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RELATIVE SPECTRA IN COMPLETE LMC-ALGEBRAS WITH APPLICATIONS

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1. Notation

Let $\mathscr{A}$ be a complex (linear associative) algebra with topology induced by a separating directed set of algebra seminorms $(\|\cdot\|_i)_{i\in I}$. Such an algebra is known as a locally multiplicatively-convex (lmc) algebra. We assume throughout that $\mathscr{A}$ is unital and complete in the sense that every net that is Cauchy in each seminorm converges.

This definition of lmc-algebra is equivalent to the existence of a local base at 0 consisting of convex idempotent sets [12]. However, we are able, in general, to replace the algebra seminorms with equivalent algebra seminorms and still have the same topology on $\mathscr{A}$. This technique proves to be useful in several examples below.

For each $i \in I$, the set

$$\mathscr{N}_i = \{a \in \mathscr{A} : \|a\|_i = 0\}$$

is seen to be an ideal of $\mathscr{A}$. $\|\cdot\|_i$ induces an algebra norm on $\mathscr{A}/\mathscr{N}_i$. Let $\mathscr{A}_i$ be the completion of the normed algebra $\mathscr{A}/\mathscr{N}_i$.

$\mathscr{A}_i$ is called the factor algebra associated with $\|\cdot\|_i$. We write π_i for the canonical homomorphism from $\mathscr{A}$ into $\mathscr{A}_i$.

We will make extensive use of the subalgebra of so-called "seminorm-bounded" elements:

$$\mathscr{B} = \left\{ a \in \mathscr{A} \mid \|a\|_{\mathscr{B}} \overset{\text{def}}{=} \sup_{i\in I} \|a\|_i < \infty \right\}$$

We use the terms "seminorm-bounded" and "bounded" synonymously.

Let $\mathscr{D}$ be a Banach algebra continuously embedded into $\mathscr{B}$. We assume $1_{\mathscr{A}} \in \mathscr{D}$. We are primarily interested in investigating when an element of $\mathscr{A}$ admits an inverse in $\mathscr{D}$. To this end, let $\mathrm{Inv}(\mathscr{A})$ denote the group of invertibles of $\mathscr{A}$ while $\mathrm{Inv}_{\mathscr{B}}(\mathscr{A})$ denotes those elements of $\mathrm{Inv}(\mathscr{A})$ that have

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inverses in $\mathscr{B}$. Similarly, $\mathrm{Inv}_{\mathscr{D}}(\mathscr{A})$ is the set of elements of $\mathrm{Inv}(\mathscr{A})$ with inverses in $\mathscr{D}$.

For $a \in \mathscr{A}$ define the spectrum relative to the subalgebra of bounded elements as

$$\alpha_{\mathscr{B}}(a) = \{\lambda \in \mathbb{C} | (\lambda - a) \notin \mathrm{Inv}_{\mathscr{B}}(\mathscr{A})\}.$$

The set $\mathbb{C} \setminus \alpha_{\mathscr{B}}(a)$ will be denoted $\rho_{\mathscr{B}}(a)$. Similarly define $\alpha_{\mathscr{D}}(a)$ and $\rho_{\mathscr{D}}(a)$.

Many of our results involve bounded linear operators on some Banach space. For such an operator a, let $\|a\|_{\mathrm{op}}$ be its operator norm and $\sigma_{\mathrm{op}}(a)$ its spectrum as a bounded operator.

2. Structure of lmc-algebras

As noted above, the topology on $\mathscr{A}$ in general can be induced by different collections of seminorms. Therefore, the collection of factor algebras for a lmc-algebra need not be unique. Under our assumption of completeness, however, $\mathscr{A}$ can be characterized by any collection of factor algebras. Recall that a directed set of topological algebras $(\mathscr{A}_i)_{i \in I}$ together with a collection of connecting continuous homomorphisms $(\pi_{i,j})_{i > j \in I}$, $\pi_{i,j} : \mathscr{A}_i \to \mathscr{A}_j$, is called a projective system if for $i, j, k \in I$ with $i > j > k$, $\pi_{j,k} \circ \pi_{i,j} = \pi_{i,k}$. Endow $\prod_{i \in I} \mathscr{A}_i$ with the product topology and coordinate-wise operations. The subalgebra

$$\varprojlim \mathscr{A}_i = \{(a_i)_{i \in I} | \pi_{i,j}(a_i) = a_j \text{ when } i > j\}$$

is called the projective (or inverse) limit of the $\mathscr{A}_i$.

Any collection of factor algebras for $\mathscr{A}$ with canonical homomorphisms forms a projective system. It is this observation that allows the characterization of complete lmc-algebras in terms of any collection of factor algebras.

THEOREM 2.1 [12]. *$\mathscr{A}$ is the projective limit of its factor algebras.*

As spectral theory in Banach algebras has been extensively researched, this theorem is the principal tool for developing much of the spectral theory of complete lmc-algebras. As a simple example we have the so-called Arens Invertibility Criterion:

COROLLARY 2.2. *If $\mathscr{A}_i$ is a collection of factor algebras of $\mathscr{A}$, then an element $a \in \mathscr{A}$ is invertible if and only if $\pi_i(a)$ is invertible in $\mathscr{A}_i$ for each $i \in I$.*

Although $\mathscr{B}$ can vary depending upon the choice of algebra seminorms, there is a restriction on those elements that can be bounded for some choice of seminorms. Surprisingly, this restriction depends only upon the algebraic structure of $\mathscr{A}$.

THEOREM 2.3. *Let $a \in \mathscr{A}$ such that $\sigma_{\mathscr{A}}(a)$ is bounded. Then there exits a collection of seminorms on $\mathscr{A}$ defining the same topology so that $a \in \mathscr{B}$.*

Proof. We employ the method in [8] used to show that for each element of a Banach algebra, the algebra can be renormed with an equivalent norm so that the spectral radius of the element is the same as the numerical radius. It suffices to consider a such that the spectrum of a is contained in the open unit disk about 0. Then, by the spectral radius formula in a Banach algebra [7], for each $i \in I$,

$$\lim_{n \longrightarrow \infty} \|a^n\|_i^{1/n} < 1.$$

This gives us for each $i \in I$, $\sup_{n \geq 0} \|a^n\|_i = M_i < \infty$. For $b \in \mathscr{A}$, set

$$q_i(b) = \sup_{n \geq 0} \|a^n b\|_i.$$

Then q_i is an algebra seminorm and is in fact equivalent to $\| \cdot \|_i$ since

$$\|b\|_i < q_i(b) \leq M_i \|b\|_i.$$

Now, for each $i \in I$, set

$$|b|_i = \sup\{q_i(bc) | c \in \mathscr{A}, q_i(c) \leq 1\}.$$

Then $| \cdot |_i$ is also an algebra seminorm. Further,

$$\begin{aligned} |b|_i &= \sup\{q_i(bc) | c \in \mathscr{A}, q_i(c) \leq 1\} \\ &\leq \sup\{q_i(b) q_i(c) | c \in \mathscr{A}, q_i(c) \leq 1\} \leq q_i(b) \leq M_i \|b\|_i. \end{aligned}$$

On the other hand, since $q_i(1_{\mathscr{A}}) = \sup_{n \geq 0} \|a^n\|_i = M_i$, we have

$$|b_i| \geq q_i\left(b \frac{1}{M_i}\right) = \frac{1}{M_i} q_i(b) \geq \frac{1}{M_i} \|b\|_i.$$

Therefore, $| \cdot |$ is equivalent to $\| \cdot \|_i$. Since $|a|_i \leq 1$ for each $i \in I$, a is bounded. ■

3. Examples

Our main example is actually a model for the remainder of the examples we exhibit. Examples 3.3 and 3.4 are in fact F-algebras (that is, complete under an invariant metric).

Example 3.1. Let $(\mathscr{X}_i)_{i\in I}$ be a projective system of Banach spaces such that each connecting homomorphism, $\pi_{i,j}$, is surjective. Let $\mathscr{X}$ be the projective limit of this system, $\mathscr{X}_b$ the collection of seminorm-bounded elements of $\mathscr{X}$. For each $i \in I$, let $\mathscr{A}_i$ be the algebra

$$\mathscr{A}_i = \left\{T_i \in B(\mathscr{X}_i)\,|\,\Pi_{i,j}T_i \stackrel{\text{def}}{=} \pi_{i,j}T_i\pi_{i,j}^{-1} \text{ is well-defined on } \mathscr{X}_j \text{ for } i > j\right\}.$$

The condition of well-definedness is equivalent to the existence of a linear map T_j such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{X}_i & \xrightarrow{\pi_{i,j}} & \mathscr{X}_j \\ \downarrow{T_i} & & \downarrow{T_j} \\ \mathscr{X}_i & \xrightarrow{\pi_{i,j}} & \mathscr{X}_j \end{array}$$

For each $i > j$, $\pi_{i,j}$ is a continuous surjective homomorphism between Banach algebras; the open mapping theorem guarantees it is open. Thus, T_j is continuous. That is, if T_i is in $\mathscr{A}_i$, then the operator $\pi_{i,j}T_i$ is bounded on $\mathscr{X}_j$.

As $\pi_{i,j}$ is a surjection, applying the open mapping theorem again yields $\delta > 0$ so that for each $y \in \mathscr{X}_j$ with $\|y\|_j < 1$ there is an $x \in \mathscr{X}_i$ with $\pi_{i,j}(x) = y$ and $\|x\|_i < \delta$. So, if $T_i^{(n)} \in \mathscr{A}_i$ converges to $T_i \in \mathscr{A}_i$, then for $y \in \mathscr{X}_j$ as above,

$$\left\|\left(\Pi_{i,j}T_i^{(n)} - \Pi_{i,j}T_i\right)y\right\| = \left\|\left(\pi_{i,j}T_i^{(n)} - \pi_{i,j}T_i\right)\left(\pi_{i,j}^{-1}y\right)\right\| \le \|\pi_{i,j}\|\left\|T_i^{(n)} - T_i\right\|\delta.$$

Thus, $\Pi_{i,j}T_i^{(n)}$ converges to $\Pi_{i,j}T_i$ in norm and the map $\Pi_{i,j}$ is continuous. Therefore, $(\mathscr{A}_i)_{i\in I}$ is a projective system of Banach algebras. Let $\mathscr{A} = \varprojlim \mathscr{A}_i$.

There is a natural way to view $\mathscr{A}$ as a collection of linear maps. More precisely, $\mathscr{A}$ can be thought of as

$$\{T : \mathscr{X} \longrightarrow \mathscr{X}\,|\,T \text{ is linear and continuous and } \pi_i T \pi_i^{-1} \text{ is well defined for all } i\}$$

For $x \in \mathscr{X}$, let Tx be the element of X given by the condition $\pi_i(Tx) = \Pi_i T(\pi_i x)$. Let $\mathscr{B}$ again be the bounded elements of $\mathscr{A}$. Let $p_i = \pi_i|_{\mathscr{X}_b}$.

PROPOSITION 3.2. (1) *If* $T \in \mathscr{B}$ *then* $T \in B(\mathscr{X}_b)$.

(2) *If in addition, for all* $i \in I$ *and all* $x_i \in \mathscr{X}_i$, *there exists* $x \in \mathscr{X}_b$ *with* $\pi_i(x) = x_i$ *and* $\|x\|_b = \|x_i\|_i$, *then for* $T \in B(\mathscr{X}_b)$ *with the property that* $p_i T p_i^{-1}$ *is well defined on* $\mathscr{X}_i$, T *extends to* $\tilde{T}$ *in* $\mathscr{A}$ *and, moreover,* $\tilde{T} \in \mathscr{B}$ *with* $\|T\|_{\text{op}} = \|\tilde{T}\|_{\mathscr{B}}$.

Proof. (a) For $x \in \mathscr{X}_b$, Tx is in $\mathscr{X}_b$ since

$$\begin{aligned}\sup_i \|Tx\|_i &= \sup_i \|\pi_i(Tx)\|_i = \sup_i \left\|\pi_i T \pi_i^{-1}(\pi_i x)\right\|_i \\ &= \sup_i \left\|(\Pi_i T)(\pi_i x)\right\|_i \leq \sup_i (\|\Pi_i T\| \cdot \|\pi_i x\|_i) \leq \|T\|_{\mathscr{B}} \cdot \|x\|_{\mathscr{X}_b}.\end{aligned}$$

This same calculation shows that $T \in B(\mathscr{X})$ with $\|T\|_{\text{op}} \leq \|T\|_{\mathscr{B}}$.

(b) We first show that T extends to a linear operator on $\mathscr{X}$. For $x \in \mathscr{X}$ let $\tilde{T}x$ be the unique element of $\mathscr{X} = \varprojlim \mathscr{X}_i$ such that $(\tilde{T}x)_i = (p_i T p_i^{-1})(\pi_i x)$. Since

$$\pi_i \tilde{T} \pi_i^{-1}(x_i) = p_i T p_i^{-1}\big(\pi_i(\pi_i^{-1}(x_i))\big) = p_i T p_i^{-1}(x_i)$$

and $p_i T p_i^{-1}$ is well defined, so is $\pi_i \tilde{T} \pi_i^{-1}$.

$\mathscr{A}$ has the subspace topology inherited from $\prod_{i \in J} \mathscr{X}_i$, so π_i is a continuous open map. Therefore, by the continuity of T, $\pi_i \tilde{T} \pi_i^{-1}$ is continuous, that is, $\tilde{T} \in \mathscr{A}$. By the main assumption in (b), for each $x_i \in \mathscr{X}_i$ with $\|x_i\|_i = 1$, there is an $x \in \mathscr{X}_b$ such that $\|x\|_{\mathscr{X}_b} = 1$. Therefore

$$\begin{aligned}\sup_{\substack{x \in \mathscr{X}_b \\ \|x\|_{\mathscr{X}_b} = 1}} \|Tx\|_{\mathscr{X}_b} &= \sup_{\substack{x \in \mathscr{X}_b \\ \|x\|_{\mathscr{X}_b} = 1}} \sup_i \|Tx\|_i \geq \sup_i \sup_{\substack{x \in \mathscr{X}_i \\ \|x\|_i = 1}} \|Tx\|_i \\ &= \sup_i \|T\|_i = \|T\|_{\mathscr{B}}\end{aligned}$$

and we have $\|T\|_{\mathscr{B}} = \|T\|_{\text{op}}$. ∎

Example 3.3. Let $1 \leq p \leq \infty$. For each $n \geq 1$, let $\mathscr{X}_n = \mathbb{C}^n$ with p-norm: $\|x\|_p = (\sum_{k=1}^n |x_k|^p)^{1/p}$ for $p < \infty$, and $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$. As this norm induces the topology of coordinatewise convergence, $\mathscr{X} = \varprojlim \mathscr{X}_n$ is topologically algebraically isomorphic to the collection of all complex sequences with the topology of coordinatewise convergence. The collection of bounded elements of this locally-convex space is isomorphic to $l^p(\mathbb{N})$. Using the construction in Example 3.1, the algebra $\mathscr{A}_n$ consists of all $n \times n$ lower-triangular matrices and $\|T\|_n = \sup\{\|Tx\|_p \,|\, x \in \mathscr{X}_n, \|x\|_p \leq 1\}$. Of course, the topology induced by $\|\cdot\|_n$ on $\mathscr{A}_n$ is equivalent to the topology of entrywise convergence. Thus $\mathscr{A}$, the inverse limit of the $\mathscr{A}_n$, is isomorphic to the algebra of all lower-triangular matrices with entrywise convergence. This algebra has the property in Proposition 3.2 (b) and therefore the collection of bounded elements of $\mathscr{A}$ is the Banach algebra of lower triangular matrices that induce bounded operators on $l^p(\mathbb{N})$. Finally, for $T \in \mathscr{A}$ we have $\pi_n(T)$ is the $n \times n$ lower-triangular matrix in the upper-left corner of T. This example illustrates the importance of the choice of seminorms for the algebra.

Example 3.4. Again choose $1 \leq p \leq \infty$. For each $t \geq 0$, let $\mathscr{X}_t = L^p[0, t)$. For $f \in \mathscr{X}_t$ and $s < t$, let $\pi_{t,s} f = f|_{[0,s)}$, the restriction of f to $[0, s)$. Then $(\mathscr{X}_t)_{t>0}$ is a projective system of Banach spaces. The inverse limit $\mathscr{X}$ is isomorphic to $L^p_{\text{loc}}([0,\infty))$, the collection of Lebesgue-measurable functions such that

$$\|f\|_K \stackrel{\text{def}}{=} \left(\int_K |f(t)|^p \, dt\right)^{1/p} < \infty$$

for all compact subsets $K \subseteq [0, \infty)$. Here, of course, the topology on $L^p_{\text{loc}}([0,\infty))$ is that given by the seminorms $\|\cdot\|_K$.

For each $t > 0$, let

$$\mathscr{A}_t = \left\{T \in B(L^p([0,t))) \middle| \pi_{t,s} T \pi_{t,s}^{-1} \text{ is well defined for } 0 < s < t\right\}.$$

Using the construction in Example 3.1, for $t > s > 0$, $\Pi_{t,s} : \mathscr{A}_t \to \mathscr{A}_s$ is defined by $\Pi_{t,s} T = \pi_{t,s} T \pi_{t,s}^{-1}$. Let $\mathscr{A} = \varprojlim \mathscr{A}_t$. For $f \in L^p_{\text{loc}}([0,\infty))$ and $t > 0$, let

$$P_t f(x) = \begin{cases} f(x) & \text{if } x \leq t \\ 0 & \text{otherwise} \end{cases}.$$

Then $\mathscr{A}$ is isormorphic to the collection of all continuous linear maps T on $L^p_{\text{loc}}([0,\infty))$ such that $P_t T = P_t T P_t$ and $P_t T|_{L^p([0,t))} \in B(L^p([0,t)))$. This algebra also satisfies the condition in Proposition 3.2 (b) and therefore the collection of bounded elements of $\mathscr{A}$ consists of all bounded linear operators on $L^p([0,\infty))$ such that $P_t T P_t^{-1}$ is well defined. For each $t > 0$, we consider $L^p([t,\infty))$ a subspace of $L^p([0,\infty))$ via the isometric embedding of extending $f \in L^p([t,\infty))$ to be zero on $[0, t)$. Then for $T \in \mathscr{B}$, we have each of these subspaces is invariant under T. In fact, as a result of Proposition 3.2, $\mathscr{B}$ is isometric to the nest algebra determined by these subspaces.

4. Spectral theory

An easy observation is that for each $a \in \mathscr{A}$, $\sigma_{\mathscr{A}}(a) \subseteq \alpha_{\mathscr{B}}(a) \subseteq \alpha_{\mathscr{D}}(a)$. As a result of Arens invertibility criterion (Corollary 2.2), we know how to calculate $\sigma(a)$ for each $a \in \mathscr{A}$:

PROPOSITION 4.1. $\sigma(a) = \bigcup_{i \in I} \sigma_{\mathscr{A}_i}(\pi_i(a))$

Since the spectrum of an element in a Banach algebra is non-empty, this result guarantees that the algebraic spectrum of any element of a complete lmc-algebra is non-empty. However, $\sigma(a)$ need be neither closed nor bounded as in the Banach algebra case.

Example 4.2. Let $\mathscr{A}$ be the complete lmc-algebra in Example 3.3. Then for each matrix $T \in \mathscr{A}$, we have $\sigma(T) = \operatorname{diag}(T)$. Let $T = (a_{mn})_{m,n=1}^{\infty}$ with $a_{nn} = n$ and $a_{mn} = 0$ if $m \neq n$. Then $\sigma(T) = \mathbb{N}$ and $\sigma(T^{-1}) = \{1/n | n \in \mathbb{N}\}$.

There are conditions on the algebra $\mathscr{A}$ that guarantee every element has compact spectrum. In particular if $\operatorname{Inv}(\mathscr{A})$ is open (i.e. $\mathscr{A}$ is a Q-algebra), then every element has compact spectrum [6]. However, none of the examples we have cited thus far have this property. In fact, N.C. Phillips [14] has shown that an lmc-algebra, $\mathscr{A}$, in which each of the seminorms satisfies the C^*-property and for which the invertibles form an open set, is actually a C^*-algebra.

The set $\alpha_{\mathscr{D}}(a)$, in general, behaves more like a Banach Algebraic spectrum than does $\sigma(a)$.

PROPOSITION 4.3. *Let a, $b \in \mathscr{A}$, $a \in \operatorname{Inv}_{\mathscr{D}}(\mathscr{A})$, $a - b \in \mathscr{D}$ and let $\|a - b\|_{\mathscr{D}} < \|a^{-1}\|_{\mathscr{D}}^{-1}$. Then $b \in \operatorname{Inv}_{\mathscr{D}}(\mathscr{A})$ and*

$$\|a^{-1} - b^{-1}\|_{\mathscr{D}} \leq \frac{\|a^{-1}\|_{\mathscr{D}}^2 \|a - b\|_{\mathscr{D}}}{1 - \|a - b\|_{\mathscr{D}} \|a^{-1}\|_{\mathscr{D}}}.$$

Hence $\alpha_{\mathscr{D}}(a)$ and $\alpha_{\mathscr{B}}(a)$ are closed.

Proof. The proof is as in the Banach algebra case. ■

Analogous to Banach algebraic spectrum, $\alpha_{\mathscr{D}}$ is in some sense upper semi-continuous. More precisely,

PROPOSITION 4.4. *Fix $a \in \mathscr{A}$. Let V be an open subset of $\mathbb{C}$ such that $\alpha_{\mathscr{D}}(a) \subseteq V$ and V^c is compact. Then there exists $\delta > 0$ so that if $b \in \mathscr{A}$ with $a - b \in \mathscr{D}$ and $\|a - b\|_{\mathscr{D}} < \delta$, then $\alpha_{\mathscr{D}}(b) \subseteq V$.*

Proof. If no such δ exists, then for each $n \geq 1$, there is a $b_n \in \mathscr{A}$ so that $a - b_n \in \mathscr{D}$, $\|a - b_n\|_{\mathscr{D}} < 1/n$ and $\lambda_n \in \alpha_{\mathscr{D}}(b_n)$ for some $\lambda_n \in V^c$. Now, some subsequence $\{\lambda_{n_k}\}$ converges to say $\lambda_0 \in V^c$ since V^c is compact. So,

$$\left\|(\lambda_{n_k} - b_{n_k}) - (\lambda_0 - a)\right\|_{\mathscr{D}} \leq |\lambda_{n_k} - \lambda_0| + \|b_{n_k} - a\|_{\mathscr{D}} \to 0.$$

But, $\lambda_0 \notin \alpha_{\mathscr{D}}(a)$ implies $(\lambda_0 - a) \in \mathrm{Inv}_{\mathscr{D}}(\mathscr{A})$. By the previous Proposition 4.3, for sufficiently large k, $(\lambda_{n_k} - b_{n_k}) \in \mathrm{Inv}_{\mathscr{D}}(\mathscr{A})$. This is a contradiction. ■

PROPOSITION 4.5. *Let $a \in \mathscr{A}$ and $\lambda_0 \notin \alpha_{\mathscr{D}}(a)$.*
(1) If $a \in \mathscr{D}$, then $\alpha_{\mathscr{D}}((\lambda_0 - a)^{-1}) = \{(\lambda_0 - \lambda)^{-1} | \lambda \in \alpha_{\mathscr{D}}(a)\}$.
(2) If $a \notin \mathscr{D}$, then $\alpha_{\mathscr{D}}((\lambda_0 - a)^{-1}) = \{(\lambda_0 - \lambda)^{-1} | \in \alpha_{\mathscr{D}}(a)\} \cup \{0\}$.

Proof. (1) is standard in Banach algebra theory. For (2), let $a \notin \mathscr{D}$, $\lambda_0 \notin \alpha_{\mathscr{D}}(a)$. If $\lambda \notin \alpha_{\mathscr{D}}(a) \cup \{\lambda_0\}$, then since

$$(\lambda - a)^{-1} - (\lambda_0 - a)^{-1} = (\lambda_0 - \lambda)(\lambda_0 - a)^{-1}(\lambda - a)^{-1},$$

we have

$$\begin{aligned}\left[(\lambda_0 - \lambda)^{-1} - (\lambda_0 - a)^{-1}\right] \cdot \left[(\lambda_0 - \lambda) + (\lambda_0 - \lambda)^2(\lambda - a)^{-1}\right] \\ = 1 + (\lambda_0 - \lambda)\Big[(\lambda - a)^{-1} - (\lambda_0 - a)^{-1} \\ -(\lambda_0 - \lambda)(\lambda_0 - a)^{-1}(\lambda - a)^{-1}\Big] \\ = 1\end{aligned}$$

Now, $(\lambda_0 - \lambda) + (\lambda_0 - \lambda)^2(\lambda_0 - a)^{-1} \in \mathscr{D}$. Thus we have $(\lambda_0 - \lambda)^{-1} \notin \alpha_{\mathscr{D}}((\lambda_0 - a)^{-1})$. That is,

$$\alpha_{\mathscr{D}}\left((\lambda_0 - a)^{-1}\right) \subseteq \left\{(\lambda_0 - \lambda)^{-1} | \lambda \in \alpha_{\mathscr{D}}(a)\right\} \cup \{0\}.$$

For the reverse inclusion, we first note that $0 \in \alpha_{\mathscr{D}}((\lambda_0 - a)^{-1})$ since $a \notin \mathscr{D}$. Now, if

$$(\lambda_0 - \lambda)^{-1} \notin \alpha_{\mathscr{D}}\left((\lambda_0 - a)^{-1}\right),$$

then $\lambda \notin \alpha_{\mathscr{D}}(a)$ since

$$(\lambda - a) = (\lambda_0 - \lambda)(\lambda_0 - a)\left[(\lambda_0 - \lambda)^{-1} - (\lambda_0 - a)^{-1}\right]$$

and the right-hand-side is invertible in $\mathscr{D}$. Hence

$$\alpha_{\mathscr{D}}\left((\lambda_0 - a)^{-1}\right) \supseteq \left\{(\lambda_0 - \lambda)^{-1} | \lambda \in \alpha_{\mathscr{D}}(a)\right\} \cup \{0\}. \quad \blacksquare$$

Several authors have shown the existence of a canonical holomorphic functional calculus for elements of a complete lmc-algebra. The construction is given here for completeness.

Let $a \in \mathscr{A}$ and let $\mathscr{H}(a)$ be the collection of all functions holomorphic on a neighborhood of $\sigma(a)$. Let $F \in \mathscr{H}(a)$. By Proposition 4.1, for each $i \in I$, we can define $F(\pi_i(a))$ using the ordinary functional calculus for elements in a Banach algebra. Since each map $\pi_{i,j} : \mathscr{A}_i \to \mathscr{A}_j$ for $i > j$ is continuous, it follows that

$$\pi_{i,j}(F(\pi_i(a))) = F(\pi_{i,j} \circ \pi_i(a)) + F(\pi_j(a)).$$

Therefore, $(F(\pi_i(a)))_{i \in I}$ is an element of $\varprojlim \mathscr{A}_i$ and by Theorem 2.1, there exists a unique $b \in \mathscr{A}$ such that $\pi_i(b) = F(\pi_i(a))$. We define $F(a) = b$. Also, from Proposition 4.1, we immediately have a spectral mapping theorem for the algebraic spectrum, $\sigma(F(a)) = F(\sigma(a))$.

It is natural to ask under what conditions $F(a) \in \mathscr{D}$ or $\mathscr{B}$. In the theory of closed unbounded operators, there is a functional calculus for functions that are holomorphic on a neighborhood of the spectrum of the operator and have a limit at infinity [9]. Applying such functions to a closed operator results in a bounded operator. In [5], Barnes used a similar construction for elements in an F-algebra. Here, we show that this method applies in our case.

Let $a \in \mathscr{A}$. Let $\mathscr{H}_{\mathscr{D}}(a)$ be the collection of all complex-valued functions, G, holomorphic on a neighborhood of $\alpha_{\mathscr{D}}(a)$ and at infinity. That is,

(1) F is holomorphic on open U with U^c compact and $\alpha_{\mathscr{D}}(a) \subseteq U$,

(2) $F(\infty) = \lim_{\lambda \to \infty} F(\lambda)$ exists.

Let Γ be a cycle in $U \setminus \alpha_{\mathscr{D}}(a)$ so that

$$\operatorname{Ind}_\Gamma(z) = \begin{cases} -1 & \text{if } z \in U^c \\ 0 & \text{if } z \in \alpha_{\mathscr{D}}(a). \end{cases}$$

Define $\hat{F}(a) \in \mathscr{A}$ by

$$\hat{F}(a) = F(\infty)1_{\mathscr{A}} + \frac{1}{2\pi i}\int_\Gamma F(\lambda)(\lambda - a)^{-1}\, d\lambda.$$

Note that the integral is defined in $\mathscr{A}$ since the function $\lambda \longrightarrow (\lambda - a)^{-1}$ is a continuous function, Γ is a rectifiable curve and $\mathscr{A}$ is complete. In fact, since $1 \in \mathscr{D}$, it is the case that $\hat{F}(a) \in \mathscr{D}$.

THEOREM 4.6 [5]. *$\hat{F}(a) = F(a)$. Furthermore, the mapping $\Psi_a : \mathscr{H}_{\mathscr{D}}(a) \to \mathscr{D}$ given by $\Psi_a(F) = F(a)$ is an algebra homomorphism.*

The following two theorems generalize results found in [5] and their proofs are merely adaptations of those found in that work.

SPECTRAL MAPPING THEOREM 4.7. *For $a \in \mathscr{A}$, and $F \in \mathscr{H}_{\mathscr{D}}(a)$, we have*

$$\alpha_{\mathscr{D}}(F(a)) = \{F(\lambda) | \lambda \in \alpha_{\mathscr{D}}(a) \cup \{\infty\}\}.$$

Proof. Let U be an open subset of $\mathbb{C}$ so that U^c is compact, $\alpha_{\mathscr{D}}(a) \subseteq U$ and F is holomorphic on U. First, since $\lim_{|\lambda| \to \infty} F(\lambda)$ exists, $\{F(\lambda) | \lambda \in \alpha_{\mathscr{D}}(a) \cup \{\infty\}\}$ is a closed subset of $\mathbb{C}$. If λ_0 is not in this set, then there is an $\varepsilon > 0$ so that

$$|F(\lambda) - \lambda_0| \geq 2\varepsilon \quad \text{for all } \lambda \in \alpha_{\mathscr{D}}(a).$$

Let $W = \{\lambda \in U | \ |F(\lambda) - \lambda_0| > \varepsilon\}$.

W is open and since $\lim_{|\lambda| \to \infty} F(\lambda)$ exists and U^c is compact, W^c is bounded. Define G on W by

$$G(\lambda) = (\lambda_0 - F(\lambda))^{-1}.$$

Then $G \in \mathscr{H}_{\mathscr{D}}(a)$ and $G(a) \cdot (\lambda - F(a)) = 1$. So, $\lambda_0 \notin \alpha_{\mathscr{D}}(F(a))$ and

$$\alpha_{\mathscr{D}}(F(a)) \subseteq F(\alpha_{\mathscr{D}}(a)) \cup F(\infty).$$

Now, let $\mu_0 \in F(\alpha_{\mathscr{D}}(a))$. Then there is $\lambda_0 \in \alpha_{\mathscr{D}}(a)$ so that $F(\lambda_0) = \mu_0$. Define G on U by

$$G(\lambda) = \begin{cases} \dfrac{F(\lambda) - F(\lambda_0)}{\lambda - \lambda_0} & \text{if } \lambda \in U \setminus \{\lambda_0\} \\ F'(\lambda_0) & \text{if } \lambda = \lambda_0. \end{cases}$$

Then $G \in \mathscr{H}_{\mathscr{D}}(a)$ and $F(\lambda_0) - F(\lambda) = (\lambda_0 - \lambda)G(\lambda)$ for $\lambda \in U$ implies

$$\mu_0 - F(a) = (\lambda_0 - a)G(a).$$

Since $(\lambda_0 - a)G(a) = G(a)(\lambda_0 - a)$, the invertibility of the right-hand-side above would imply that of $\lambda_0 - a$. Thus, $(\mu - F(a)) \notin \mathrm{Inv}_{\mathscr{D}}(\mathscr{A})$ and

$$F(\alpha_{\mathscr{D}}(a)) \subseteq \alpha_{\mathscr{D}}(F(a)).$$

Since $\alpha_{\mathscr{D}}(F(a))$ is closed and $\lim_{|\lambda| \to \infty} F(\lambda) = F(\infty)$ exists, we also have that $F(\infty) \in \alpha_{\mathscr{D}}(F(a))$. ■

Using the functional calculus and the above Spectral Mapping Theorem, we are able to prove an important property of $\alpha_{\mathscr{D}}(a)$.

THEOREM 4.8. *$\alpha_{\mathscr{D}}(a)$ is bounded if and only if $a \in \mathscr{D}$.*

Proof. If $a \in \mathscr{D}$, then $\alpha_{\mathscr{D}}(a) = \sigma_{\mathscr{D}}(a)$ which is compact. Now, suppose $\alpha_{\mathscr{D}}(a)$ is bounded. Let R_1 be such that $\alpha_{\mathscr{D}}(a)$ is contained in the interior of $D_{R_1} = \{\lambda \in \mathbb{C} : |\lambda| < R_1\}$. Fix R_1 and R_2 so that $R_2 > R > R_1$. Let the open set U be defined by $U = \{z \colon |z| \neq R\}$. Define the function

$$F(\lambda) = \begin{cases} \lambda, & |\lambda| < R \\ 0, & |\lambda| > R. \end{cases}$$

Notice that $F \in \mathscr{H}_{\mathscr{D}}(a)$ and $F(\infty) = 0$. Let δ be the positively oriented boundary of D_{R_1} and τ the negatively oriented boundary of D_{R_2}. Then

$$\begin{aligned} F(a) &= \frac{1}{2\pi i}\int_{\delta+\tau} F(\lambda)(\lambda - a)^{-1}\,d\lambda = \frac{1}{2\pi i}\int_{\delta} F(\lambda)(\lambda - a)^{-1}\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\delta} \lambda(\lambda - a)^{-1}\,d\lambda. \end{aligned}$$

So, we have

$$\pi_i(F(a)) = \frac{1}{2\pi i}\int_{\delta} \lambda(\lambda - \pi_i(a))^{-1}\,d\lambda.$$

Also, the function $G(\lambda) = \lambda$ is holomorphic on $\sigma_i(\pi_i(a))$ and

$$\pi_i(a) = G(\pi_i(a)) = \frac{1}{2\pi i}\int_{\delta} \lambda(\lambda - \pi_i(a))^{-1}\,d\lambda = \pi_i(F(a)).$$

Hence, $F(a) = a \in \mathscr{D}$. ∎

In [1], Allen investigates spectrum of an element of a locally-convex algebra relative to the collection of topologically bounded elements [1, Definition 2.1]. In that work, it is shown that in the complete locally multiplicatively-convex case, the topologically bounded elements are precisely those elements with bounded algebraic spectrum. Combining this observation with Theorems 2.3 and 4.8, we see that this collection consists of all elements of the algebra that are bounded under any collection of seminorms generating the topology.

The functional calculus Ψ is only defined for a small class of holomorphic functions. These functions must be holomorphic off of a compact subset of $\mathbb{C}$ and have a limit at infinity. However, one can show that under different conditions, the algebraic functional calculus yields bounded elements. For example, a Hille-Yosida type theorem, both for (C_0)-semigroups and for analytic semigroups is true in this setting (see [10]). The proofs of these theorems are nearly identical to their counterparts in the theory of closed operators and are omitted here.

5. Relationships between $\sigma(a)$, $\alpha_{\mathscr{B}}(a)$ and $\alpha_{\mathscr{D}}(a)$

It is well-known that if $\mathscr{B}$ is a Banach algebra with closed subalgebra $\mathscr{D}$ and a is in $\mathscr{D}$, then $\partial\sigma_{\mathscr{D}}(a) \subseteq \partial\sigma_{\mathscr{B}}(a)$ [7]. A similar property holds in complete lmc-algebras if we assume $\mathscr{D}$ is a closed subalgebra of $\mathscr{B}$.

PROPOSITION 5.1. *Assume $\mathscr{D}$ is closed in $\mathscr{B}$. Then for $a \in \mathscr{A}$, we have $\partial\alpha_{\mathscr{D}}(a) \subseteq \partial\alpha_{\mathscr{B}}(a)$.*

Proof. The proof is similar to the Banach algebra case. Since the interior of $\alpha_{\mathscr{B}}(a)$ is a subset of the interior of $\alpha_{\mathscr{D}}(a)$, it suffices to show $\partial\alpha_{\mathscr{D}}(a) \subseteq \alpha_{\mathscr{B}}(a)$.

Let $\lambda_0 \in \partial\alpha_{\mathscr{D}}(a)$ and suppose $\lambda_0 \notin \alpha_{\mathscr{B}}(a)$. Let $\{\lambda_n\} \subseteq (\alpha_{\mathscr{D}}(a))^c$ be a sequence of complex numbers converging to λ_0. By the continuity of the resolvent function $R_a\colon \rho_{\mathscr{B}}(a) \to \mathscr{B}$ where $R_a(\lambda) = (\lambda - a)^{-1}$, we have

$$(\lambda_n - a)^{-1} = R_a(\lambda_n) \to R_a(\lambda_0) = (\lambda_0 - a)^{-1}.$$

But $\mathscr{D}$ closed in $\mathscr{B}$ implies $(\lambda_0 - a)^{-1} \in \mathscr{D}$. This is a contradiction. ∎

THEOREM 5.2. *Let $\mathscr{B}$ be the Banach algebra of bounded elements of $\mathscr{A}$, $\mathscr{D}$ a closed subalgebra. Then for every connected component, E, of $(\alpha_{\mathscr{B}}(a))^c$, either $E \subseteq \alpha_{\mathscr{D}}(a)$ or $E \subseteq (\alpha_{\mathscr{D}}(a))^c$. That is, $\alpha_{\mathscr{D}}(a)$ can be formed by filling in some of the "holes" of $\alpha_{\mathscr{B}}(a)$. In particular, if $a \in \mathscr{B} \setminus \mathscr{D}$, then $\alpha_{\mathscr{D}}(a)$ has compact complement.*

Proof. Let E be a component of $(\alpha_{\mathscr{B}}(a))^c$. Write $E = (E \cap \alpha_{\mathscr{D}}(a)) \cup (E \cap (\alpha_{\mathscr{D}}(a))^c)$. Notice that

$$\begin{aligned}(E \cap \alpha_{\mathscr{D}}(a)) \cap \overline{\left(E \cap (\alpha_{\mathscr{D}}(a))^c\right)} &\subseteq (E \cap \alpha_{\mathscr{D}}(a)) \cap \left(\overline{E} \cap \overline{(\alpha_{\mathscr{D}}(a))^c}\right) \\ &= E \cap \partial\alpha_{\mathscr{D}}(a) \\ &\subseteq E \cap \alpha_{\mathscr{B}}(a) = \emptyset.\end{aligned}$$

Therefore, $E \cap \alpha_{\mathscr{D}}(a)$ is both closed and open in E. So, either $E = \emptyset$ in which case $E \subseteq (\alpha_{\mathscr{D}}(a))^c$ or $E \cap \alpha_{\mathscr{D}}(a) = E$, that is $E \subseteq \alpha_{\mathscr{D}}(a)$.

If $a \in \mathscr{B} \setminus \mathscr{D}$, then by Theorem 4.8, $\alpha_{\mathscr{D}}(a)$ is unbounded. Hence, it contains the unbounded component of $(\alpha_{\mathscr{B}}(a))^c$ and $\alpha_{\mathscr{D}}(a)$ has compact complement. ■

COROLLARY 5.3. *Assume $\mathscr{D}$ is closed in $\mathscr{B}$. If $a \in \mathscr{A}$ has $\alpha_{\mathscr{D}}(a)$ with empty interior, then $\alpha_{\mathscr{D}}(a) = \alpha_{\mathscr{B}}(a)$.*

Proof. Since $\alpha_{\mathscr{D}}(a)$ has empty interior, $\alpha_{\mathscr{D}}(a) = \partial\alpha_{\mathscr{D}}(a)$. By Theorem 5.2, $\alpha_{\mathscr{D}}(a) \subseteq \alpha_{\mathscr{B}}(a)$. ■

Several results in Banach algebra theory and operator theory pertain to the connectedness of spectrum [4], [16]. We state a theorem here for which many of these results are corollaries.

THEOREM 5.4. *Fix an element $a \in \mathscr{A}$. If Θ is a bounded component of $\alpha_{\mathscr{D}}(a)$, then $\Theta \cap \overline{\sigma_A(a)} \neq \emptyset$.*

Proof. Suppose that Δ and Γ are disjoint nonempty open and closed subsets of $\alpha_{\mathscr{D}}(a)$ such that Δ is bounded, $\alpha_{\mathscr{D}}(a) = \Delta \cup \Gamma$ and $\overline{\sigma_{\mathscr{A}}(a)} \subseteq \Gamma$. Let U be a bounded open neighborhood of Δ such that $\overline{U}$ does not intersect Γ. Let $V = (\overline{U})^c$, a neighborhood of Γ. Then
(1) $U \cap V = \emptyset$,
(2) $(U \cup V)^c$ is compact.
Define the function F by

$$F(\lambda) = \begin{cases} 1, & \lambda \in U, \\ 0, & \lambda \in V. \end{cases}$$

Then, $F \in \mathscr{H}_{\mathscr{D}}(a)$. By the Spectral Mapping Theorem 4.7, $F(a) \neq 0$. However, by the Spectral Mapping Theorem in a Banach algebra [7] and the fact that $F \equiv 0$ on some neighborhood of $\sigma_{\mathscr{A}}(a)$ we have $F(a) = 0$. This is a contradiction.

Now suppose $\Theta \cap \overline{\sigma_{\mathscr{A}}(a)} = \emptyset$. Let $\delta = d(\Theta, \overline{\sigma_{\mathscr{A}}(a)})$. Corollary 1 on Page 83 of [13] implies there is a closed and open subset $\Omega \subseteq \alpha_{\mathscr{D}}(a)$ such that $\Theta \subseteq \Omega$ and $d(\omega, \Theta) < \delta$ for $\omega \in \Omega$. But $\Omega \cap \overline{\sigma_{\mathscr{A}}(a)} \neq \emptyset$ by the previous argument and this is a contradiction. ■

COROLLARY 5.5. (1) *If $\alpha_{\mathscr{D}}(a) \setminus \sigma(a)$ is bounded, then every component of $\alpha_{\mathscr{D}}(a)$ intersects $\overline{\sigma(a)}$.*

(2) *If $a \in \mathscr{D}$, then every component of $\alpha_{\mathscr{D}}(a)$ intersects $\sigma(a)$. Hence, for $a \in \mathscr{D}$, the connectedness of $\sigma(a)$ implies that of $\alpha_{\mathscr{D}}(a)$.*

(3) *Every bounded component of $\alpha_{\mathscr{D}}(a)$ intersects $\alpha_{\mathscr{B}}(a)$.*

(4) *If $\alpha_{\mathscr{D}}(a) \setminus \alpha_{\mathscr{B}}(a)$ is bounded, then every component of $\alpha_{\mathscr{D}}(a)$ intersects $\alpha_{\mathscr{B}}(a)$.*

(5) *Any isolated point of $\alpha_{\mathscr{D}}(a)$ is in $\sigma(a)$ (hence in $\alpha_{\mathscr{B}}(a)$).*

(6) *If $\alpha_{\mathscr{D}}(a)$ is totally disconnected, then $\alpha_{\mathscr{D}}(a) = \alpha_{\mathscr{B}}(a) = \overline{\sigma(a)}$.*

Proof. (1), (3), and (4) are clear. (2) holds by Theorem 4.8. (5) holds since isolated points are components. (6) holds because a space is totally disconnected if each of its points is a component. ■

These observations allow us to say more about the structure of $\alpha_{\mathscr{D}}(f(a))$ in certain cases. Suppose $a \in \mathscr{A}$ and f is holomorphic on a neighborhood of $\alpha_{\mathscr{D}}(a)$ such that $f(a) \in \mathscr{D}$. Then $\alpha_{\mathscr{D}}(f(a))$ is a compact subset of $\mathbb{C}$. By the Spectral Mapping Theorem for the algebraic spectrum and the previous result, if Δ is a connected component of $\alpha_{\mathscr{D}}(f(a))$, then Δ has non-trivial intersection with $\overline{\sigma(f(a))} = \overline{f(\sigma(a))}$. The continuity of f gives us $f(\overline{\sigma(a)}) \subseteq \overline{f(\sigma(a))} \subseteq \overline{f(\overline{\sigma(a)})}$. Thus, if $\overline{\sigma(a)}$ is connected, then so is $\overline{\sigma(f(a))}$ and we have:

Corollary 5.6. *Let $a \in \mathscr{A}$ and f holomorphic on $\alpha_{\mathscr{D}}(a)$ such that $f(a) \in \mathscr{D}$. Then, if $\overline{\sigma(a)}$ is connected, so is $\alpha_{\mathscr{D}}(f(a))$.*

6. Interpretation of results in a Banach algebra

Although the following results can be proved using Banach algebra techniques, we include their statements here as corollaries of the previous Propositions. Throughout this section, $\mathscr{B}$ is a unital Banach algebra, a special case of a complete lmc-algebra in which the bounded elements comprise the entire algebra.

We can restate Theorems 4.8 and 5.2.

Corollary 6.1. *If $\mathscr{D}$ is a closed subalgebra of $\mathscr{B}$ such that $1_{\mathscr{B}} \in \mathscr{D}$, then for $a \in \mathscr{B}$, $a \in \mathscr{D}$ if and only if $\alpha_{\mathscr{D}}(a)$ is bounded.*

Corollary 6.2. *Let $\mathscr{D}$ be a closed subalgebra of $\mathscr{B}$ with $1_{\mathscr{B}} \in \mathscr{D}$. Let $a \in \mathscr{B}$. Then for every connected component, Δ, of $(\alpha_{\mathscr{B}}(a))^c$, either $\Delta \subseteq \alpha_{\mathscr{D}}(a)$ or $\Delta \subseteq (\alpha_{\mathscr{D}}(a))^c$. In particular, if $a \in \mathscr{B} \setminus \mathscr{D}$, then $\alpha_{\mathscr{D}}(a)$ has compact complement.*

Combining Corollaries 6.1 and 6.2, we have:

Corollary 6.3. *Let $\mathscr{D}$ be a closed subalgebra of $\mathscr{B}$ with $1_{\mathscr{B}} \in \mathscr{D}$. If $a \in \mathscr{B}$ with $\|1 - a\| < 1$ then $a^{-1} \in \mathscr{D}$ if and only if $a \in \mathscr{D}$.*

Proof. If $a \in \mathscr{D}$, then by hypothesis, $\sigma_{\mathscr{D}}(a)$ is contained in $\{\lambda \in \mathbb{C} \mid |\lambda - 1| < 1\}$ and a is invertible in $\mathscr{D}$. If $a \notin \mathscr{D}$, then since $\sigma_{\mathscr{B}}(a)$ is contained in $\{\lambda \in \mathbb{C} \mid |\lambda - 1| < 1\}$, 0 is in the unbounded component, E, of $(\sigma_{\mathscr{B}}(a))^c$. Since $\alpha_{\mathscr{D}}(a)$ is unbounded, it includes E and hence, $a^{-1} \notin \mathscr{D}$. ■

COROLLARY 6.4. *Let $\mathscr{D}$ be a closed subalgebra of $\mathscr{B}$ with $1_{\mathscr{B}} \in \mathscr{D}$. Suppose $a, b \in \mathscr{B}$ with $a \in \mathrm{Inv}(\mathscr{B})$ and satisfying* (i) $\|1 - a^{-1}b\| < 1$ *and* (ii) $(a^{-1}b)^{-1} \in \mathscr{D}$. *Then $a \in \mathrm{Inv}_{\mathscr{D}}(\mathscr{B})$ if and only if $b \in \mathrm{Inv}_{\mathscr{D}}(\mathscr{B})$.*

Proof. $a^{-1} \in \mathscr{D}$ implies b is invertible and $b^{-1} = (a^{-1}b)^{-1}a^{-1} \in \mathscr{D}$. Conversely, conditions (i) and (ii) give by Corollary 6.3 that $a^{-1}b \in \mathscr{D}$. Thus, if $b \in \mathscr{D}$ then $a^{-1} = (a^{-1}b)b^{-1} \in \mathscr{D}$. ∎

7. Applications

Example 3.1 *revisited.* Let $\mathscr{A}$ be the algebra in Example 3.1.

PROPOSITION 7.1. *For $T \in \mathscr{A}$, T is a closed operator on $\mathscr{X}_b$.*

Proof. Suppose $\{f_n\}$ is a sequence in $\mathscr{X}_b$ that converges to f in $\mathscr{X}_b$ and that Tf_n is in $\mathscr{X}_b$ and converges to g in $\mathscr{X}_b$. Then since $\pi_i f_n$ converges to $\pi_i f$, we have

$$\begin{aligned}\pi_i(Tf_n) &= (\Pi_i T)(\pi_i f_n) = \pi_i T \pi_i^{-1}(\pi_i f_n) \longrightarrow \pi_i T \pi_i^{-1}(\pi_i f)\\ &= (\Pi_i T)(\pi_i f) = \pi_i(Tf).\end{aligned}$$

On the other hand $\pi_i(Tf_n) \longrightarrow \pi_i g$. Therefore, $Tf = g$. ∎

For $T \in \mathscr{A}$, we would now like to compare its spectrum as an element of the complete lmc-algebra and its spectrum as a closed operator. To this end, we have:

PROPOSITION 7.2. *For $T \in \mathscr{A}$, $\alpha_{\mathscr{B}}(T) \setminus \sigma_{\mathrm{op}}(T) \subset \bigcup_{i \in I} \sigma_p(\Pi_i T)$ where $\sigma_p(S)$ is the point spectrum of an operator S.*

Proof. Let $\lambda \in \alpha_{\mathscr{B}}(T) \setminus \sigma_{\mathrm{op}}(T)$. Then $(\lambda - T)^{-1}$ exists in $B(\mathscr{X}_b)$ but for some $i \in I$, $p_i(\lambda - T)^{-1}p_i^{-1}$ is not well defined. That is, there is $x \in \mathscr{X}_b$, $x \neq 0$ so that $p_i(x) = 0$ but $p_i(\lambda - T)^{-1}(x) \neq 0$. Set $y_i = p_i(\lambda - T)^{-1}(x)$; then

$$(\lambda - \pi_i T \pi_i^{-1})y = \pi_i(\lambda - T)\pi_i^{-1}p_i(\lambda - T)^{-1}(x) = \pi_i x = 0.$$

That is $\lambda \in \sigma_p(\Pi_i T)$. ∎

COROLLARY 7.3. *Let $T \in \mathscr{B}$. Suppose $\bigcup_{i \in I} \sigma_p(\Pi_i T)$ has empty interior. Then $\sigma_{\mathrm{op}}(T) = \alpha_{\mathscr{B}}(T)$.*

Proof. Since $\mathcal{B}$ is a closed subalgebra of $B(\mathcal{X}_b)$, we have that $\partial\alpha_{\mathcal{B}}(T) \subseteq \partial\sigma_{\text{op}}(T)$. This implies either the two sets are equal or they differ by a set with nonvoid interior. ■

Example 3.3 *revisited.* Let $\mathcal{A}$ be the algebra of lower triangular matrices as in Example 3.3. We assume $p \in [1, \infty]$ is fixed and that $\|T\|_n$ is the operator norm of the $n \times n$ upper left-hand corner of T acting on $l^p(n)$. Then as noted earlier, the topology on $\mathcal{A}$ induced by these seminorms is that of entrywise convergence. Also observed earlier was that the Banach algebra $\mathcal{B}$ is the closed subalgebra of $B(l^p(\mathbb{N}))$ consisting of operators induced by lower-triangular matrices (with respect to the standard basis). The next two propositions are easily proved and are stated without proof.

PROPOSITION 7.4. *If $T \in \mathcal{A}$, then T is a closed operator on $l^p(\mathbb{N})$ and $\sigma_{\mathcal{A}}(T) = \text{diag}(T)$.*

PROPOSITION 7.5. *$\mathcal{B}$ is an inverse closed subalgebra of $B(l^p)$.*

The latter well-known result actually follows from previous considerations. If $T \in \mathcal{B}$ has no inverse in $\mathcal{B}$, then $0 \in \alpha_{\mathcal{B}}(T)$. Since $\bigcup_{n=1}^{\infty}\sigma_p(T_n)$ is a countable set, by Corollary 7.3, $\alpha_{\mathcal{B}}(T) = \sigma_{\text{op}}(T)$ and so T also has no inverse as a bounded operator.

Let T be a lower triangular matrix inducing a bounded operator on $l^p(\mathbb{N})$. In [17], it is proved that every component of the spectrum of T as a bounded operator intersects the diagonal. We have shown a generalization of this result and provide the statement below.

THEOREM 7.6. *Let $T \in \mathcal{A}$. Then every non-empty bounded relatively open and closed subset of $\sigma_{\text{op}}(T)$ contains a diagonal element of T. In particular, if T is strictly lower-triangular (i.e $a_{nn} = 0$) and induces a bounded operator on $l^p(\mathbb{N})$, then $\sigma_{\text{op}}(T)$ is connected.*

For now, let $p = \infty$ and $\mathcal{A}$ the algebra of lower-triangular matrices with corresponding seminorms. Let $\mathcal{D}$ be the collection of matrices $T \in \mathcal{A}$ such that T maps every convergent sequence to a convergent sequence. Such matrices are called conservative and have been studied extensively. It is well known that $\mathcal{D}$ is a closed subalgebra of $\mathcal{B}$, the lower-triangular matrices inducing bounded operators on $l^\infty(\mathbb{N})$ [18]. Applying Theorem 5.4, we state a result for which Theorem 1 of [16] is a corollary.

THEOREM 7.7. *Let $T \in \mathcal{A}$. Then every non-empty bounded relatively closed and open subset of $\alpha_{\mathcal{D}}(T)$ intersects the diagonal of T. In particular, there does not exist a conservative matrix, T, with 0 an isolated point in $\alpha_{\mathcal{D}}(T)$.*

Furthermore, Corollary 6.2 gives us an unusual test for whether or not a matrix is conservative.

THEOREM 7.8. *If $T \in \mathscr{B}$, then T is conservative if and only if for some λ in the unbounded component of $(\alpha_{\mathscr{B}}(T))^c$, $(\lambda - T)^{-1}$ is conservative.*

Now suppose A and B are infinite lower triangular matrices inducing bounded operators on l^∞. A is said to be stronger than B if for every sequence x such that Bx is convergent, Ax is also convergent. If A is stronger than B and B is stronger than A, then A and B are said to be equipotent. It is know that A is stronger than B if and only if BA^{-1} is conservative [18]. From this and Corollary 6.4, we have:

THEOREM 7.9. *Suppose A and B are lower triangular matrices inducing bounded operators on $l^\infty(\mathbb{N})$ and A invertible as a matrix. If $\|1 - BA^{-1}\| < 1$ and B is stronger than A, then A and B are equipotent.*

Proof. By Corollary 6.3, BA^{-1} is conservative if and only if AB^{-1} is conservative. ■

Example 3.4 *revisited.* Fix $p \in [1, \infty]$ and let $\mathscr{A}$ be the algebra obtained in Example 3.4. Combining Propositions 7.1 and 7.2 we already have the following result.

THEOREM 7.10. *For $T \in \mathscr{A}$, T is a closed operator on $L^p([0, \infty))$. Moreover,*

$$\alpha_{\mathscr{B}}(T) \setminus \sigma_{\mathrm{op}}(T) \subseteq \bigcup_{x>0} \sigma_p((P_x T)).$$

Using this theorem, we can prove that several large classes of integral operators possess connected spectrum.

Suppose $k(x,t)$ is measurable on $(0, \infty) \times (0, \infty)$ and that $\int_0^x k(x,t) f(t)\, dt$ is finite almost everywhere for each $f \in L^p_{\mathrm{loc}}([0, \infty))$. We define $T = \mathrm{Int}(k)$ by

$$Tf(x) = \mathrm{Int}(k) f(x) = \int_0^x k(x,t) f(t)\, dt.$$

We refer to k as the kernel associated with the integral operator T. If $Tf \in L^p_{\mathrm{loc}}([0, \infty))$ for all $f \in L^p_{\mathrm{loc}}([0, \infty))$, then $T \in \mathscr{A}$ and is therefore a closed operator on $L^p([0, \infty))$. In [4] it is conjectured that if T as above is a bounded operator on $L^p([0, \infty))$, then it has connected spectrum. In that work, Barnes exhibits several classes of kernels for which the conjecture holds. Applying some of our results, we are able to enlarge these classes.

For now, assume $1 < p < \infty$. Suppose k is a kernel such that

$$\left(\int_0^x |k(x,t)|^q \, dt\right)^{1/q} \in L^p_{\text{loc}}([0,\infty))$$

where $1/p + 1/q = 1$. If the left-hand-side is actually in $L^p([0,\infty))$ for all $f \in L^p([0,\infty))$ then $T = \text{Int}(k)$ is known as a Hille-Tamarkin operator. In the general case, we say $T = \text{Int}(k)$ is a local Hille-Tamarkin operator.

THEOREM 7.11. *If T is a local Hille-Tamarkin operator, then*:
(1) *If T is bounded on $L^p([0,\infty))$, then $\sigma_{\text{op}}(T)$ is connected.*
(2) *In general, T has no non-zero eigenvalues.*

Proof. Since T is a local Hille-Tamarkin operator, $P_x T$ is a Hille-Tamarkin operator on $L^p([0,x))$ for each $x > 0$. It is well known that such operators are compact [11]. For each $x > 0$, $0 \le t \le x$, the operator $P_x T$ has

$$M_t = \{f \in L^p([0,x)) | f(y) = 0 \text{ for almost all } y > t\}$$

as a closed invariant subspace. These spaces form a continuous chain in $L^p([0,x))$. That is, for each $t \in (0,x)$,

$$\overline{\text{span}}\left(\bigcup_{0<y<t} M_y\right) = M_t,$$

$$\bigcap_{y>t} M_y = M_t.$$

In [15], it is shown that a compact operator on a Banach space with a continuous chain of closed invariant subspaces is quasinilpotent. That is, $\sigma_{\text{op}}(P_x T) = \{0\}$. It is easy to see that

$$\mathscr{A}_x = \left\{T \in B(L^p([0,t))) | \pi_{x,y} T \pi_{x,y}^{-1} \textit{ is well defined for } 0 < y < x\right\}$$

is a closed subalgebra of $B(L^p([0,x)))$ and so $\sigma_{\mathscr{A}_x}(P_x T) = \{0\}$. By Proposition 4.1, $\sigma_{\mathscr{A}}(T) = \{0\}$. If T is bounded, Corollary 5.5 (2) gives us $\alpha_{\mathscr{B}}(T)$ is connected. To conclude $\sigma_{\text{op}}(T)$ is connected, we observe that by Proposition 7.2,

$$\alpha_{\mathscr{B}}(T) \setminus \sigma_{\text{op}}(T) \subseteq \{0\}.$$

This of course implies the two sets are equal and $\sigma_{\text{op}}(T)$ is connected. To prove (ii) we note that any eigenvalues of T are in $\sigma_{\mathscr{A}}(T)$. ■

The following kernels induce local Hille-Tamarkin operators:

(1) (i) $k(x, t) = \Sigma_{k=1}^{n} \phi_k(x)\psi_k(t)$ where $\phi_k \in L^p([0, \infty))$ and $\psi_k \in L^q_{\text{loc}}([0, \infty))$.

(2) (ii) $k(x, t)$ locally essentially bounded.

Now, suppose $p = 1$. An integral operator $T = \text{Int}(k)$ is said to be a local Hille-Tamarkin operator on $L^1_{\text{loc}}([0, \infty))$ if

$$\int_0^y \operatorname*{ess\,sup}_{t \geq 0} |k(x, t)| \, dx < \infty \quad \text{for each } y > 0$$

THEOREM 7.12. *Suppose T is a local Hille-Tamarkin operator on $L^1_{\text{loc}}([0, \infty))$. Then T has no non-zero eigenvalues. If in addition, T is bounded on $L^1([0, \infty))$, then $\sigma_{\text{op}}(T)$ is connected.*

Proof. In [11], it is shown that if S is a Hille-Tamarkin operator on $L^1([0, x])$, then S^2 is compact. Hence $P_x T^2 = (P_x T)^2$ is compact. We can therefore apply the argument in the previous proof to conclude that $\sigma_{\mathscr{A}}(T^2) = \{0\}$. By the Spectral Mapping Theorem, we have $\sigma_{\mathscr{A}}(T) = \{0\}$. The rest of the theorem follows just as in the previous proof. ■

It is well known that bounded convolution operators on $L^p([0, \infty))$ have connected spectrum. In fact, the spectrum can be calculated using the Laplace Transform. It is known that operators induced by kernels of the form $k(x, t) = h(x, t) j(x - t)$ have connected spectrum whenever h is essentially bounded and $j \in L^1([0, \infty))$ [4]. We can get more by applying our technique.

THEOREM 7.13. *Let $1 \leq p < \infty$. Let $T = \text{Int}(k)$ by induced by a kernel of the form $k(x, t) = h(x, t) j(x - t)$ where $h \in L^\infty_{\text{loc}}([0, \infty))$ and $j \in L^1_{\text{loc}}([0, \infty))$. Then T has no non-zero eigenvalues and if T is a bounded operator on $L^p_{\text{loc}}([0, \infty))$, then it has connected spectrum.*

Proof. Theorem 11 of [4] implies that for each $x > 0$, $\sigma_{\text{op}}(P_x T) = \{0\}$. Duplicating the proofs of the previous two theorems, we are done. ■

Applications of Banach Algebra Results. Let X be a Banach lattice. A bounded operator T on X is said to be positive if $Tf \geq 0$ whenever $f \geq 0$. $T \in B(X)$ is regular if it can be written $T = (T_1 - T_2) + i(T_3 - T_4)$ where T_i is positive for $i = 1, 2, 3, 4$. As we see below, it is not necessarily true that every bounded operator is regular. The regular operators form a unital algebra and can be given a complete norm, denoted $\| \cdot \|_r$, that is continuous with respect to the operator norm.

The L^p-spaces are standard examples of Banach lattices. It is well known that when $p = 1$ or $p = \infty$, every bounded operator is regular. In a recent paper by Arendt and Voigt [3] it is proved that for $1 < p < \infty$, the Banach algebra of regular operators is not dense in $B(L^p([0,\infty)))$. We can use this result to add to an observation made in [2] that certain operators have no regular resolvents.

THEOREM 7.14. *Let $1 < p < \infty$. If T is a bounded operator on $L^p([0,\infty))$ such that the distance from T to the collection of regular operators is positive and $\sigma_{\mathrm{op}}(T)$ is polynomially-convex, then the distance from $(\lambda - T)^{-1}$ to the regular operators is positive for all $\lambda \in (\sigma(T))^c$.*

Proof. Let $\mathscr{B} = B(L^p([0,\infty)))$ and let $\mathscr{D}$ be the closure of the algebra of regular operators in the operator norm. Then as noted, $\mathscr{D} \neq \mathscr{B}$. Since $T \in \mathscr{B} \setminus \mathscr{D}$, applying Corollary 6.2, we are done. ■

As a final application of the Banach algebra results, let X be a Banach space. Suppose M is a closed subspace of X. Then the collection of all bounded operators T such that $TM \subseteq M$ is a closed subalgebra of $B(X)$. Therefore as a simple application of Corollary 6.2, we have:

THEOREM 7.15. *If $T \in B(X)$, then T has M as an invariant subspace if and only if $(\lambda - T)^{-1}$ has M as an invariant subspace for some λ in the unbounded component of $(\sigma_{\mathrm{op}}(T))^c$. In particular, if $\sigma_{\mathrm{op}}(T)$ is polynomially-convex, then T has a non-trivial invariant subspace if and only if $(\lambda - T)^{-1}$ does for some λ.*

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