (and using the same notation) it suffices to prove

$$K \doteq \int_0^\delta \int_s^\delta g(t)g(s) \cdot m\left(g(t)g(s) \cdot |J|^{-1}\right) dt\,ds \le C.$$

assuming $\|g\|_{q',2} = 1$ and $g \ge 0$. Since $d = 1$ here, we have $\nu(t) = \gamma_q(\sqrt{t})/[2\sqrt{t}]$. So

$$\nu(t) \le \gamma'(t^{q/2}),\ t \le 1.$$

Hence,

$$m(u) = \frac{1}{\nu^{-1}(1/u)} \le \left((\gamma')^{-1}(1/u)\right)^{-2/q},\ u \ge u_0 > 0.$$

Now

$$K \le \int_{B_1} + \int_{B_2} \doteq K_1 + K_2,$$

where

$$B_1 = \{(s,t) \in \Delta\colon g(t)g(s) \ge 1\}$$

and

$$B_2 = \Delta \setminus B_1 = \{(s,t) \in \Delta : 1 > g(t)g(s) \geq 0\}.$$

As in the proof of Theorem 3.2,

$$K_2 \leq \int_0^\delta \int_s^\delta m(|J|^{-1})\, dt\, ds \leq \int_0^\delta \int_s^\delta \left[(\gamma')^{-1}(\gamma'(t) - \gamma'(s))\right]^{-2/q} dt\, ds$$
$$\leq \int_0^\delta \int_s^\delta \left(\sqrt{t^2 - s^2}\right)^{-2/q} dt\, ds < \infty \quad \text{since } q > 1.$$

Also,

$$K_1 \leq \int_{B_1} g(t)g(s)\left[(\gamma')^{-1}\left(\frac{\gamma'(t) - \gamma'(s)}{g(t)g(s)}\right)\right]^{-2/q} dt\, ds$$
$$\leq \int_{B_1} g(t)g(s)\left[\beta^{-1}\left(\frac{\beta(t) - \beta(s)}{g(t)g(s)}\right)\right]^{-2/q} dt\, ds$$
$$\leq \int_{B_1} g(t)g(s)\left[\log\left(\frac{g(t)g(s)}{e^{-t^{-a}} - e^{-s^{-a}}}\right)\right]^{2/qa} dt\, ds$$

by applying Lemma 4.1 with $\alpha(t) = \gamma'(t)$ and $\beta(t) = e^{-t^{-a}}$. Thus,

$$K_1 \lesssim \int_{B_1} g(t)g(s)[\log(g(t)g(s))]^{2/qa}\, dt\, ds$$
$$+ \int_\Delta g(t)g(s)(t^{-a})^{2/qa}\, dt\, ds$$
$$+ \int_\Delta g(t)g(s)|\log(1 - e^{t^{-a} - s^{-a}})|^{2/qa}\, dt\, ds$$
$$\doteq K_3 + K_4 + K_5.$$

By Hölder's inequality for Lorentz spaces (Theorem 3.4 in [O]),

$$K_3 \lesssim \int_{B_1} g(t)g(s)[g(t)g(s)]^{2\varepsilon/qa}\, dt\, ds$$
$$\leq \left(\int_0^\delta g(t)^{1+2\varepsilon/qa}\, dt\right) = \|g\|_\rho^c$$
$$\leq C\|g\|_{q',\infty}^c \cdot \|1\|_{r,\rho}^c$$
$$\leq C\|g\|_{q',2}^c = C,$$

where $\rho = 1 + 2\varepsilon/(qa)$ and $\varepsilon > 0$ is chosen so small that $\rho < q'$, and r is chosen so that $1/\rho = 1/q' + 1/r$. To estimate K_4 we use Lorentz norm inequalities of R. O'Neil [O], following [So].

$$\begin{aligned} K_4 &= \int_\Delta g(t)g(s)\cdot t^{-2/q}\,dt\,ds \\ &\le \int_\Delta g(t)g(s)\cdot(t-s)^{-1/q}\cdot t^{-1/2q}\cdot s^{-1/2q}\,dt\,ds \\ &= \int_0^\delta g(s)s^{-1/2q}\cdot\int_s^\delta g(t)t^{-1/2q}\cdot(t-s)^{-1/q}\,dt\,ds \end{aligned}$$

$$\begin{aligned} (4.11)\qquad &\le \|g(s)\cdot s^{-1/2q}\|_{L^{(2q)',2}(ds)} \\ &\quad\cdot\left\|\int_s^\delta g(t)t^{-1/2q}(t-s)^{-1/q}\,dt\right\|_{L^{2q,2}(ds)} \end{aligned}$$

$$(4.12)\qquad \le C\|g(t)\cdot t^{-1/2q}\|^2_{(2q)',2}$$

$$(4.13)\qquad \le C\|g\|^2_{q',2}\cdot\|t^{-1/2q}\|^2_{2q,\infty}$$

$$\le C\|g\|^2_{q',2} = C.$$

The inequalities (4.11), (4.12), and (4.13) are consequences of Theorems 3.5, 2.6, and 3.4 in [O], respectively. Finally, by Hölder's inequality and Lemma 4.5, if $1 < \rho < q'$ and $1/\rho = 1/q' + 1/r$, then

$$\begin{aligned} K_5 &\le \left(\int_\Delta (g(t)g(s))^\rho\,dt\,ds\right)^{1/\rho}\cdot\left(\int_\Delta |\log(1-e^{t^{-a}-s^{-a}})|^{2\rho'/qa}\,dt\,ds\right)^{1/\rho'} \\ &\le C\|g\|^2_\rho \le C\|g\|^2_{q',\infty}\cdot\|1\|^2_{r,\rho} \le C\|g\|^2_{q',2} = C. \ \square \end{aligned}$$

Proof of Theorem 3.9. As in the proof of Theorem 3.7, it is enough to prove

$$K \doteq \int_0^\delta\int_s^\delta g(t)g(s)\cdot m\left(g(t)g(s)|J|^{-1}\right)dt\,ds \le C,$$

assuming $\|g\|_{q',b'} = 1$ with $1 < q < \infty$ and $b' = 2(k+1)q/[(k+1)q-2]$. Now

$$\begin{aligned} \nu(t) &= \frac{\gamma_q(\sqrt{t})}{2\sqrt{t}} = \frac{1}{2}(t^{q/2})^{1-2/q}\cdot\gamma(t^{q/2}) \\ &\le C\gamma'\left((t^{q/2})^{(k+1-2/q)/(k-1)}\right),\quad t\le 1, \end{aligned}$$

by the hypothesis (3.11) with $a = 1 - 2/q$. So

$$m(u) = \frac{1}{\nu^{-1}\left(\frac{1}{u}\right)} \leq \left[(\gamma')^{-1}\left(\frac{1}{Cu}\right)\right]^{-2(k-1)/[q(k+1-2/q)]}$$
$$\lesssim \left[(\gamma')^{-1}\left(\frac{1}{u}\right)\right]^{-2(k-1)/[(k+1)q-2]}, \quad u \geq u_0 > 0.$$

Hence

$$K \lesssim \int_{B_1} g(t)g(s) \cdot \left[(\gamma')^{-1}\left(\frac{\gamma'(t) - \gamma'(s)}{g(t)g(s)}\right)\right]^{-2(k-1)/[(k+1)q-2]} dt\, ds$$
$$+ \int_{B_2} \left[(\gamma')^{-1}(\gamma'(t) - \gamma'(s))\right]^{-2(k-1)/[(k+1)q-2]} dt\, ds$$
$$\doteq K_1 + K_2,$$

where B_1 and B_2 are as in the proof of Theorem 3.7. By (3.10) and Lemma 4.1 with $\alpha(t) = \gamma'(t)$ and $\beta(t) = t^{k-1}$, we obtain

$$K_1 \leq \int_{B_1} g(t)g(s)\left(\frac{t^{k-1} - s^{k-1}}{g(t)g(s)}\right)^{[1/(k-1)][-2(k-1)]/[(k+1)q-2]} dt\, ds.$$

Since

$$t^{k-1} - s^{k-1} \geq (t - s) \cdot t^{(k-2)/2} \cdot s^{(k-2)/2} > 0$$

on $\Delta = \{(s,t)\colon 0 < s < t < \delta\}$, we have

$$K_1 \leq \int_0^\delta g(s)^{\rho_1} \cdot s^{-\rho_2} \int_s^\delta g(t)^{\rho_1} \cdot t^{-\rho_2} \cdot (t - s)^{-\rho_3}\, dt\, ds,$$

where $\rho_1 = (k+1)q/[(k+1)q - 2]$, $\rho_2 = (k-2)/[(k+1)q - 2]$, and $\rho_3 = 2/[(k+1)q - 2]$. By Theorems 3.5 and 2.6 in [O],

$$K_1 \leq \|g(s)^{\rho_1} \cdot s^{-\rho_2}\|_{L^{r,2}(ds)} \cdot \left\|\int_s^\delta g(t)^{\rho_1} t^{-\rho_2} \cdot (t - s)^{-\rho_3}\, dt\right\|_{L^{r',2}(ds)}$$
$$\leq C\|g(s)^{\rho_1} \cdot s^{-\rho_2}\|_{r,2} \cdot \|g(t)^{\rho_1} \cdot t^{-\rho_2}\|_{r,2} \cdot \|t^{-\rho_3}\|_{1/\rho_3,\infty}$$
$$\leq C\|g(s)^{\rho_1} \cdot s^{-\rho_2}\|_{r,2}^2,$$

if $1/r' = 1/r + \rho_3 - 1$, i.e., $r = [(k+1)q - 2]/[(k+1)q - 3]$. Finally, by

Theorem 3.4 in [O],

$$\begin{aligned}\|g(s)^{\rho_1}\cdot s^{-\rho_2}\|_{r,2} &\le C\|g(s)^{\rho_1}\|_{q'/\rho_1,2}\cdot\|s^{-\rho_2}\|_{1/\rho_2,\infty}\\ &\le C\|g\|^{\rho_1}_{q',2\rho_1} = C\|g\|^{\rho_1}_{q',b'} = C,\end{aligned}$$

since $1/r = \rho_1/q' + \rho_2 < 1$ and $b' = 2\rho_1$. So $K_1 \le C$. The estimation of K_2 is similar. □

References

[B] J.G. Bak, *Restrictions of Fourier transforms of flat curves and surfaces*, Ph.D. Thesis, University of Wisconsin-Madison, 1988.

[BMO] J.-G. Bak, J. D. McMichael, and D. M. Oberlin, *Convolution estimates for some measures on flat curves*, J. Funct. Anal. **101** (1991), 81–96.

[C] M. Christ, *On the restriction of the Fourier transform to curves: endpoint results and the degenerate case*, Trans. Amer. Math. Soc. **287** (1985), 223–238.

[DM] S. W. Drury and B. P. Marshall, *Fourier restriction theorems for degenerate curves*, Math. Proc. Camb. Philos. Soc. **101** (1987), 541–553.

[F] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. **124** (1970), 9–36.

[H] R. A. Hunt, *On $L(p,q)$ spaces*, Enseign. Math. **12** (1966), 249–275.

[JT] M. Jodeit and A. Torchinsky, *Inequalities for Fourier transforms*, Studia Math. **37** (1971), 245–276.

[O] R. O'Neil, *Convolution operators and $L(p,q)$ spaces*, Duke Math. J. **30** (1963), 129–142.

[R] M. M. Rao, *Interpolation, ergodicity and martingales*, J. Math. Mech. **16** (1966), 543–567.

[So] C. Sogge, *A sharp restriction theorem for degenerate curves in* $\mathbf{R}^2$, Amer. J. Math. **109** (1987), 223–228.

[S] E. M. Stein, *Oscillatory integrals in Fourier analysis*, Beijing Lectures in Harmonic Analysis, Ann. of Math. Studies no. 112, Princeton, Univ. Press, Princeton, N.J., 1986, pp. 307–355.

[SW] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, New Jersey, 1971.

[T] P. Tomas, *Restriction theorems for the Fourier transform*, Proceedings of Symposia in Pure Mathematics, vol. 35, part 1, Amer. Math. Soc., Providence, R.I., 1979, pp. 111–114.

[W] S. Wainger, *Problems in harmonic analysis related to curves and surfaces with infinitely flat points*, Lecture Notes in Math. no 1384, Springer-Verlag, New York, 1989, pp. 91–110.

[Z] A. Zygmund, *On Fourier coefficients and transforms of two variables*, Studia Math. **50** (1974), 189–201.

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