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COMPLEMENTED HILBERTIAN SUBSPACES IN REARRANGEMENT INVARIANT FUNCTION SPACES

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Introduction

A classical paper of Rodin and Semenov [RS] studies the closed subspace generated by Rademacher elements in a symmetric space X (defined on the interval $[0,1]$) and gives a necessary and sufficient condition on X for this subspace to be isomorphic to the space l_2. Let M be the Orlicz function defined by $M(u) = e^{u^2} - 1$, and L_M be the associated Orlicz space on $[0,1]$, $\| \ \|_M$ its norm. Then this condition reads:

$$\exists C < \infty, \forall f \in L_\infty([0,1]), \quad \|f\|_X \leq C\|f\|_M \tag{1}$$

or equivalently the closure $\mathscr{G}$ of $L_\infty([0,1])$ in L_M is (algebraically) included in X.

When this condition is realized, this l_2 subspace is complemented in X if moreover X is (algebraically) included in the dual Orlicz space $\mathscr{G}^* = L_{M_*}([0,1])$ (M_* is the Young conjugate of the function M; it is given, up to equivalence, by $M_*(t) \simeq t\sqrt{\log et}$), or equivalently:

$$\exists C < \infty \, \forall f \in X, \ \|f\|_{M_*} \leq C\|f\|_X. \tag{2}$$

This was shown independently by Rodin and Semenov (1979) [RS2] and Lindenstrauss and Tzafriri (1979) [LT2]. These results were extended by Bravermann (1982) [B] in a short note, showing that if a sequence (X_i) of independent individually distributed random variables spans l_2 in the rearrangement invariant space X, then the variable X_i belong to L_2 and $\mathscr{G} \subset X$. If moreover (X_i) span a complemented closed space, then $X \subset \mathscr{G}^*$.

In the first two sections of this paper we show that, roughly speaking, condition (1) characterizes when the r.i. space X contains a subspace isomorphic to l_2 (in short "hilbertian subspace") while conditions (1) and (2) characterize the situation where X contains a complemented hilbertian subspace. This gives an answer to a question of E. M. Semenov (as reformulated in [T]).

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However, as is well known, there exist Orlicz functions φ which are not majorized by M, (resp. nor majorizing M_*) but such that L_φ contains a hilbertian (resp. complemented hilbertian) subspace; the elements of the l_2 basis being disjoint elements of L_φ (in fact this can be obtained in the sequence space l_φ—see [LT]). Thus we have to discard this case by an additional hypothesis, to be able to obtain (1) and (2) as necessary conditions. We add also an order continuity hypothesis on X (note that if X is not order-continuous, it contains l_∞, hence l_2).

The characterization of the existence of hilbertian subspace by condition (1) was already obtained by E. V. Tokarev [T], using a result of Gaposhkin [G] and Rosenthal's embedding theorem for hilbertian subspaces of L_1. The proof given here is reformulated in order to extend to the case of l_p-subspaces ($1 < p \leq 2$), and to introduce to the proof of the complemented case. This second case was partially solved by Tokarev in the same paper, in fact for subspaces belonging to a particular class $K_0(X)$ (in our terminology, when the unit ball of the space is X-equiintegrable). In the case where X is q-concave, for a $q < 2$, this implies the conclusion of our Th. 4, since all hilbertian subspaces of X are then in the class $K_0(X)$. It is possible in fact to prove directly (using Lemma 10 below) that, under our hypotheses, if X contains a complemented hilbertian subspace, then it contains another one which belongs to the class $K_0(X)$. This gives an alternative proof of our result (Theorem 4), which however does not lead to the quantitative version we give in §3.

In this section, given the additional assumption that X does not contain c_0, we show the equivalence of the existence of a factorization

$$l_2 \xrightarrow{i} X \xrightarrow{\pi} l_2$$

of the identity of the Hilbert space with $\|\pi\|\,\|i\| \leq C$ and of the existence of two variables $A.G \in X$, $B.G \in X'$, where G is a normal Gaussian variable independent from (A, B), and $\langle A, B\rangle = 1$, $\|A.B\|_X \|B.G\|_{X'} \leq C$.

Note also that the analogous problem, for finite dimensional spaces l_2^n was considered in the chapter 9 of [Ka], where a criterion is given for a general Banach lattice *not* to contain l_2^n's uniformly complemented. If one defines the constants $d_n(X)$ to be the least constants such that $\Sigma\|f_i\| \leq d_n\|\Sigma f_i\|$ for all disjoint families of n vectors in X, this criterion reads:

$$\liminf d_n.(\log n)^{-1/2} = 0.$$

In the case of r.i. spaces, say on $[0,1]$ it is easy to see that under this condition, l_2 cannot be a sublattice of X, and X does not embed algebraically in L_{M_*}; i.e., the condition (2) is violated. However this criterion seems to be generally stronger than the negation of condition (2).

In the last two sections, we investigate when a rearrangement invariant function space X is isomorphic to its Hilbert valued extension $X(l_2)$. (This question for spaces with unconditional basis is studied in [KaW]). For an Orlicz space this happens exactly when its Boyd indices are non trivial, i.e., when it is reflexive (Section 4). For a q-concave, $q < 2$, r.i. space over $[0, 1]$, a necessary and sufficient condition is that the lower Boyd index of the space is non-trivial (strictly greater than 1) (Section 5). Note that, by known results, the non-triviality of both Boyd indices is equivalent (for order-continuous r.i. spaces) to the fact that X has an unconditional basis (in fact, that the Haar basis is unconditional): see [LT2], Theorems 2c6 and 2c11. This could perhaps give a way to generalize our results to more general r.i. spaces.

We refer to [LT2] for basic facts about rearrangement invariant function spaces. The definition of r.i. space we consider is that of [LT2]; more precisely the function space X (over $I = [0, a]$ or $= [0, \infty)$) is r.i. iff i): for every $x \in X$, its rearrangement x^* has same norm, and ii): either simple integrable functions are dense in X, or X has Fatou property ([LT2], p. 30). If $(\Omega, \mathscr{A}, \mu)$ is an arbitrary measure space, $X(\Omega, \mathscr{A}, \mu)$ is the space of measurable functions on Ω whose rearrangements are in $X(I)$.

A reference for ultrapowers and ultraproducts in [H].

1. l_p-subspaces of rearrangement invariant spaces

If X is a Banach lattice, we denote by X' the Nakano dual of X, i.e., the space of σ-order continuous elements x^* of the dual: $x_n \in X$, $x_n \downarrow 0 \Rightarrow \langle x_n, x^* \rangle \to 0$. When X is order continuous then $X' = X^*$ and X embeds isometrically in X''.

Proposition 1. *Let X be an order continuous rearrangement invariant function space, and $1 < p \leq 2$. Assume that X does not contain the space l_p as sublattice. Then X contains l_p as subspace iff X'' contains a p-stable random variable (a Gaussian variable when $p = 2$).*

Remark 2. Let $X([0, 1])$ be the restriction of the space X to the interval $[0, 1]$ (in the case it is defined on $[0, \infty)$). Let $\mathscr{S}_p$, resp $\mathscr{G}$ be the closure of $L_\infty([0, 1])$ in the weak L_p space $L_{p,\infty}$, resp. in the Orlicz space $L_M([0, 1])$ $(M(u) = e^{u^2} - 1)$. Then X'' contains a p-stable, resp. Gaussian variable iff the space $\mathscr{S}_p$, resp. $\mathscr{G}$ is (algebraically) included in $X([0, 1])$.

This remark is simply a consequence of the classical estimation of the tail of a p-stable random variable γ_p (see [F]): $\mathbf{P}(|\gamma_p| > t) \simeq t^{-p}$ for $t \to \infty$. Hence $L_{p,\infty}([0, 1])$ is simply the space of functions which are majorized, up to a rearrangement, by an homothet of $|\gamma_p| + 1$. In the Gaussian case, note that the Orlicz space $L_M([0, 1])$ coincides with the Lorentz space $L_{M,\infty}([0, 1])$

(which consists of functions which are majorized by a function equimeasurable with an homothet of $|G| + 1$, G a normal Gaussian variable): see [LT2], Thm. 2b4 and its proof.

Remark 3. When X is order continuous, the Fatou property ($X = X''$) is equivalent to the fact that X does not contain c_0 as subspace. In this case, again with the hypothesis that X does not contain l_p as sublattice, X contains l_p iff it contains a p-stable (resp. Gaussian) variable.

Proof of Prop. 1. (a) We prove first the necessity.

Let E be a l_p subspace of the r.i. space X (defined for instance on $[0, \infty)$ equipped with Lebesgue measure λ). Then the norm on X is equivalent to the $L_1(U)$ norm on elements of E, for some measurable subset U of finite measure. For, if not, there is a sequence $(f_n)_n$ in the unit sphere of E such that $\forall a > 0$, $\int_0^a |f_n|\, d\lambda \to 0$ as $n \to \infty$. Using the order continuity of X, we deduce that $\forall k$, $\| \, |f_k| \wedge |f_n| \, \|_X \to 0$ as $n \to \infty$; after suitable extraction we obtain a l_p-basis $(f_n)_n$ in E and a disjoint sequence $(f'_n)_n$ which are equivalent for the norm of X, a contradiction.

We use now the following fact, which is a consequence of the paper [DCK] by Dacunha-Castelle and Krivine on subspaces of L_1 (see also [A]); in the hilbertian case, as noticed in [T], it is also a consequence of Gaposhkin's result on the central limit theorem for sequences of functions ([G], Thm. 1.5.1) and of Rosenthal's theorem on subspaces of L_p ([R], Thm. 1). Every infinite dimensional l_p subspace of a space $L_1(U)$ contains a normalized l_p sequence of functions whose distributions are asymptotically conditionally p-stable (with same parameter) and conditionally independent. More precisely, there exist a superspace $L_1(U \times S, \lambda \otimes \sigma)$ (where σ is a probability), a function Y in $L_1(U \times S, \lambda \otimes \sigma)$ which has conditional p-stable distribution (i.e., $Y(\omega, \cdot\,)$ is a p-stable variable for a.e. $\omega \in U$), and a normalized sequence (f_n) in $L_1(U)$ which converges "weakly conditionally in distribution" to Y, in the sense that for every bounded continuous function $\varphi \in C_b(\mathbf{R})$,

$$\lim_{n \to \infty} \varphi(f_n) = \int_S \varphi(Y(\cdot\,, s))\, d\sigma(s)$$

where the limit is taken in the $\sigma(L_\infty, L_1)$ sense; or, equivalently,

$$\forall f \in L_1(U), \quad (f, f_n) \xrightarrow{\text{dist}} (f, Y)$$

where "dist" refers to the usual weak convergence of probability distributions (or more generally of finite measures). This second definition is only apparently stronger than the first one, in the case of L_1-bounded sequences. We refer to [BR] for an extensive study of this kind of distributional convergence (which we call in short "wcd" hereafter).

We shall use the whole information given by this result only in §2, and use here only the fact that the distribution of (f_n) converges to that of Y. Using [LT2], remark following 1.b.18, it is easy to see that $Y \in X''$ (with $\|Y\|_{X''} \leq \liminf_{n\to\infty} \|f_n\|_X$). But Y is equimeasurable with a function of the form $A \otimes \gamma_p$, where $A \in L_1^+(U)$ and γ_p is a p-stable variable (defined on (S, σ)). Since conditional expectation operators act on X'', we see that

$$\alpha \mathbf{1}_U \otimes \gamma_p \in X''(U \times S) \quad \left(\text{where } \alpha = \frac{1}{\lambda(U)} \int_U A \, d\lambda\right),$$

hence $\gamma_p \in X''(S, \sigma)$.

(b) The sufficiency of the condition results in the hilbertian case from the Rodin-Semenov theorem (see [LT2], Thm. 2b4). In the l_p-case, we have to work a little bit more. Suppose that X'' contains a p-stable random variable γ_p. It is well known that γ_p can be realized as a product $G \otimes \gamma^{1/2}$, where G is a normal Gaussian variable, and γ is a positive $(p/2)$-stable variable. Consider in $X(\Omega \times S)$ the sequence $(G_n \otimes Y_n)_{n=1}^\infty$, where the G_n are independent normal Gaussian variable, and the Y_n are independent truncated square roots of $(p/2)$-stable positive variables (Y_n^2 is equimeasurable with $\gamma \mathbf{1}_{\{|\gamma| \leq n\}}$). For all $f \in X''(\Omega \times S)$, we have

$$f + G_n \otimes Y_n \xrightarrow{\text{dist}} f + \Gamma,$$

where Γ is a p-stable variable independent from f (to fix ideas, let us consider that $\Gamma \in X''(T)$). Hence

$$\liminf_{n\to\infty} \|f + G_n \otimes Y_n\|_{X''(\Omega\times S)} \geq \|f + \Gamma\|_{X''(\Omega\times S\times T)}.$$

But if f takes itself the form $G \otimes h$, G Gaussian in $L^0(\Omega)$, $h \in L^0(S)$, then

$$\begin{aligned}\|G \otimes h + G_n \otimes Y_n\| &= \left\|G \otimes \left(h^2 + Y_n^2\right)^{1/2}\right\| \leq \left\|G \otimes \left(h^2 + \gamma\right)^{1/2}\right\| \\ &= \|G \otimes h + G_n \otimes \gamma^{1/2}\| = \|G \otimes h + \Gamma\|;\end{aligned}$$

thus we have in fact the equality

$$\lim_{n\to\infty} \|f + G_n \otimes Y_n\|_{X''(\Omega\times S)} = \|f + \Gamma\|_{X''(\Omega\times S\times T)}.$$

The end of the reasoning is now a matter of folklore, inspired from [KM] (see also the proof of Lemma 10). Let $E = \overline{\operatorname{span}}[G_n \otimes Y_n]_{n\geq 1}$. The preceding shows that the spreading model generated by the sequence $(G_n \otimes Y_n)_n$ over

E is isometric to the space generated by E and a sequence $(\Gamma_n)_n$ of independent p-stable random variables (in a suitable extension $X''(\Omega \times S \times T^{\mathbf{N}})$); i.e.,

$$\forall x \in E, \left\| x + \sum_{j=1}^{n} a_j \Gamma_j \right\| = \lim_{k_n \to \infty} \lim_{k_{n-1} \to \infty} \cdots \lim_{k_1 \to \infty} \left\| x + \sum_{j=1}^{n} a_j G_{k_j} \otimes Y_{k_j} \right\|.$$

We have moreover that, for all $k_1 < k_2 < \ldots k_n$,

$$\left\| x + \sum_{j=1}^{n} a_j G_{k_j} \otimes Y_{k_j} + a_{n+1} G_m \otimes Y_m + \sum_{j > n+1} a_j \Gamma_j \right\| \tag{3}$$
$$\xrightarrow[m \to \infty]{} \left\| x + \sum_{j=1}^{n} a_j G_{k_j} \otimes Y_{k_j} + \sum_{j > n} a_j \Gamma_j \right\|$$

and this convergence is uniform for $\sum |a_k|^p \leq 1$. Note that in (3), the lefthand side is simply

$$\left\| x + \sum_{j=1}^{n} a_j G_{k_j} \otimes Y_{k_j} + a_{n+1} G_m \otimes Y_m + \left(\sum_{k > m+1} |a_k|^p \right)^{1/p} \Gamma_2 \right\|,$$

while the righthand side equals

$$\left\| x + \sum_{j=1}^{n} a_j G_{k_j} \otimes Y_{k_j} + a_{m+1} \Gamma_1 + \left(\sum_{k > m+1} |a_k|^p \right)^{1/p} \Gamma_2 \right\|.$$

If we choose n_{k+1} such that the difference of the two sides in (3) is less than $\varepsilon 2^{-k}$ (for all (a_k) with $\sum |a_k|^p \leq 1$), this procedure gives a subsequence $(G_{n_k} \otimes Y_{n_k})$ which is $(1+\varepsilon)/(1-\varepsilon)$ isomorphic to l_p. $\square$

2. Complemented hilbertian subspaces of r.i. spaces

The main result of this section is the following Theorem 4 on complemented embeddability of l_2 in an r.i. space. In the remainder of this section we associate to such a complemented hilbertian subspace conditionally Gaussian variables in both the spaces X and X', which will be the main tool in Sections 3 and 5.

We introduce two notations:

If X is an r.i. space over $[0, \infty)$, we denote by $X([0,1])$ its restriction to the interval $[0,1]$.

If f, g are two functions in X, resp. X', we denote by $\langle f, g \rangle$ the duality bracket, i.e., the integral $\int fg \, d\lambda$.

THEOREM 4. *Let X be an order continuous rearrangement invariant function space* (*over* $[0,1]$ *or* $[0,\infty)$), *not containing* l_2 *as complemented sublattice. Then X contains a complemented hilbertian subspace iff both X' and X'' contain a Gaussian variable*, *or equivalently* $\mathscr{G} \subset X([0,1]) \subset \mathscr{G}'$.

Proof of Theorem 4. The sufficiency of this condition results from the fact that it implies that Rademacher functions span a complemented hilbertian subspace (see [LT2],Thm. 2b4). Now we prove the necessity. By the proof of Proposition 1, it suffices to prove that both X and X' must contain an hilbertian subspace which is strongly embedded (i.e., on which the topology of X, resp. X', agrees with the $L_1(A)$-topology, relative to some integrable subset A of $[0,\infty)$). Assume that this is not the case. Consider a projection P: $X \to X$, the range of which is an hilbertian subspace, generated by a sequence $(g_n)_{n=1}^{\infty}$, which is equivalent to the natural l_2-basis. Since $X^* = X'$, the projection P takes the form

$$Pf = \sum_{n=1}^{\infty} \langle f, h_n \rangle g_n$$

where $(h_n)_{n=1}^{\infty}$ is a sequence in X' which is biorthogonal to the sequence $(g_n)_{n=1}^{\infty}(\langle g_n, h_m\rangle = \delta_{nm})$ and which is clearly equivalent to the l_2-basis. Suppose for instance that X' does not contain a strongly embedded hilbertian subspace. Then for all $N \geq 1$, there exists some $h \in \overline{\operatorname{span}}(h_k)_{k \geq N}$ such that $\|h\|_{X'} = 1$ and $\|\mathbf{1}_{[0,N]} h\|_{L_1} < 2^{-N}$. Proceeding inductively, we obtain a sequence of functions $\bar{h}_n = \sum_{j \in J_n} \alpha_j^{(n)} h_j$ in X', which are disjoint successive blocks of the h_j, satisfying $\|\bar{h}_n\|_{X'} = 1$, $\|\mathbf{1}_{[0,n]} h_n\|_{L_1} < 2^{-n}$. We have $\|\boldsymbol{\alpha}^{(n)}\|_2 := (\sum_{j \in J_n} |\alpha_j^{(n)}|^2)^{1/2} \sim 1$. Choose $\boldsymbol{\beta}^{(n)} := (\beta_j^{(n)})_j \in l_2$ with $\|\boldsymbol{\beta}^{(n)}\|_2 = \|\boldsymbol{\alpha}^{(n)}\|_2^{-1}$ and $\sum_j \beta_j^{(n)} \alpha_n^{(n)} = 1$, and set $\bar{g}_n := \sum_{j \in J_n} \beta_j^{(n)} g_j$. Then $\bar{P}$: $X \to X$: $\bar{P}f = \sum_n \langle f, \bar{h}_n \rangle \bar{g}_n$ is another projection onto an hilbertian subspace of X, with moreover $\bar{h}_n \to 0$ locally in measure as $n \to \infty$ (i.e., in measure on every integrable subset of $[0,\infty)$). Since the unit ball of every r.i. space is bounded in measure, so is the sequence $(\bar{g}_n)_n$, and hence $\bar{g}_n \bar{h}_n \to 0$ locally in measure as $n \to \infty$. On the other hand $\int \bar{g}_n \bar{h}_n \, d\lambda = \langle \bar{g}_n, \bar{h}_n \rangle = 1$. The same is true when X is supposed not to contain a strongly embedded hilbertian subspace.

So we can suppose w.l.o.g. that the sequence $(g_n . h_n)_n$ (which is bounded in L_1) is not L_1-equiintegrable. By passing if necessary to a subsequence, we can suppose that there exist a $\delta > 0$ and disjoint sets (A_n) so that: $\int_{A_n} |f_n| \, |g_n| \, d\lambda = c_n > \delta$. Now, the formula

$$Qf := \sum_n c_n^{-1} \big\langle f, \mathbf{1}_{A_n} |h_n| \big\rangle \mathbf{1}_{A_n} |g_n|$$

defines a bounded projection in X, whose range is a sublattice isomorphic to

l_2, which provides a contradiction. In fact, choosing unimodular elements $u, v \in L_\infty$ with $\mathbf{1}_{A_n}|h_n| = \mathbf{1}_{A_n}uh_n$, $\mathbf{1}_{A_n}|g_n| = \mathbf{1}_{A_n}vg_n$, we have

$$Qf = v\sum_n c_n^{-1}\mathbf{1}_{A_n}P(\mathbf{1}_{A_n}uf) = vR(uf)$$

Up to the coefficients c_n^{-1}, the operator R is "block-diagonally" extracted from P, hence bounded by a well-known argument due to Tonge (see [LT], Prop. 1c8), and so is Q. On the other hand the (unconditional) basic sequences $g_n' := \mathbf{1}_{A_n}g_n$ and $h_n' := \mathbf{1}_{A_n}h_n$ are dominated by the (l_2) sequences (g_n), resp. (h_n). For, denoting by (ε_n) a sequence of independent Bernouilli random variables, we have for every sequence (λ_n) of scalars (with finite support):

$$\begin{aligned}\Big\|\sum_n \lambda_n g_n'\Big\|_X &= \Big\|\mathbf{E}_\varepsilon\Big|\sum_n \varepsilon_n\lambda_n g_n'\Big|\Big\|_X \le \Big\|\mathbf{E}_\varepsilon\Big|\sum_n \varepsilon_n\lambda_n g_n\Big|\Big\|_X \\ &\le \mathbf{E}_\varepsilon\Big\|\sum_n \varepsilon_n\lambda_n g_n\Big\|_X \sim \Big\|\sum_n \lambda_n g_n\Big\|_X\end{aligned}$$

where the first inequality is a consequence of the pointwise inequality

$$\mathbf{E}_\varepsilon\Big|\sum_n \varepsilon_n\lambda_n g_n'\Big| \le \mathbf{E}_\varepsilon\Big|\sum_n \varepsilon_n\lambda_n g_n\Big|$$

(as long as $|g_n'| \le |g_n|$ pointwise), while the second one is simply the triangular inequality. Thus the sequences (g_n') and $(c_n^{-1}h_n')$ are both dominated by the l_2-basis, and since they are biorthogonal, they are in fact equivalent to the l_2-basis. □

PROPOSITION 5. *Let X be an r.i. space over $\Omega = [0,1]$ or $[0,\infty)$, satisfying the conditions of Theorem* 4. *For every projection P: $X \to X$ with hilbertian range E, there exist an l_2-basic sequence (x_n) in E and a biorthogonal l_2-basic sequence (x_n') in the range of P^* such that*:

(i) *the sequence (x_n) converges wcd to a conditionally gaussian variable $A \otimes G \in X''(\Omega \times [0,1])$;*

(ii) *the sequence (x_n') converges wcd to a conditionally gaussian variable $B \otimes G \in X'(\Omega \times [0,1])$;*

(iii) $\langle A, B\rangle > 0$.

Proof. The reasoning of the proof of Thm. 4 shows that if P is defined by $Pf = \sum_{n=1}^\infty \langle f, h_n\rangle g_n$, where (g_n) is a l_2-sequence, there exists an integer n_0 such that $(g_n)_{n\ge n_0}$ and $(h_n)_{n\ge n_0}$ span closed spaces of X, resp. X' whose topology coincides with that of some $L_1(A)$ (A of finite measure). By

Gaposhkin's Theorem there are sequences of successive disjoint l_2-normalized blocks x_n (resp. $B \otimes G$). We have $A \otimes G \in X''(\Omega \times [0,1])$ and $B \otimes G \in X'(\Omega \times [0,1])$. But in fact Gaposhkin's result is that, after extraction of a subsequence, every system of block coefficients $\boldsymbol{\alpha}^{(n)} := (\alpha_k^{(n)})_k$ with $\|\boldsymbol{\alpha}^{(n)}\|_2 = 1$ and $\|\boldsymbol{\alpha}^{(n)}\|_\infty \to 0$ gives rise to this wcd convergence. So we may take l_2-conjugate systems of block-coefficients for the x_n's and the x'_n's, and obtain that $\langle x_n, x'_n \rangle = \delta_{nm}$, i.e., these sequences are biorthogonal.

It is clear that we may suppose that $A, B \geq 0$, so to prove (iii), it suffices to prove that the functions A and B are not disjoint.

Suppose at the contrary that A and B are disjoint, i.e. that there exists a set $U \in \mathscr{A}$ such that $\mathbf{1}_{U^c} A = 0$ and $\mathbf{1}_U B = 0$. We then have

$$\mathbf{1}_{U^c} x_n \xrightarrow{\text{wcd}} 0 \quad \text{and} \quad \mathbf{1}_U x'_n \xrightarrow{\text{wcd}} 0;$$

a fortiori this convergence happens in distribution, hence in measure. Thus the L_1-bounded sequences $\mathbf{1}_U x_n x'_n$ and $\mathbf{1}_{U^c} x_n x'_n$ converge to zero in measure, but at least one of them does not converge to zero in norm (since $\int x_n x'_n \, d\lambda = 1$). As in the proof of Thm. 4, we can then exhibit two biorthogonal sequences (y_n) and (y'_n) in X resp. X', which are dominated by (x_n), resp. (x'_n), (hence equivalent to the l_2-basis), and give rise to a projection $X \to X$ whose range is a complemented hilbertian sublattice of X, a contradiction. □

Before the end of this section, and in close relation with Prop. 5, we give a result on the projection onto the span of conditionally independent Gaussian variables, which will be used in Sections 3 and 5.

PROPOSITION 6. *Let X be an r.i. function space over a product space $(\Omega \times S, \mathscr{A} \overline{\otimes} \Sigma, \mu \otimes \sigma)$. Let G be a normal Gaussian variable, and (G_n) a sequence of independent normal Gaussian variables in $L^0(S, \Sigma, \sigma)$.*

(a) *If there are A, B in $L^0(\Omega, \mathscr{A}, \mu)$ such that $A \otimes G \in X$, $B \otimes G \in X'$ with $\langle A, B \rangle = 1$, then the sequence $(A \otimes G_n)_n$ spans in X a C-complemented closed space, where $C = \|A \otimes G\|_X \|B \otimes G\|_{X'}$.*

(b) *Suppose that X is order-continuous. Then conversely if $A \in L^0(S, \Sigma, \sigma)$ is such that the sequence $(A \otimes G_n)_{n=1}^\infty$ spans in $X(\Omega \times S)$ a complemented closed subspace, then there exists a function $B \in L^0(\Omega, \mathscr{A}, \mu)$ such that $B \otimes G \in X'(\Omega \times \Sigma)$, $\langle A, B \rangle = 1$, and*

$$\|A \otimes G\|_X \|B \otimes G\|_{X'} \leq C.$$

In particular, X' contains a Gaussian variable.

Proof. (a) The sequences $(A \otimes G_n)$ and $(B \otimes G_n)$ span isometric copies of l_2 in X, resp X'. We define a projection R: $X \to X$ by: $Rf = \sum_{n=1}^\infty \langle f,$

$B \otimes G_n\rangle A \otimes G_n$. The norm of R is evaluated as follows:

$$\begin{aligned}
\|Rf\|_X &= \Big\| \sum_n \langle f, B \otimes G_n\rangle A \otimes G_n \Big\|_X \\
&= \Big(\sum_n |\langle f, B \otimes G_n\rangle|^2 \Big)^{1/2} \|A \otimes G\|_X \\
&= \Big(\sum_n \alpha_n \langle f, B \otimes G_n\rangle \Big) \|A \otimes G\|_X \quad \text{for some } \alpha_n \in \mathbf{R},\ \sum_n \alpha_n^2 = 1 \\
&= \Big\langle f, \sum_n \alpha_n B \otimes G_n \Big\rangle \|A \otimes G\|_X \\
&\le \|f\|_X \Big\| \sum_n \alpha_n B \otimes G_n \Big\|_{X'} \|A \otimes G\|_X \\
&= \|f\|_X \|B \otimes G\|_{X'} \|A \otimes G\|_X
\end{aligned}$$

Hence $\|Rf\| \le C\|f\|$.

(b) Now we prove the converse. We can w.l.o.g. suppose that (S, Σ, μ) is the product space $\mathbf{R}^{\mathbf{N}}$ equipped with the standard Gaussian measure γ, and that G_n is the nth coordinate map $\mathbf{R}^{\mathbf{N}} \to \mathbf{R}$. The orthogonal group $O(n)$ acts on $\mathbf{R}^n$, leaving the n-dimensional gaussian measure invariant; let us consider that $O(n)$ acts on $\mathbf{R}^{\mathbf{N}}$, by changing only the n first coordinates. Each element U of $O(n)$ gives rise to an isometry of X, again denoted by U, and defined by

$$U.f(\omega, (x_1, \dots, x_n), x_{n+1}, \dots) = f(\omega, U^*(x_1, \dots x_n), x_{n+1} \dots).$$

Note that if (u_{ij}) is the matrix of U (relatively to the natural basis) and $f = \sum_{j=1}^n \lambda_j \otimes G_j$ (where the λ_i are $\mathscr{A}$-measurable functions) then $U.f = \sum_i (\sum_j u_{ij} \lambda_j) \otimes G_i$.

Let $E = \overline{\operatorname{span}}[A \otimes G_n]_{n=1}^\infty$ and P be a given projection from X onto E, and set

$$R_n = \int_{O(n)} U^* P U \, d\sigma_n(U),$$

where σ_n is the normalized Haar measure on the compact group $O(n)$. R_n is clearly a projection onto E, invariant under the action of $O(n)$ (i.e., $\forall V \in O(n)$, $R_n V = V R_n$) and of norm $\|R_n\| \le \|P\|$. Note that $R_n R_m = R_m R_n = R_{m \wedge n}$.

The set $(R_n)_{n=1}^\infty$ is relatively compact in the weak operator topology (due to the reflexivity of E), and has a unique cluster point R (because if $f \in X$

depends only on ω and the n first coordinates, then $\forall m \geq n$, $R_m f = R_n f$), which is invariant under the action of $O(n)$ for all $n \in \mathbf{N}$. Let us write:

$$Rf = \sum_{i=1}^{\infty} \langle f, h_i \rangle A \otimes G_i$$

where $h_i \in X^*(\Omega \times \mathbf{R}^{\mathbf{N}})$ (note that $X' = X^*$ as X is supposed to be order continuous). Then for every $U \in O(n)$,

$$\begin{aligned} RUf &= \sum_{i=1}^{\infty} \langle Uf, h_i \rangle A \otimes G_i \\ &= \sum_{i=1}^{\infty} \langle f, U^* h_i \rangle A \otimes G_i = \sum_{i=1}^{\infty} \langle f, h_i \circ U \rangle A \otimes G_i \end{aligned}$$

and, on the other hand, (u_{ij}) being the matrix of U,

$$URf = \sum_{i=1}^{\infty} \left(\sum_j u_{ij} \langle f, h_j \rangle A \otimes G_j \right) = \sum_{i=1}^{\infty} \left\langle f, \sum_j u_{ij} h_j \right\rangle A \otimes G_j$$

whence we obtain

$$h_i \circ U = \sum_{j=1}^{n} u_{ij} h_j.$$

Considering the sequence $(h_i)_{i=1}^{\infty}$ as a measurable map $h\colon \Omega \times \mathbf{R}^{\mathbf{N}} \to \mathbf{R}^{\mathbf{N}}$, and associating to $U \in O(n)$ the bijection $\tilde{U}$ of $\Omega \times \mathbf{R}^{\mathbf{N}}$ acting only on the n first coordinateO]s in $\mathbf{R}^{\mathbf{N}}$ as U, we have thus the functional equation: $h \circ \tilde{U} = U \circ h$.

If U belongs to the subgroup $\Gamma_{1,n}$ of $O(n)$, whose elements leave the first coordinate inchanged, we obtain in particular: $h_1 = h_1 \circ \tilde{U}$, hence $h_1 = \int_{\Gamma_{1,n}} h_1 \circ \tilde{U} \, d\sigma_{1,n}(U)$, where $\sigma_{1,n}$ is the Haar measure on $\Gamma_{1,n}$, hence h_1 is clearly a (measurable) function of $\omega, x_1, \Sigma_{j=2}^{n} x_j^2, x_{n+1}, \ldots$. Let $\mathscr{F}_n$ be the measure-complete σ-algebra generated by $\mathscr{A}, G_1, \Sigma_{j=2}^{n} G_j^2, G_{n+1} \ldots$ and $\mathscr{F}_\infty = \bigcap_n \mathscr{F}_n$. Then h_1 is $\mathscr{F}_n$-measurable for all n, hence $\mathscr{F}_\infty$-measurable. But $\mathscr{F}_\infty = \bar{\sigma}(\mathscr{A}, G_1)$ (the measure-complete σ-algebra generated by $\mathscr{A}$ and G_1): for, if $f \in L_1$ is of the form $f = g \cdot \varphi(G_2, \ldots G_n)$, where $g \in L_1$ is $\sigma(\mathscr{A}, G_1)$-measurable, and φ is a continuous bounded function on $\mathbf{R}^n$, set $f_k = g \cdot \varphi(G_{2+kn}, \ldots G_{(k+1)n})$; then $\mathbf{E}^{\mathscr{F}_\infty} f_k = \mathbf{E}^{\mathscr{F}_\infty} f$ by symmetry; hence

$$\mathbf{E}^{\mathscr{F}_\infty} f = \mathbf{E}^{\mathscr{F}_\infty} \frac{f_0 + \cdots + f_k}{k} \xrightarrow[k \to \infty]{} g \mathbf{E} \varphi(G_2, \ldots G_n)$$

by the law of large numbers applied to the $f_k = \varphi(G_{2+kn}, \ldots G_{(k+1)n})$.

Finally $h_1(\omega, x_1, \dots) = H(\omega, x_1)$ a.e., and, using the relation $h \circ \tilde{U} = U \circ h$ for U the orthogonal symmetry exchanging the coordinate x_1 and x_k, we have

$$h_k(\omega, x_1, \dots x_k \dots) = H(\omega, x_k) \text{ a.e. } (\forall k \geq 2).$$

Using now the relation $h \circ \tilde{U} = U \circ h$ for U being the central symmetry, we obtain

$$(1) \qquad H(\omega, -x) = -H(\omega, x) \text{ for a.e. } (\omega, x);$$

and if U is the transformation

$$(x_1, x_2) \mapsto \left(\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_1 - x_2}{\sqrt{2}} \right)$$

we obtain

$$(2) \qquad H\left(\omega, \frac{x_1 \pm x_2}{\sqrt{2}}\right) = \frac{H(\omega, x_1) \pm H(\omega, x_2)}{\sqrt{2}} \quad \text{for a.e. } \omega, x_1, x_2.$$

Then H is a.e. equivalent to a function $\tilde{H}$, whose partial functions $\tilde{H}_\omega$: $x \mapsto \tilde{H}(\omega, x)$ are of class C^1 for a.e. ω. (if φ is a centrally symmetric C^1 function with compact support and integral 1, it is straightforward that $H_\omega(x) = \sqrt{2} H_\omega * \varphi(x/\sqrt{2})$ for a.e. (ω, x)). Then, for a.e. $\omega, \tilde{H}$ verifies (1) and (2) for all x, x_1, x_2, and by a standard reasoning, $\tilde{H}(\omega, x) = B(\omega) \cdot x$.

Coming back to the projection R, we see that it can be written as

$$Rf = \sum_{i=1}^{\infty} \langle g, B \otimes G_i \rangle A \otimes G_i$$

Note that $A \in X$, $B \in X'$, with $\|A\|_X \leq \|A \otimes G\|_X / \mathbf{E}|G|$ and $\|B\|_{X'} \leq \|B \otimes G\|_{X'} / \mathbf{E}|G|$. From $R(A \otimes G_i) = A \otimes G_i$ we obtain $\langle A, B \rangle = 1$. □

3. Quantitative version of the preceding results

We say that the space l_2 is *C-representable as complemented subspace of X* if the identity map of l_2 is C-factorizable through X, i.e., there exist linear operators i: $l_2 \to X$ and π: $X \to l_2$, such that $\pi \circ i = \mathrm{id}_{l_2}$ and $\|\pi\| \, \|i\| \leq C$.

In this section we prove the following improvement of Theorem 4 (with a slight reinforcement of the hypotheses on the r.i. space):

THEOREM 7. *Let X be a rearrangement invariant function space, not containing c_0 as subspace nor l_2 as complemented sublattice. Then l_2 is C-representable in X as complemented subspace iff there exist variables $A \otimes G$,* *resp.* $B \otimes G$ *in* $X(\Omega \times [0,1])$, *resp.* $X'(\Omega \times [0,1])$ *where* G *is a normalized Gaussian variable*, $\|A \otimes G\|_X \|B \otimes G\|_{X'} \leq C$ *and* $\langle A, B\rangle = 1$.

The proof of Theorem 7 involves several lemmas. We give first some preliminary material.

A sequence $(x_k)_k$ in X is *X-equiintegrable* iff it satisfies the conditions

$$(4) \qquad \lim_{M\to\infty} \sup_k \left\| x_k \mathbf{1}_{\{|x_k|>M\}} \right\|_X = 0; \qquad \inf_{\mu(A)<\infty} \sup_k \| \mathbf{1}_{A^c} x_k \|_X = 0.$$

As is well known (see for instance [W]) in an r.i. space not containing c_0, every sequence (x_n) has a subsequence x_{n_k} for which there is a splitting $x_{n_k} = x'_k + x''_k$, where each x'_k is disjoint from the corresponding x''_k, the elements x''_k are disjoint and the sequence $(x'_k)_k$ is X-equiintegrable.

We shall also use repeatedly in the subsequent proofs the following well known fact, which we will call the "Bessaga-Pelczynski perturbation principle": if (x_n) is a basic unconditional sequence which spans a complemented subspace in X, and (y_n) is a sequence in X such that $\|x_n - y_n\| \to 0$, then a subsequence (y_{n_k}) is equivalent to (x_{n_k}), and spans a subspace which is complemented in X. In fact, if $P\colon X \to \overline{\mathrm{span}}[x_n]$ is a given projection, and $\pi\colon \overline{\mathrm{span}}[x_n] \to \overline{\mathrm{span}}[x_{n_k}]$ is the natural projection, then, if (n_k) is sufficiently lacunary, the restriction J of πP to $F = \overline{\mathrm{span}}[y_{n_k}]$ is an isomorphism onto $E = \overline{\mathrm{span}}[x_{n_k}]$, and $Q = J^{-1}\pi P$ is a projection. Note that this construction is of almost isometric nature, i.e. we can obtain that the tails $(y_{n_k})_{k\geq m}$ are $(1+\varepsilon_m)$-equivalent to $(x_{n_k})_{k\geq m}$ (with $\varepsilon_m \to 0$) and, if (x_n) is K-suppression unconditional, the projection Q_m onto $F_m = \overline{\mathrm{span}}[y_{n_k}]_{k\geq m}$ to be of norm $\leq (1+\varepsilon_m)\|P\|K$.

In the same spirit we state the following very elementary fact, in order to avoid further repetitions:

LEMMA 8. *Let F be a complemented hilbertian subspace of X, $P\colon X \to F$ a projection, $S\colon F \to X$ a bounded operator. If $PS\colon F \to F$ is an (into) isomorphism, then SF is itself a complemented hilbertian subspace of X (and S an isomorphism).*

Proof. $S|_F$ is an isomorphism, so $G = SF$ is hilbertian; $J = P|_G$ is an isomorphism from G into F. Let π be a projection $F \to P(G)$. Then $Q = J^{-1}\pi P$ is a projection $X \to G$. □

The following lemma precises Proposition 5 in the case $X \not\supset c_0$, and will be given a quantitative version by Lemma 12:

LEMMA 9. *Let X satisfy the hypotheses of Theorem 7. Then for every complemented hilbertian subspace E of X there is a sequence which is X-equiin-*

tegrable, converges wcd to a conditionally Gaussian variable and spans a complemented hilbertian subspace, and is arbitrarily close in measure to E.

Proof. Let $(x_n)_n$ be an l_2-basis of E, and P be the projection onto E.

First we may suppose that the elements of E live on a μ-finite subset U of Ω: if there is no μ-integrable subset U such that the norm of $X(\Omega)$ and that of $X(U)$ are equivalent on U, then using the order continuity of X we construct recursively a sequence y_n of disjoint blocks on the basis x_n and a sequence of disjoint elements y'_n of X such that $\|y_n - y'_n\| \to 0$; by the Bessaga-Pelczynski perturbation principle, a subsequence of (y'_n) spans a complemented hilbertian space, a contradiction. Then, reasoning as in §1, we may suppose that the norms of X and of $L_1(U)$ are equivalent on E, and that the x_n converge (weakly conditionally in distribution) to a conditionally Gaussian variable.

By passing if necessary to a subsequence we have a disjoint splitting $x_k = x'_k + x''_k$, where the elements x''_k are disjoint and the sequence $(x'_k)_k$ is X-equiintegrable.

Let E', resp. E'' be the closed subspaces of X spanned by the sequence $(x'_k)_k$, resp. $(x''_k)_k$. If $x = \sum_{k=1}^n \alpha_k x_k$, set $S'x = \sum_k \alpha_k x'_k$ and $S''x = \sum_k \alpha_k x''_k$; note that $x = S'x + S''x$, and that S' and S'' extend to bounded operators from E to E', resp. E to E'', since the disjoint basic sequence (x''_n) is dominated by (x_n); i.e., $\|\sum_k \alpha_k x''_k\| \le C \, \|\sum_k \alpha_k x_k\|$ where C is the unconditionality constant of $(x_n)_n$ (see the proof of Thm. 4). Then there exists M and a subspace E_1 of E of finite codimension (spanned by the x_k, $k \ge N_1$) such that $\forall x \in E_1$, $\|x\| \le M\|PS'x\|$. For if not, there is a sequence $(y_n)_n$ in E, with $\|y_n\| = 1$, $\|PS'y_n\| \to 0$ and $y_n \to 0$ weakly. Note that $y_n - PS''y_n = Py_n - PS''y_n = PS'y_n \to 0$. Again, using the Bessaga-Pelczynski perturbation principle, we find a subsequence $(z_n)_n$ of (y_n), such that the sequence $(PS''z_n)_n$ is basic and equivalent to (z_n). In particular $\|PS''z\| \ge \delta\|z\|$ for all $z \in \overline{\operatorname{span}}[z_n]$. Since $\overline{\operatorname{span}}[z_n]$ is complemented in E, we deduce by Lemma 5 that $\overline{\operatorname{span}}[S''z_n]$ is complemented in X, and $(S''z_n)$ is equivalent to the l_2-basis. Hence X contains l_2 as complemented sublattice, a contradiction.

Thus we find E_1 on which $\|y\| \simeq \|Py'\|$, hence $\simeq \|y'\|$. Let $E'_1 := \{y'/y \in E_1\}$. Then E_1 is hilbertian, and moreover is complemented by Lemma 8. Note also that we have

$$x''_n \xrightarrow[n\to\infty]{\text{dist}} 0.$$

So $(x''_n)_n$ converges to zero in measure, and $(x'_n)_n$ converges wcd to a conditionally Gaussian variable (the same as for $(x_n)_n$). □

The key for obtaining a quantitative version of Lemma 9 is the following:

LEMMA 10. *Let X be an order continuous r.i. function space and $(x_n)_n$ a X-equiintegrable sequence which converges wcd to a non-zero conditionally* *gaussian variable. Then for every $\varepsilon > 0$ there exists a subsequence $(x_{n_i})_i$ which is $(1 + \varepsilon)$ equivalent to the l_2 basis and such that the unit ball of $F = \overline{\operatorname{span}}[x_{n_i}]_i$ is X-equiintegrable.*

Proof. Choose a sequence of reals $\varepsilon_n > 0$ with $\sum_{n=1}^{\infty}\varepsilon_n \leq \varepsilon$.

The sequence (x_n) converges wcd to $A \otimes G$, where $G \in L_0([0,1])$ is a normal gaussian variable and $A \in L_0(\Omega)$ is such that $A \otimes G \in X(\Omega \times [0,1])$. We may suppose that $L_0([0,1])$ contains an auxiliary normal Gaussian variable G', independent from G.

Throughout the proof of this lemma, if $x \in X$, we denote by x^* the non-increasing rearrangement of $|x|$; this is an element of $X([0,\infty))$ or $X([0,1])$.

Now we construct a subsequence $(x_{n_i})_{i \geq 1}$ of $(x_n)_n$ such that for all k we have the following property, denoted by (H_k):

$$\left\| \left(\sum_{i=1}^{k} \lambda_i x_{n_i} + \rho A \otimes G \right)^* - \left(\sum_{I=1}^{k} \lambda_i^2 + \rho^2 \right)^{1/2} (A \otimes G)^* \right\|_X$$
$$\leq \left(\sum_{i=1}^{k} \varepsilon_i \right) \left(\sum_{i=1}^{k} \lambda_i^2 + \rho^2 \right)^{1/2} \|A \otimes G\|_X \quad \forall \lambda_1, \dots \lambda_k, \rho \in \mathbf{R}.$$

Suppose that we have found the first k terms $n_1 < n_2 < \dots n_k$ (possibly $k = 0$). For every $\lambda_1, \dots \lambda_k, \rho \in \mathbf{R}$ we have

$$\sum_{i=1}^{k} \lambda_i x_{n_i} + \rho A \otimes G' + \lambda_{k+1} x_n \xrightarrow[n \to \infty]{\text{dist}} \sum_{i=1}^{k} \lambda_i x_{n_i} + \rho A \otimes G' + \lambda_{k+1} A \otimes G$$

which has the same distribution as

$$\sum_{i=1}^{k} \lambda_i x_{n_i} + \left(\lambda_{k+1}^2 + \rho^2\right)^{1/2} A \otimes G.$$

This implies the convergence Lebesgue-a.e. of the non-increasing rearrangements:

$$\left(\sum_{i=1}^{k} \lambda_i x_{n_i} + \rho A \otimes G' + \lambda_{k+1} x_n \right)^*$$
$$\xrightarrow{\text{a.e.}} \left(\sum_{i=1}^{k} \lambda_i x_{n_i} + \left(\lambda_{k+1}^2 + \rho^2\right)^{1/2} A \otimes G \right)^*.$$

As the sequence (x_n) is X-equiintegrable, we deduce that this convergence holds also in the sense of the X-norm. Now let

$$F_n(\lambda_1,\dots\lambda_{k+1},\rho) = \left(\sum_{i=1}^{k}\lambda_i x_{n_i} + \rho A\otimes G' + \lambda_{k+1}x_n\right)^*$$

$$F_\infty(\lambda_1,\dots\lambda_{k+1},\rho) = \left(\sum_{i=1}^{k}\lambda_i x_{n_i} + \left(\lambda_{k+1}^2+\rho^2\right)^{1/2} A\otimes G\right)^*.$$

The $X([0,\infty))$ valued functions F_n are equicontinuous on the compact set

$$K_{k+2} = \left\{\left(\sum_{i=1}^{k+1}\lambda_i^2+\rho^2\right)^{1/2} \le 1\right\} \subset \mathbf{R}^{k+2};$$

this is a straightforward consequence of the Lipschitz inequality of Lorentz-Shimogaki [LS],

$$\forall u,v\in X,\quad \|u^*-v^*\|_X \le \|u-v\|_X,$$

which is true for every r.i. space. Thus the convergence $F_n \to F_\infty$ holds uniformly on K_{k+2}. We choose $n_{k+1} > n_k$ such that $\|F_{n_{k+1}} - F_\infty\|_\infty \le \varepsilon_{k+1}\|A\otimes G\|_X$, and we obtain

$$\left\|\left(\sum_{i=1}^{k}\lambda_i x_{n_i} + \rho A\otimes G' + \lambda_{k+1}x_n\right)^* - \left(\sum_{i=1}^{k}\lambda_i x_{n_i} + \left(\lambda_{k+1}^2+\rho^2\right)^{1/2} A\otimes G\right)^*\right\|_X \le \varepsilon_{k+1}\left(\sum_{i=1}^{k}\lambda_i^2+\rho^2\right)^{1/2}\|A\otimes G\|_X$$

This inequality together with (H_k) implies (H_{k+1}). Now if $\Sigma_i\lambda_i^2<\infty$, we apply (H_k) with $\rho=0$, pass to the limit on k, and obtain finally

$$\left\|\left(\sum_{i=1}^{\infty}\lambda_i x_{n_i}\right)^* - \left(\sum_{i=1}^{\infty}\lambda_i^2\right)^{1/2}(A\otimes G)^*\right\|_X \le \varepsilon\left(\sum_{i=1}^{\infty}\lambda_i^2\right)^{1/2}\|A\otimes G\|_X.$$

This implies that the sequence (x_{n_i}) is $(1+\varepsilon)$-equivalent to the l_2 basis.

Moreover, the proof of this inequality gives in fact that

$$\left\| \left(\sum_{i=k}^{\infty} \lambda_i x_{n_i} \right)^* - \left(\sum_{i=k}^{\infty} \lambda_i^2 \right)^{1/2} (A \otimes G)^* \right\|_X \leq \delta_k \left(\sum_{i=k}^{\infty} \lambda_i^2 \right)^{1/2} \|A \otimes G\|_X$$

where $\delta_k = \sum_{i=k}^{\infty} \varepsilon_i$. This implies that, if $(\sum_{i=1}^{\infty} \lambda_i^2)^{1/2} \leq 1$,

$$\left\| \mathbf{1}_U \left(\sum_{i=1}^{\infty} \lambda_i x_{n_i} \right) \right\|_X \leq \left\| \mathbf{1}_U \left(\sum_{i=1}^{k} \lambda_i x_{n_i} \right) \right\|_X + \delta_{k+1} \|A \otimes G\|_X$$
$$+ \left\| \mathbf{1}_{[0, \mu(U)]} (A \otimes G)^* \right\|_X \quad \forall U \subset \Omega .$$

In the case where $\mu(\Omega) = 1$, this implies that the unit ball of $F = \overline{\operatorname{span}}[x_{n_i}]$ is X-equiintegrable. In the case $\mu(\Omega) = \infty$, we may suppose that μ is σ-finite; let $(\Omega_p)_p$ be an increasing sequence of μ-integrable subsets whose union is Ω. Note that each sequence $(\mathbf{1}_{\Omega_p^c} x_n)_n$ is X-equiintegrable and wcd converging to $\mathbf{1}_{\Omega_p^c} A \otimes G$. Then a diagonal argument allows us to obtain a subsequence (x_{n_i}) such that for every k, p with $k \geq p$,

$$\left\| \left(\sum_{i=k}^{\infty} \lambda_i \mathbf{1}_{\Omega_p^c} x_{n_i} \right)^* - \left(\sum_{i=k}^{\infty} \lambda_i^2 \right)^{1/2} \left(\mathbf{1}_{\Omega_p^c} A \otimes G \right)^* \right\|_X \leq \delta_k \left(\sum_{i=k}^{\infty} \lambda_i^2 \right)^{1/2} \|\mathbf{1}_{\Omega_p^c} A \otimes G\|_X$$

Hence if $(\sum_{i=1}^{\infty} \lambda_i^2)^{1/2} \leq 1$ and $q \geq p$:

$$\left\| \mathbf{1}_{\Omega_q^c} \sum_{i=1}^{\infty} \lambda_i x_{n_i} \right\| \leq \sum_{i=1}^{p} |\lambda_i| \, \|\mathbf{1}_{\Omega_q^c} x_{n_i}\| + \left\| \mathbf{1}_{\Omega_p^c} \sum_{i=p}^{\infty} \lambda_i x_{n_i} \right\|$$
$$\leq \sqrt{p} \bigvee_{i=1}^{p} \|\mathbf{1}_{\Omega_q^c} x_{n_i}\| + (1 + \varepsilon) \|\mathbf{1}_{\Omega_p^c} A \otimes G\|_X$$

Letting $q \to \infty$ and then $p \to \infty$ we obtain the second condition in (4) for the X-equiintegrability of the sequence (x_{n_i}). □

Remark 11. We can choose the subsequence (x_{n_i}) such that every normalized weakly null sequence (z_l) in $\overline{\operatorname{span}}[x_{n_i}]$ converges wcd to the same conditionally Gaussian variable.

It is sufficient to prove this fact when (z_l) is a sequence of successive l_2-normalized blocks on the x_{n_i} (since every weakly null sequence (z_l) has a subsequence (z_{l_i}) which can be approximated in X-norm, and a fortiori weakly conditionally in distribution, by such a sequence of disjoint successive blocks on the basis (x_n)).

It is clear by the preceding proof that for each $V \in \mathscr{A}$, we can choose the subsequence (x_{n_i}) such that for all $k \geq k_V$, for every block z built on the x_{n_i}, $i \geq k$, we have

$$\|(\mathbf{1}_V z)^* - \|z\|_2(\mathbf{1}_V A \otimes G)^*\| \leq \delta_k \|z\|_2 \|\mathbf{1}_V A \otimes G\|.$$

By a diagonal argument, this can be done for all V in a countable subset $\Gamma$of $\mathscr{A}$. If the measure space $(\Omega, \mathscr{A}, \mu)$ is separable, this shows that for every $V \in X$, and every sequence of successive disjoint l_2-normalized blocks (z_l) on the (x_{n_i}),

$$\mathbf{1}_V z_l \xrightarrow{\text{dist}} \mathbf{1}_V A \otimes G,$$

which shows that

$$z_l \xrightarrow{\text{wcd}} A \otimes G.$$

In the non-separable case, use the fact that the x_n live in a separable sublattice $X(\Omega, \mathscr{B}, \mu)$ and that X is rearrangement invariant (V can be put down into a fixed separable superspace $(\Omega, \mathscr{C}, \mu)$ of $(\Omega, \mathscr{A}, \mu)$ by a measure-preserving transformation leaving elements of $\mathscr{B}$ invariant). $\square$

LEMMA 12. *Let X satisfy the hypotheses of Theorem 7. If l_2 is C-representable as a complemented subspace of X, then for every $\varepsilon > 0$ there is a special $C(1 + \varepsilon)$-factorization of the identity of l_2 through X which maps the basis of l_2 onto a sequence which is X-equiintegrable and converges wcd to a conditionally Gaussian variable.*

Proof. (A) Let

$$l_2 \xrightarrow{i} X \xrightarrow{\pi} l_2$$

be a factorisation of id_{l_2} through X, with $\|\pi\| \|i\| \leq C$. Then $E = i(l_2)$ is a complemented hilbertian subspace of X. We show first that we may suppose that the l_2 basis of E converges wcd to a conditionally Gaussian variable. As at beginning of the proof of Lemma 9, we find in E a normalized sequence $x_n = i(y_n)$ which converges wcd to a conditionally gaussian variable. Strictly speaking, this wcd convergence of (x_n) was obtained only on a certain μ-integrable set U, such that the norms of X and that of $L_1(U)$ are equivalent on E_0; but, if $(\Omega_p)_p$ is an increasing sequence of μ-integrable subsets containing U, whose union contains all the supports of the x_n, we can by a diagonal argument construct the sequence (x_n) converging wcd on each set Ω_p to a conditionally Gaussian variable. Then the sequence $(x_n)_n$ is wcd convergent on the whole of Ω to a conditionally Gaussian variable. The y_n's may be taken as successive norm one blocks on the basis of l_2, hence forming isometrically a l_2-basis. Let $H = \overline{\operatorname{span}}[y_n]_{n=1}^{\infty}$ and Q be the orthogonal projection of l_2 onto H. Then $\mathrm{id}_H = (Q \circ \pi) \circ i|_H$ gives a C-representation of H (itself isometric to l_2) as complemented subspace of X, with l_2 basis (x_n).

(B) Note that every subsequence of (x_n) gives raise to a C-representation of l_2 as complemented subspace of X. Thus we may suppose that we have the splitting $x_n = x_n' + x_n''$ into X-equiintegrable and disjoint part. Let $E = \overline{\operatorname{span}}[x_n]$, $E' = \overline{\operatorname{span}}[x_n']$, $E'' = \overline{\operatorname{span}}[x_n'']$ and $S'\colon E \to E'$, resp. $S''\colon E \to E''$ the natural operators ($S'x_n = x_n'$, $S''x_n = x_n''$). By the proof of Lemma 9, we may suppose that (x_n') converges wcd to a non-zero conditionally Gaussian variable. After extracting a subsequence if necessary, we may suppose (by Lemma 10 and Remark 11) that the unit ball of E' is X-equiintegrable, and that every weakly null sequence in E' converges wcd to a conditionally gaussian variable. As the x_n'' are asymptotically disjoint from the unit ball of E', (due to the X-equiintegrability of this unit ball), it is easy to see that the restriction of S' and S'' to $E_n = \overline{\operatorname{span}}[x_k]_{k \geq n}$ satisfy $\|S'|_{E_n}\| \leq 1 + \varepsilon(n)$, $\|S''|_{E_n}\| \leq 1 + \varepsilon(n)$, with $\varepsilon(n) \to 0$ as $n \to \infty$. Given $\varepsilon > 0$, we may suppose $\varepsilon(n) \leq \varepsilon$.

Let

$$l_2 \xrightarrow{\ i\ } X \xrightarrow{\ \pi\ } l_2$$

be a C-factorization of id_{l_2} through X, the image by i of the natural basis (e_n) of l_2 being (x_n). Let $P = i\pi$ be the induced projection from X onto $E = i(l_2)$. Due to Lemma 8, and the hypothesis on X, the operator PS'' is strictly singular. Thus there exists a sequence (u_n) of successive normalized blocks on the basis (e_n) such that $PS''y_n \to 0$ as $n \to \infty$, where $y_n = i(u_n)$. Since i is an isomorphism into X, we have in fact $\pi S''y_n \to 0$ as $n \to \infty$. But $u_n - \pi S'i(u_n) = \pi(y_n) - \pi S'y_n = \pi S''y_n \to 0$, hence (after extraction) we can suppose that the restriction to $H = \overline{\operatorname{span}}[u_n]$ of the operator $I - \pi S'i$ is of norm $\leq \varepsilon$. Let Q be the orthogonal projection from l_2 onto H. We have

$$\|(I - Q\pi S')|_H\| = \|(Q(I - \pi S'))|_H\| \leq \varepsilon.$$

Then $J = Q\pi S'i|_H$ is invertible, and $\|J^{-1}\| \leq 1/(1 - \varepsilon)$. Set $i' = S'J^{-1}\colon H \to X$ and $\pi' = Q\pi\colon X \to H$. We have $\pi'i' = \mathrm{id}_H$ and

$$\|i'\|\,\|\pi'\| \leq (1 - \varepsilon)^{-1}\|S'\|\,\|i\|\,\|\pi\| \leq C(1 - \varepsilon)^{-1}(1 + \varepsilon).$$

Finally $(z_n) = (S'iJ^{-1}u_n)$ provides a $C(1 - \varepsilon)^{-1}(1 + \varepsilon)$ representation of l_2 as a complemented subspace of X, and by the choice of (x_n), the sequence (z_n) is X-equiintegrable and converges wcd to a conditionally Gaussian variable. □

LEMMA 13. *Under the hypotheses of Theorem 7, if l_2 is C-representable as a complemented subspace of X, there exists a sequence $(A \otimes G_n)_n$ (of conditionally i.i.d. gaussian variables in $X(\Omega \times [0,1])$) whose closed linear span is C-complemented in X.*

Proof. (A) We make first a little digression about complemented spreading models in ultrapowers.

Let X be a Banach space and $(x_k)_{k=1}^{\infty}$ a sequence without converging subsequence. Let $\mathscr{U}$ be a non trivial ultrafilter over the index set $\mathbf{N}$.

The sequence $(x_k)_k$ defines an element ξ_1 of the ultrapower $\tilde{X}_1 = X^{\mathbf{N}}/\mathscr{U}$, then an element ξ_2 of $\tilde{X}_2 = X_1^{\mathbf{N}}/\mathscr{U}, \ldots$ We define recursively a sequence $(\tilde{X}_k)_k$ of successive ultrapowers and a sequence $(\xi_k)_k$, $\xi_k \in \tilde{X}_k$. Thus we have

$$\|\alpha_1\xi_1 + \cdots + \alpha_k\xi_k\|_{\tilde{X}_k} = \lim_{n_k, \mathscr{U}} \lim_{n_{k-1}, \mathscr{U}} \cdots \lim_{n_1, \mathscr{U}} \|\alpha_1 x_{n_1} + \cdots + \alpha_k x_{n_k}\|_X$$
$$\forall \alpha_1, \ldots \alpha_k \in \mathbf{R}.$$

The whole sequence $(\xi_k)_k$ can be considered as living in the same space $\tilde{X} = \prod_k \tilde{X}_k/\mathscr{U}$ (which is an ultrapower of X) and spans in $\tilde{X}$ the so-called spreading model associated to the sequence $(x_k)_k$ and the ultrafilter $\mathscr{U}$.

Suppose now that $(x_n)_n$ is the image of a 1-symmetric basis $(e_n)_n$ of a reflexive space Z under a C-factorization of id_Z through X. For each k, set $Z_k = \mathrm{span}[e_1, \ldots e_k]$; for each multiindex $(n_1, \ldots n_k) \in \mathbf{N}^k$, let $\sigma_{n_1,\ldots n_k}\colon Z_k \to Z$ be defined by $\sigma_{n_1,\ldots n_k}(e_l) = e_{n_l}$; set $Z_{n_1,\ldots,n_k} = \sigma_{n_1,\ldots n_k}(Z_k)$ and let $r_{n_1,\ldots,n_k}$ be the natural projection $Z \to Z_{n_1,\ldots,n_k}$. Set

$$\pi_{n_1,\ldots n_k} = \sigma_{n_1,\ldots n_k}^{-1} \circ r_{n_1,\ldots,n_k} \circ \pi \quad \text{and} \quad i_{n_1,\ldots n_k} = i \circ \sigma_{n_1,\ldots n_k};$$

we obtain a factorization $\pi_{n_1,\ldots n_k} \circ i_{n_1,\ldots n_k}$ of id_{Z_k} through $\mathrm{span}[x_{n_1}, \ldots x_{n_k}]$ with $\|\pi_{n_1,\ldots n_k}\| \, \|i_{n_1,\ldots n_k}\| \leq C$. Thus we have a C-factorization $\tilde{\pi}_k \circ \tilde{\imath}_k$ of id_{Z_k} through $\tilde{X}_k$ by setting

$$\tilde{\imath}_k z = \text{class of the family } \left(i_{n_1,\ldots n_k} z\right)_{(n_1,\ldots n_k) \in \mathbf{N}^k}$$

and

$$\tilde{\pi}_k \tilde{x} = \lim_{n_k, \mathscr{U}} \lim_{n_{k-1}, \mathscr{U}} \cdots \lim_{n_1, \mathscr{U}} \pi_{n_1,\ldots,n_k}(x_{n_1,\ldots n_k})$$

when $\tilde{x}$ is the class of the family $(x_{n_1,\ldots n_k})_{(n_1,\ldots n_k) \in \mathbf{N}^k}$. (Here the limits are norm limits in finite dimensional spaces.) The image by $\tilde{\pi}_k$ of the basis of Z_k is the sequence $(\xi_1, \ldots \xi_k)$. We thus obtain a C-factorization $\tilde{\pi} \circ \tilde{\imath}$ of id_Z through $\tilde{X}$ by letting $\tilde{\imath}\tilde{z}$ be class of the family $(\tilde{\imath}_k r_{1,2,\ldots k} z)_k$ and $\tilde{\pi}\tilde{x} =$

$\lim_{k,\mathscr{U}} \tilde{\pi}_k \tilde{x}_k$ when $\tilde{x}$ is the class of the family $(\tilde{x}_k)_k$, where the limits are taken, say, coordinatewise. The image of the basis of Z by $\tilde{\pi}$ is the "fundamental sequence of the spreading model" $(\xi_n)_n$.

(B) Now we use this ultrapower construction starting with the hilbertian X-equiintegrable sequence given by Lemma 12. In this case the elements $(\xi_k)_{k=1}^{\infty}$ (constructed above) live in the band $\tilde{X}_{eq}$ of the ultrapower $\tilde{X}$ whose elements are defined by X-equiintegrable families of elements of X. As is well known, this band is nothing but a space $X(\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{\mu})$, where $(\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{\mu})$ is (a Stone representation of) the ultrapower of the measure space $(\Omega, \mathscr{A}, \mu)$ (or, equivalently, $L_1(\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{\mu}) = [L_1(\Omega, \mathscr{A}, \mu)]_{eq}^{\sim}$). Moreover, the distribution of the sequence $(\xi_k)_k$ is the limit of that of the (x_k); to be more explicit,

$$(x, x_{n_1}, x_{n_2}, \dots x_{n_k}) \xrightarrow[n_1, \mathscr{U}; n_2, \mathscr{U}; \dots n_k, \mathscr{U}]{\text{dist}} (x, \xi_1, \dots \xi_k) \quad \forall x \in X,$$

where the convergence in distribution is evaluated against bounded continuous functions on $\mathbf{R}^{k+1}$ [DC]. But we have

$$(x_{n_1}, x_{n_2}, \dots x_{n_k}) \xrightarrow[n_1 \to \infty; n_2 \to \infty; \dots n_k \to \infty]{\text{wcd}} (Y_1, Y_2, \dots Y_k)$$

where the $Y_j \in L_1(\Omega \times S)$ are conditionally Gaussian, independent and identically distributed. Hence the sequence $(\xi_k)_k$ is conditionally equivalent in distribution to a sequence of conditionally independent equidistributed Gaussian variables, i.e.,

$$\forall x \in X, \forall k \in \mathbf{N}\colon (x, \xi_1, \dots \xi_k) \overset{\text{dist}}{\sim} (x, A \otimes G_1, A \otimes G_2, \dots A \otimes G_k)$$

where $A \in L_0(\Omega, \mathscr{A}, \mu)$ and $(G_k)_k$ is a sequence of independent normal Gaussian variables, defined on $([0,1], \lambda)$. We can suppose that $A > 0$ on a subset U of Ω. Let $\mathscr{B}_k$ be the λ complete σ-algebra generated by the variable G_k and $\mathscr{C}$ that generated by $\mathscr{A}_U$ (the trace of $\mathscr{A}$ on U) and the variables $\xi_1, \dots \xi_k$. There is an isomorphism $\mathscr{T}$ of measure algebras from $(\mathscr{C}, \tilde{\mu})$ onto

$$\left(\mathscr{A}_U \,\overline{\otimes}\, \mathscr{B}_1 \dots \overline{\otimes}\, \mathscr{B}_k \,\overline{\otimes} \dots, \mu \otimes \lambda \dots \otimes \lambda \dots\right),$$

which is the identity on $\mathscr{A}$ and maps ξ_j on $A \otimes G_j$. This isomorphism $\mathscr{T}$ generates a lattice isometry from the space $X(\mathscr{C}, \mu)$ onto

$$X\left(U \times [0,1]^{\mathbf{N}}, \mathscr{A}_U \otimes \bigotimes_k \mathscr{B}_k, \mu \otimes \lambda^{\otimes \mathbf{N}}\right);$$

hence the sequence $(Y_k)_k = (A \otimes G_k)_k$ spans a complemented closed space in the last space. By a standard isomorphism theorem, we may replace $[0,1]^{\mathbf{N}}$ by $[0,1]$.

(C) More precisely we have $Y_k = \pi(e_k)$ for a certain $C(1+\varepsilon)$-factorization $\pi \circ i$ of id_{l_2} through $X(\Omega \times [0,1])$. As (Y_n) is isometrically equivalent to the basis of l_2, this means that $\overline{\mathrm{span}}[Y_n]_n$ is $C(1+\varepsilon)$-complemented in $X(\Omega \times [0,1])$. A more careful treatment would show that the tail $(z_n)_{n \geq m}$ of the sequence $(z_n)_n$ constructed in the proof of Lemma 12 provides in fact a $C(1+\varepsilon_m)$-representation of l_2 as complemented subspace of X, with $\varepsilon_m \to 0$ as $m \to \infty$; as a consequence, the sequence $(Y_l)_l$ spans in fact a C-complemented space. □

Proof of Theorem 7. Theorem 7 is now an immediate consequence of Lemma 13 and Proposition 6. □

4. On the isomorphism $X \simeq X(l_2)$ for Orlicz spaces

It is well known (see [LT2], Prop. 2d4), that a separable r.i. function space X with nontrivial Boyd indices is isomorphic (as a Banach space) to its vectorial extension $X(l_2)$. In this section we show that the converse statement is true for Orlicz function spaces.

First we fix some notations. If X is a Banach lattice of measurable functions on $(\Omega, \mathscr{A}, \mu)$, then $X(l_2)$ denotes the space of $\mathscr{A}$-measurable functions $f\colon \Omega \to l_2$ (l_2 being equipped with its Borel σ-field) for which the scalar function $\omega \mapsto \|f(\omega)\|_{l_2}$ belongs to X, while $X[l_2]$ is the closure in $X(l_2)$ of the algebraic tensor product $X \otimes l_2$ (consisting of finite sums $\sum_{i=1}^n f_i \otimes x_i$, $f_i \in X$, $x_i \in l_2$). Our $X[l_2]$ is denoted by $X(l_2)$ in [LT2], while, when $X = X''$, the space $X(l_2)$ coincides with that denoted by $\widetilde{X(l_2)}$ in [LT2]. When X is order continuous, $X(l_2) = X[l_2]$.

If $\varphi\colon \mathbf{R}_+ \to \mathbf{R}_+$ is an Orlicz function, i.e., a convex increasing function with $\varphi(0) = 0$, then $L_\varphi(\Omega, \mathscr{A}, \mu)$ is, as usual, the space of measurable functions $f\colon \Omega \to \mathbf{R}$ such that

$$\varphi(c|f|) \in L_1(\Omega, \mathscr{A}, \mu) \quad \text{for some } c > 0,$$

and we denote by M_φ the closure in L_φ of the space of simple μ-integrable functions.

We have the following result.

THEOREM 14. *The following assertions are equivalent*:

(i) *L_φ is reflexive.*
(ii) *L_φ is isomorphic to $L_\varphi[l_2]$.*
(iii) *L_φ is isomorphic to $L_\varphi(l_2)$.*
(iv) *M_φ is isomorphic to $M_\varphi(l_2)$.*

This is a consequence of the following result.

PROPOSITION 15. *An Orlicz function space never contains a complemented subspace isomorphic to $l_1(l_2)$.*

Proof. Suppose that $X = L_\varphi$ contains $l_1(l_2)$ as complemented subspace. Then its bidual X^{**} contains $(l_1(l_2))^{**} = (l_\infty(l_2))^*$ as complemented subspace. The structure of the dual of an Orlicz space is well known (see [An] for the case where the measure space is finite, [Fer] for the general case). We have:

$$L_\varphi(\Omega, \mathscr{A}, \mu)^* = L_{\varphi_*}(\Omega, \mathscr{A}, \mu) \oplus L_1(S, \Sigma, \sigma)$$

where φ_* is the Young conjugate of φ. (The L_1-component is null if φ verifies the Δ_2-condition; in the other case it is non-separable). Thus the bidual is given by

$$X^{**} = L_\varphi(\Omega, \mathscr{A}, \mu) \oplus L_1(T, \mathscr{T}, \tau) \oplus L_\infty(S, \Sigma, \sigma)$$

where again $L_1(T)$ is either null (if φ satisfies the Δ_2^*-condition) or non-separable. On the other hand $(l_\infty(l_2))^*$ contains $l_\infty^*(l_2)$ as complemented subspace. Here $l_\infty^*(l_2)$ can be defined in an abstract way by using Krivine functional calculus (see [LT2], 1d1), but in fact it identifies with the Banach space M whose elements are the sequences (μ_n) in $(l_\infty)^*$ such that

$$\|(\mu_n)\| := \operatorname{Sup}\left\{ \sum_{i=1}^{N} \langle \mu_n, x_n \rangle / N \geq 1, x_n \in l_\infty, \forall n = 1, \dots N; \left\| \sum_{n=1}^{N} x_n^2 \right\|_\infty \leq 1 \right\}$$

is finite. If $\mu_n = 0$ for $n > N$, then $(\mu_n)_n$ identifies to $\Sigma_{i=1}^N \mu_n \otimes e_n$ ((e_n) is the natural basis of l_2). This space in turn isometrically identifies to a subspace of $(l_\infty(l_2))^*$ by setting

$$\langle (\mu_n)_n, (x_{i,n})_{i,n} \rangle = \sum_n \langle \mu_n, (x_{i,n})_i \rangle$$

for every $(\mu_n)_n \in M$ and $(x_{i,n})_{i,n} \in l_\infty(l_2)$. If $F \in (l_\infty(l_2))^*$, we define $PF \in l_\infty^*(l_2)$ by $PF = (\mu_n)_n$ where $\forall x \in l_\infty$, $\langle \mu_n, x \rangle = \langle F, x \otimes e_n \rangle$; then P is a natural projection from $(l_\infty(l_2))^*$ onto $l_\infty^*(l_2)$.

Note that l_∞^* is a L_1-space containing a 1-complemented sublattice isomorphic to a $l_1(\Gamma)$ space, defined on an index set Γ which has cardinality $2^{\mathfrak{c}}$, where $\mathfrak{c}$ is the cardinality of the continuum ($\Gamma = \beta\mathbf{N}$). We obtain thus that $l_1(\Gamma)(l_2)$ embeds isomorphically as a complemented subspace E in $X^{**} = L_\varphi \oplus L_1(T) \oplus L_\infty(S)$.

Let $(e_{\gamma,n})_{\gamma \in \Gamma, n \in \mathbf{N}}$ be the $l_1(\Gamma)(l_2)$ basis in X^{**}. For each $\gamma \in \Gamma$, let $E_\gamma = \overline{\operatorname{span}}[e_{\gamma,n}]_{n \geq 1}$. We also denote by Q the given projection $X^{**} \to E$, and by P_1 (resp. P_∞, P_0) the natural projections $X^{**} \to L_1(T)$ (resp. $X^{**} \to L_\infty(S)$, $X^{**} \to L_\varphi(\Omega)$). For all $\varepsilon > 0$, there exists $y_\gamma \in E_\gamma$ with $\|y_\gamma\| = 1$ and

$\|QP_1 y_\gamma\| < \varepsilon$, $\|QP_\infty y_\gamma\| < \varepsilon$: if not, e.g., QP_1 would be an isomorphism into. In particular P_1: $E_\gamma \to P_1 E_\gamma$ and Q: $P_1(E_\gamma) \to QP_1(E_\gamma)$ are isomorphisms; let J be the inverse isomorphism of the last one.

$QP_1(E_\gamma)$ is a hilbertian subspace of E, where E is isomorphic to $l_1(l_2)$: hence it contains a further space Z, which is complemented in E, by a projection π. Then JZ would be a hilbertian subspace of $P_1(E_\gamma)$, hence of $L_1(T)$, and would be complemented in X^{**}, and a fortiori in $L_1(T)$, by the projection $J\pi Q$; this is impossible. The same reasoning works with P_∞ in place of P_1.

Then we have

$$\forall (\alpha_\gamma)_\gamma \in \mathbf{R}^{(\Gamma)}, \left\| Q(P_1 + P_\infty)\Big(\sum_\gamma \alpha_\gamma y_\gamma\Big)\right\| \le \varepsilon \sum_\gamma |\alpha_\gamma| \le \varepsilon C \left\| \sum_\gamma \alpha_\gamma y_\gamma \right\|$$

where C is the equivalence constant of E with $l_1(l_2)$. Hence

$$\left\| QP_0\Big(\sum_\gamma \alpha_\gamma y_\gamma\Big)\right\| \ge (1 - \varepsilon C)\left\| \sum_\gamma \alpha_\gamma y_\gamma \right\|$$

(since $Qy_\gamma = y_\gamma$, $\forall\gamma$)). Then

$$\left\| \sum_\gamma \alpha_\gamma P_0 y_\gamma \right\| = \left\| P_0\Big(\sum_\gamma \alpha_\gamma y_\gamma\Big)\right\| \ge (1 - \varepsilon C)\|Q\|^{-1}\left\| \sum_\gamma \alpha_\gamma y_\gamma \right\|;$$

i.e., $(P_0 y_\gamma)_\gamma$ spans a subspace of $L_\varphi(\Omega)$ isomorphic to $l_1(\Gamma)$. If $(\mathscr{A}, \mu)$ is countably generated as measure algebra, this is impossible, since the density character of $L_\varphi(\Omega)$ is at most $\mathfrak{c}$, while that of $l_1(\Gamma)$ is $2^{\mathfrak{c}}$. In the general case, we can find a sub-σ algebra $\mathscr{B}$ of $\mathscr{A}$, such that $(\mathscr{B}, \mu)$ is countably generated, and such that the elements of the $(l_1(l_2))$-basis in $L_\varphi(\Omega, \mathscr{A}, \mu)$ are $\mathscr{B}$-measurable. □

Remark. The same result is true (with same proof) for M_φ in place of L_φ.

Proof of Theorem 14. If L_φ is reflexive, then $L_\varphi = M_\varphi$ and $L_\varphi(l_2) = L_\varphi(l_2) = M_\varphi(l_2)$, hence the assertions (ii)–(iv) are the same, and in fact a consequence of [LT2], prop. 2d4.

Conversely, suppose that one of the conditions (ii), (iii), or (iv) is verified. We show first that φ verifies the condition Δ_2^* (i.e., is equivalent to a p-convex Orlicz function, for some $p > 1$). If not, then by [L] there exists a complemented subspace F of L_φ, (resp. M_φ) isomorphic to l_1 and spanned by disjoint positive functions $(f_i)_{i=1}^\infty$ (in fact, indicator functions). There is a positive projection P with range F. If $(e_i)_i$ denotes the natural basis of l_2, then $(f_i \otimes e_i)_i$ span in $L_\varphi[l_2]$ (resp. $M_\varphi(l_2)$) a subspace E isomorphic to

$l_1(l_2)$, and $P \otimes \mathrm{id}_{l_2}$ defines a projection from $L_\varphi(l_2)$ (resp $M_\varphi(l_2)$) onto E. The boundedness of $P \otimes \mathrm{id}_{l_2}$ is a consequence of the positivity of P: if $(h_i)_i \subset L_0$ with $(\sum_i |h_i|^2)^{1/2} \in L_\varphi$, then

$$\Big(\sum_i |Ph_i|^2\Big)^{1/2} = \bigvee_{\substack{\sum |\alpha_i|^2 \le 1 \\ \alpha_i \in \mathbf{Q}}} \sum_i \alpha_i Ph_i = \bigvee_{\substack{\sum |\alpha_i|^2 \le 1 \\ \alpha_i \in \mathbf{Q}}} P\sum_i \alpha_i h_i \le P\Big(\sum_i |h_i|^2\Big)^{1/2}.$$

hence $l_1(l_2)$ appears as a complemented subspace of L_φ (resp M_φ), which is impossible.

If now φ does not verify the Δ_2-condition, then M_φ contains c_0 as complemented sublattice spanned by disjoint elements (again indicator functions) and L_φ contains l_∞ as complemented sublattice; again with positive projection. Then:

$$c_0(l_2) \underset{c}{\subset} M_\varphi(l_2), \quad l_\infty[l_2] \underset{c}{\subset} L_\varphi[l_2], \quad l_\infty(l_2) \underset{c}{\subset} L_\varphi(l_2)$$

and we deduce

$$c_0(l_2) \underset{c}{\subset} M_\varphi, \quad \text{or} \quad l_\infty[l_2] \underset{c}{\subset} L_\varphi, \quad \text{or} \quad l_\infty(l_2) \underset{c}{\subset} L_\varphi$$

Dualizing and using the fact that $l_\infty^*(l_2) \subset_c (l_\infty[l_2])^*$ and $\subset_c (l_\infty(l_2))^*$, we obtain

$$l_1(l_2) \underset{c}{\subset} M_\varphi^* = L_{\varphi_*} \quad \text{or} \quad l_\infty^*(l_2) \underset{c}{\subset} L_\varphi^* = L_{\varphi_*} \oplus L_1(S).$$

The first assertion is impossible by Prop. 15, and the second one also, by the proof of Prop. 15. □

5. On the isomorphism $X(l_2) \approx X$ for q-concave r.i. spaces ($q < 2$)

The main result of this section is the following:

THEOREM 16. *Let X be a q-concave ($q < 2$) rearrangement invariant space over $\Omega = [0,1]$. Then $X(l_2)$ is isomorphic to X iff X has non-trivial lower Boyd index ($p_X > 1$).*

The following criterion will be used:

LEMMA 17. *Let Y be an r.i. space (over $\Omega = [0,1]$ or $[0,\infty]$); G a normalized Gaussian variable (defined on the probability space $(S, \Sigma, \mathbf{P})$). Denote by Z the space of measurable functions $f \in L_0(\Omega)$ such that $f \otimes G$*

belongs to $Y''(\Omega \times S)$. *Then* Y *has nontrivial upper Boyd index iff it is (algebraically) included in* Z.

Proof. (a) Suppose that Y has non trivial upper Boyd index $q_Y < \infty$. Let

$$U_k = \{k \leq |G| < k+1\}.$$

Then $\mathbf{P}(U_k) \leq P[|G| \geq k] \leq C \cdot e^{-k^2/2}$. Let $f \in Y$; then $\forall \rho \leq 1$, $\forall q > q_Y$, $\|D_\rho f\|_Y \leq C_q \rho^{1/q} \|f\|_Y$ (where D_ρ is the usual dilation operator: $D_\rho f(t) = f(t/\rho)$). Hence

$$\|f \otimes \mathbf{1}_{U_k}\|_{Y(\Omega \times S)} \leq C e^{-k^2/2q} \|f\|_{Y(\Omega)}$$

and $\Sigma_k f \otimes (\mathbf{1}_{U_k} G)$ converges in $Y(\Omega \times S)$; i.e., $f \otimes G \in Y(\Omega \times S)$.

Conversely suppose that $\|f\|_Z \leq C\|f\|_Y$ for all $f \in Y$. To prove that $q_Y < \infty$, it suffices to prove that $\|D_a\|_{Y \to Y} < 1$ for some $a > 0$. But setting $a_k = \mathbf{P}(U_k)$,

$$\|D_{a_k} f\|_Y = \|f \otimes \mathbf{1}_{U_k}\|_Y \leq \frac{1}{k}\|f \otimes G\|_Y \leq \frac{C}{k}\|f\|_Y;$$

hence for sufficiently large k, we are done. □

To prove Theorem 16, we are led to use the following proposition, the proof of which is inspired from that of Theorem 5.6 of [JMST], relative to r.i. spaces (over $[0,1]$) which embed complementably in a given q-concave r.i. function space. Recall that a quasi-norm $\|\ \|$ on a space X verifies $\|x_1 + x_2\| \leq \gamma(\|x_1\| + \|x_2\|)$ for some $\gamma > 0$ and all $x_1, x_2 \in X$.

PROPOSITION 18. *Let* X *be a* $\frac{1}{2}$-*convex*, 1-*concave quasi-Banach r.i. function space, over* $[0,1]$, Y *a (Banach) r.i. function space (over* $[0,1]$). *We suppose that the* 2-*convexified spaces* $X^{(2)}$ *and* $Y^{(2)}$ *are in duality (*$X^{(2)}$ *is the Köthe dual of* $Y^{(2)}$). *Let* Z *be another (Banach) r.i. function space, algebraically included in* Y. *Suppose that there exist positively linear bounded operators*

$$T: Y_+ \to Z_+ \quad \text{and} \quad S: X_+ \to X_+$$

such that for every f *in* Y_+ *and* g *in* X_+, $\langle (Tf)^{1/2}, (Sg)^{1/2} \rangle \geq \langle f^{1/2}, g^{1/2} \rangle$, *and* T *is order continuous. Then* $Y = L_1$ *or* $Y = Z$.

Before proving Prop. 18, we state several lemmas.

LEMMA 19. *Let* $X(\Omega)$ *be a concave (quasi-Banach) Köthe function space over a finite measure space* $(\Omega, \mathscr{A}, \mu)$, *containing algebraically* $L_1(\Omega)$, *and* T: $L_1^+(S) \to X_+(\Omega)$ *be a positively linear bounded operator. Then there exists a* *measurable* μ-a.e. *positive function* ψ *defined on* Ω *such that* T *factorizes through* $L_1(\psi) = L_1(\Omega, \mathscr{A}, \psi \cdot \mu)$, *i.e.*,

$$\forall f \in L_1^+(S), \quad \int \psi \cdot Tf\, d\mu \le C \int f\, d\mu,$$

and

$$\forall g \in L_1^+(\psi), \quad \|g\|_X \le C \int \psi \cdot g\, d\mu.$$

We thus have $T = i \circ \tilde{T}$, where i: $L_1(\psi) \to X$ is the identity map and $\tilde{T}$: $L_1^+ \to L_1(\psi)$ acts as T.

LEMMA 20. *Let* $Y = Y(\Omega)$ *be a (Banach) Köthe function space over* $(\Omega, \mathscr{A}, \mu)$, *and* T: $Y_+(\Omega) \to L_1^+(S)$ *be an order continuous positively linear bounded operator. There exists an element* φ *of* Y'_+ *such that* T *factorizes through* $L_1^+(\varphi)$, *i.e.*,

$$\forall f \in Y_+, \quad \|Tf\|_1 \le C \cdot \int \phi \cdot f\, d\mu \le C' \|f\|_Y.$$

We have thus $T = \tilde{T} \circ j$, where j: $Y \to L_1(\varphi)$ is the identity map, and $\tilde{T}$: $L_1^+(\Omega) \to L_1^+$ is formally the same operator as T (i.e., coincides with T on $Y_+(\Omega)$).

Lemma 19 and 20 are special cases of Krivine factorization theorem, and the proof of Lemma 19 is close to that of Lemma 5.7 of [JMST].

To obtain Lemma 19, we suppose $\|T\| \le 1 - \varepsilon$ and separate in the space $L_\infty(\Omega)$ the convex sets $L_\infty^+(\Omega)$ (which has non empty interior) and $C_1 - C_2$, where

$$C_1 = \{f \in L_\infty^+(\Omega)/f \le Tz + h,\ z \in L_1^+(S),\ \|z\|_1 \le 1,\ h \in L_1^+(\Omega),\ \|h\|_1 \le \varepsilon\}$$

and

$$C_2 = \{f \in L_\infty^+(\Omega) \cap X_+/\|f\|_X > \gamma\}$$

(γ is the quasi-norm constant), by a positive element of L_∞^*. To obtain Lemma 20, the same reasoning works, if $\|T\| \le 1$, now with $C_1 = L_\infty^+(\Omega) \cap B_Y$ (B_Y is the unit ball of Y) and $C_2 = \{f \in L_\infty^+(\Omega) \cap Y_+/\|Tf\|_1 > 1\}$. We use the order continuity of T to pass from the case $f \in L_\infty^+(\Omega) \cap Y_+$ to the case $f \in Y_+$.

LEMMA 21. *Let Y, Z be r.i. (Banach) function spaces over* $[0,1]$, *with* $Z \neq L_\infty([0,1])$. *Suppose that there exists a positively linear bounded operator* T: $Y_+ \to Z_+$. *For all* $n \geq 1$ *and* i, $1 \leq i \leq 2^n$, *set* $x_{n,i} = \mathbf{1}_{[(i-1)\cdot 2^{-n}, i\cdot 2^{-n}]}$ *and* $y_n = \max_{1 \leq i \leq 2^n} Tx_{n,i}$. *Suppose* $\inf_n \|y_n \mathbf{1}_{\{y_n \leq R\}}\|_Z > 0$ *for some* R. *Then the* Y*-norm dominates the* Z*-norm; i.e., there exists a constant* V *such that* $\forall f \in Y \cap Z$, $\|f\|_Z \leq V\|f\|_Y$.

This lemma is the positive analog of Lemma 5.2 of [JMST], with plainly analogous proof.

Finally we shall use the following positive version of Thm. 2.1 of [JMST].

LEMMA 22. *Let Z be an m-concave Banach lattice (with constant M). For every finite positively K-symmetric sequence $(y_i)_{i=1}^n$ in Z_+, and every choice of positive scalars $(a_i)_{i=1}^n$, we have*

$$\frac{1}{D}\left\|\sum_{i=1}^n a_i y_i\right\| \leq \max\left[\left(\mathbf{E}_\pi\left\|\max_{i=1}^n |a_{\pi(i)} y_i|\right\|^m\right)^{1/m}, \frac{1}{n}\left\|\sum_{i=1}^n y_i\right\|_Z\left(\sum_{i=1}^n a_i\right)\right]$$
$$\leq K\left\|\sum_{i=1}^n a_i y_i\right\|$$

(*where $D = D(K, m, M)$ does not depend on the sequence $(y_i)_i$*).

(By $\mathbf{E}_\pi$ we mean the average for π belonging to the group $\mathbf{S}_n$ of permutations over $\{1\ldots, n\}$; the sequence $(y_i)_i$ is said to be positively K-symmetric if there exists C such that for every nonnegative reals a_i, $i = 1, \ldots n$, and each permutation $\pi \in \mathbf{S}_n$, we have $\|\Sigma a_i y_{\pi(i)}\| \leq K\|\Sigma a_i y_i\|$).

This lemma can be proven following the method of the proof of Thm. 2.1 of [JMST], or it can also be formally deduced by applying this theorem in $L_2(Z^{(2)})$, where $Z^{(2)}$ is the 2-convexification of Z, to the sequence $(\varepsilon_i \otimes y_i^{1/2})$, where the ε_i are independent Bernouilli variables, and using Maurey-Khintchine inequalities [LT2, 1.d.6].

Proof of Proposition 18. (A) The first step is an interpolation procedure, as in the proof of Thm. 5.6 in [JMST] (but we cannot dualize the operators now).

Let j_X be the natural injection from $L_1([0,1])$ onto X. We apply Lemma 19 to the operator $S \circ j_X$: $L_1^+ \to X$ and obtain a measurable function $\psi > 0$ (a.e. on $[0,1]$) such that S: $L_1^+ \to L_1^+(\psi)$ is bounded and the identity map: $j_{\psi,X}$: $L_1^+(\psi) \to X$ is also bounded. This implies that the identity map: $L_2(\psi) \to X^{(2)}$ is bounded, and so is its conjugate $Y^{(2)} \to L_2(1/\psi)$; hence the identity map $i_{Y,1/\psi}$: $Y \to L_1(1/\psi)$ is bounded. A fortiori the identity map $i_{Z,1/\psi}$: $Z \to L_1(1/\psi)$ is bounded. We apply now Lemma 20 to the operator

$i_{Z,1/\psi} \circ T$: $Y_+ \to L_1^+(1/\psi)$ and obtain a measurable function $\varphi \geq 0$ on $[0,1]$ such that T: $L_1^+(\varphi) \to L_1^+(1/\psi)$ is bounded, and the same for the identity map: $i_{Y,\varphi}$: $Y \to L_1(\varphi)$. There exists a measurable subset E of $[0,1]$, such that φ is bounded from above and below (i.e., $1/M \leq \varphi \leq M$) on E. The spaces $Y(E)$ and $X(E)$ (consisting of functions of Y, resp. X, with support in E) are lattice isomorphic to Y, resp. X, by the same dilation operator D (if $|E| = a$ we may suppose $E = [0,a]$ and take $Df(t) = f(at)$, $\forall t \in [0,1]$). That D is bounded on $X(E)$ is a consequence of the $\frac{1}{2}$-convexity of X. This same operator takes $L_1(\varphi)(E) \approx L_1(E)$ onto $L_1([0,1])$. Replacing T and S by $a^{-1/2}T \circ D^{-1}$, resp. $a^{-1/2}S \circ D^{-1}$ (this normalization conserves the "duality inequality"), we may suppose that $E = [0,1]$ and $L_1(\varphi) = L_1([0,1])$. We have then the following commutative diagrams:

$$\begin{array}{ccc} Y_+ & \xrightarrow{T} & Z_+ \\ {\scriptstyle i_Y}\downarrow & & \downarrow{\scriptstyle i_{Z,1/\psi}} \\ L_1^+ & \xrightarrow{T} & L_1^+(1/\psi) \end{array} \quad \text{and} \quad \begin{array}{ccc} L_1^+ & \xrightarrow{S} & L_1^+(\psi) \\ {\scriptstyle j_X}\downarrow & & \downarrow{\scriptstyle j_{\psi,X}} \\ X_+ & \xrightarrow{S} & X_+ \end{array}$$

Fix $0 < \theta < 1$; we introduce the Calderón-Lozanovski interpolation spaces:

$$Y_\theta = [Y, L_1]_\theta, \qquad Z_\theta = [Z, L_1(1/\psi)]_\theta, \qquad \overline{Y}_\theta = [Y, L_1(1/\psi)]_\theta,$$
$$X_\theta = [X, L_1]_\theta, \qquad \overline{X}_\theta = [X, L_1(\psi)]_\theta.$$

As is well known (this is an easy application of the formula $x^\theta y^{1-\theta} = \inf_{t>0}((1-\theta)t^\theta x + \theta t^{-(1-\theta)}y)$) T is bounded as an operator from Y_θ^+ into Z_θ^+, and so is S: $X_\theta^+ \to \overline{X}_\theta^+$.

The 2-convexified spaces $X_\theta^{(2)}$ and $Y_\theta^{(2)}$ are in duality; for, we have $X_\theta^{(2)} = [X, L_1]_\theta^{(2)} = [X^{(2)}, L_2]_\theta$ and similarly $Y_\theta^{(2)} = [Y^{(2)}, L_2]_\theta$, and by [Lo],

$$Y_\theta^{(2)\prime} = \left[Y^{(2)}, L_2\right]_\theta' = \left[Y^{(2)\prime}, L_2'\right]_\theta = \left[X^{(2)}, L_2\right]_\theta = X_\theta^{(2)}.$$

Similarly $\overline{X}_\theta^{(2)}$ and $\overline{Y}_\theta^{(2)}$ are in duality. The inequality $\langle (Tf)^{1/2}, (Sg)^{1/2}\rangle \geq \langle f^{1/2}, g^{1/2}\rangle$ extends to the case where $f \in Y_\theta^+$, $g \in X_\theta^+$: use the fact that

$$Tf = T(\sup\{f' | f' \text{ simple}, 0 \leq f' \leq f\}) \geq \sup\{Tf' | f' \text{ simple}, 0 \leq f' \leq f\}$$

and similarly for Sg.

We deduce that $\|Tf\|_{Z_\theta} \geq \delta_\theta \|f\|_{Y_\theta}$ for all $f \in Y_\theta^+$. For, if $f \in Y_\theta^+$ we can find $g \in X_\theta^+$ with $\|g\|_{X_\theta} = 1$ and $\langle f^{1/2}, g^{1/2} \rangle \geq (1-\varepsilon)\|f^{1/2}\|_{Y_\theta^{(2)}} = (1-\varepsilon)\|f\|_{Y_\theta}^{1/2}$. We have

$$\begin{aligned}\langle g^{1/2}, g^{1/2} \rangle &\leq \left\langle (Tf)^{1/2}, (Sg)^{1/2} \right\rangle \leq \left\|(Tf)^{1/2}\right\|_{\overline{Y}_\theta^{(2)}} \left\|(Sg)^{1/2}\right\|_{\overline{X}_\theta^{(2)}} \\ &\leq \left\|(Tf)^{1/2}\right\|_{Z_\theta^{(2)}} \left\|(Sg)^{1/2}\right\|_{\overline{X}_\theta^{(2)}} = \|Tf\|_{Z_\theta}^{1/2} \|Sg\|_{\overline{X}_\theta}^{1/2} \\ &\leq \|Tf\|_{Z_\theta}^{1/2} \|S\|_{X_\theta \to \overline{X}_\theta}^{1/2} \|g\|_{X_\theta}^{1/2};\end{aligned}$$

hence $\|Tf\|_{Z_\theta} \geq (1-\varepsilon)^2 \|S\|_{\overline{X}_\theta \to X_\theta}^{-1} \|f\|_{Y_\theta}$.

We thus have $\|Tf\|_{Z_\theta} \simeq \|f\|_{Y_\theta}$ for all $f \in Y_\theta^+$ (but T need not to be an isomorphism for the metric structures of Y_θ^+ and Z_θ^+).

(B) We are now in position to apply either Lemma 21 to $T\colon Y \to Z$ or Lemma 22 to Z_θ. We suppose first that $Z \neq L_\infty$. Then the reasoning is very similar to that of [JMST], Thm. 5.6. For $n \geq 1$ and $1 \leq i \leq 2^n$ set $y_{n,i} = \mathbf{1}_{[(i-1)2^{-n}, i2^{-n}]}$ and $z_n = \sup_{i=1}^{2^n} Ty_{n,i}$. Note that $z_n \leq T\mathbf{1}$.

Case I. There exists R such that $\inf_n \|\mathbf{1}_{\{z_n \leq R\}} z_n\|_Z > 0$. Then by Lemma 21, there exists K such that $\forall f \in Z_+$, $\|f\|_Z \leq K\|f\|_Y$; the converse inequality holds by hypothesis, hence $Y = Z$.

Case II. There exist sequences $n_l \to \infty$ and $R_l \to \infty$ such that $\|\mathbf{1}_{\{z_{n_l} \leq R_l\}} z_{n_l}\|_{Z''} \to 0$ on $l \to \infty$. Then, after extracting if necessary, $z_{n_l} \to 0$ a.e. as $l \to \infty$. Since $z_{n_l} \leq T\mathbf{1} \in L_1(1/\psi)$, we have $\lim_{l\to\infty} \|z_{n_l}\|_{L_1(1/\psi)} = 0$ by Lebesgue's Theorem.

Then, by the interpolation inequality, $\|z_{n_l}\|_{Z_\theta} \to 0$ as $l \to \infty$. If $y = \sum_{i=1}^{2^n} b_i y_{n,i}$ is a simple nonnegative dyadic function, the norm of Ty in Z_θ is estimated by applying Lemma 22 to the $1/\theta$-concave lattice Z_θ and to the finite sequence $(Ty_{n,i})_{1 \leq i \leq 2^n}$ (which is positively equivalent to $(y_{n,i})_{1 \leq i \leq 2^n}$, hence positively symmetric), we obtain

$$\|y\|_{Y_\theta} \leq C\left[\|y\|_\infty \|z_n\|_{Z_\theta} \vee \|y\|_{L_1([0,1])}\right]$$

where the constant does not depend on n. For a fixed dyadic function y, we make $n = n_l$ and let $l \to \infty$, deducing that $\|y\|_{Y_\theta} \leq C\|y\|_1$, which implies that $Y_\theta = L_1([0,1])$ (algebraically). Dualizing, we obtain $L_\infty = Y_\theta' = [L_1, Y]_\theta' = [L_\infty, Y']_\theta = Y'^{1/\theta}$ (the $1/\theta$-convexification of Y'); hence $Y' = L_\infty$, thus $Y = L_1$.

(C) We left aside in the preceding the case $Z = L_\infty$. In this case we may replace in the hypotheses the space Z by the interpolation space $[Y, Z]_{1/2} = Y^{(2)}$. We conclude that either $Y = Y^{(2)}$, which implies that $Y \doteq L_\infty$, in contradiction with the hypothesis, or $Y = L_1$. □

Proof of Theorem 16. The "if" part of Thm. 16 results from [LT2, Prop. 2d4]. Conversely, let X be a q-concave ($q < 2$) r.i. space over $\Omega = [0,1]$, and suppose that $X(l_2)$ embeds into X as a complemented subspace. Denote by $X_{1/2}, X'_{1/2}$ the 1/2-concavifications of X and X'; note that $X_{1/2}$ is a 1/2-convex, concave quasi-Banach space, while $X'_{1/2}$ is convex (up to renorming we may suppose that the quasi-norm on $X'_{1/2}$ is a norm). Let Z be the r.i. Banach function space over $[0,1]$ defined by $f \in Z$ iff $f \otimes G^2 \in X'_{1/2}$, where, as before, G is a normal gaussian variable defined on the probability space $(S, \Sigma, \mathbf{P})$, equipped with the norm $\|f\|_Z = \|f \otimes G^2\|_{X'_{1/2}(\Omega\times S)}$. As conditional expectation operators are defined on the r.i. space $X'_{1/2}$, we see that $Z \subset X'_{1/2}$ (with $\|f\|_Z \geq (\mathbf{E}|G|)\cdot\|f\|_{X'_{1/2}}$). We shall construct two bounded, positively linear operators S: $X^+_{1/2} \to X^+_{1/2}$ and T: $X'^+_{1/2} \to Z^+$ verifying the "duality inequality"

$$\forall f \in X^+_{1/2}, \forall g \in X'^+_{1/2}, \left\langle (Tf)^{1/2}, (Sg)^{1/2}\right\rangle \geq \langle f^{1/2}, g^{1/2}\rangle$$

(since T is order continuous). An application of Prop. 18 then shows that $Z = X'_{1/2}$; hence $X' = Z^{(2)}$, and, by Lemma 17, X' has nontrivial upper Boyd index. By [LT2], Prop. 2b2, X has a non-trivial lower Boyd index.

(A) *Construction of S.* Let

$$X(l_2) \xrightarrow{V} X \xrightarrow{\pi} X(l_2)$$

be a C-representation of $X(l_2)$ as a complemented subspace of X. For all n, we denote by $(x_{i,n})_{i=1,\dots 2^n}$ the dyadic partition of $[0,1]$ of order n, $x_{i,n} = [(i=1)2^{-n}, i2^{-n}]$, and by (e_m) the natural basis of l_2. Let X_n be the sublattice of $X(\Omega)$ generated by the $x_{i,n}$, $i = 1,\dots 2^n$.

Let us first remark that in X, the unit ball of any hilbertian subspace H is necessarily X-equiintegrable. For, if not, there exist a norm one sequence (y_k) in H, a disjoint sequence (y'_k) in X, and a real $\delta > 0$, with $|y'_k| \leq |y_k|$ and $\|y'_k\| \geq \delta$. Let A_k be the support of y'_k. Since H is reflexive, we may suppose (up to extracting) that

$$y_k \xrightarrow[k\to\infty]{w} y_\infty \,(\text{weakly});$$

then

$$z_k = y_k - y_\infty \xrightarrow{w} 0,$$

and $z'_k = \mathbf{1}_{A_k} z_k$ verifies $\liminf_{k\to\infty} \|z'_k\|_X \geq \delta$; so we can suppose w.l.o.g.

$$y_k \xrightarrow[k\to\infty]{w} 0,$$

hence (after extraction) that (y_k) is equivalent to the l_2-basis. Then

$$\begin{aligned}\left\|\sum \alpha_k y'_k\right\| = \left\|\mathbf{E}_\varepsilon\left|\sum \alpha_k \varepsilon_k y'_k\right|\right\| &\le \left\|\mathbf{E}_\varepsilon\left|\sum \alpha_k \varepsilon_k y_k\right|\right\| \\ &\le \mathbf{E}_\varepsilon\left\|\sum \alpha_k \varepsilon_k y_k\right\| \sim \left(\sum |\alpha_k|^2\right)^{1/2}\end{aligned}$$

but by q-concavity of X,

$$\left\|\sum \alpha_k y'_k\right\|_X \ge c_q\left(\sum \|\alpha_k y'_k\|_X^q\right)^{1/q} \ge c_q\delta\left(\sum |\alpha_k|^q\right)^{1/q}$$

which is a contradiction since $q < 2$.

For fixed i and n, the sequence $(V(x_{i,n} \otimes e_m))_{m=1}^\infty$ generates a subspace $E_{i,n}$ of X which is complemented and isomorphic to l_2. By the preceding, the unit ball of $E_{i,n}$ is X-equiintegrable, and in particular the L_1-norm and the X-norm are equivalent on this subspace. Thus there exist in the unit ball of $E_{i,n}$ a sequence $(u_{i,n}^m)_m$ which converges w.c.d. to a (non-zero) conditionally Gaussian variable $A_{i,n} \otimes G$. Doing the same for each $i = 1, \ldots 2^n$, we find 2^n sequences $(u_{i,n}^m)_m$, each converging w.c.d. to a variable $A_{i,n} \otimes G$. Hence we have the following joint w.c.d. convergence:

$$\left(u_{i,n}^{m_i}\right)_{i=1}^{2^n} \xrightarrow[m_1\to\infty;\, m_2\to\infty;\,\ldots\, m_{2^n}\to\infty]{\text{wcd}} (A_{i,n} \otimes G_i)_{i=1}^{2^n}$$

where now the G_i are independent normal gaussian variables. Reasoning as in §3, we obtain 2^n sequences $(v_{i,n}^m)_{m\ge 1}$, $i = 1, \ldots 2^n$, in $X(\Omega \times S)$ which are jointly equimeasurable with the sequences $(A_{i,n} \otimes G_i^m)_{m\ge 1}$ and generating a C-representation of $X_n(l_2)$ as a complemented subspace of $X(\Omega \times S)$. Again an argument using a measure preserving transform of $\Omega \times S$ allows to replace the $(v_{i,n}^m)_{m\ge 1, 1\le i\le 2^n}$ by the $(A_{i,n} \otimes G_i^m)$. We have thus a factorization of the identity of $X_n(l_2)$:

$$X_n(l_2) \xrightarrow{j_n} X(\Omega \times S) \xrightarrow{\pi_n} X_n(l_2)$$

with $j_n(x_{i,n} \otimes e_m) = A_{i,n} \otimes G_i^m$ and $\|j_n\|\,\|\pi_n\| \le C$.

There is no relation between the systems $(A_{i\,n})_{i=1,\ldots 2^n}$ for different values of n. However j_n induces naturally a C-representation of $X_{n-1}(l_2)$ in $X(\Omega \times S)$,

$$X_{n-1}(l_2) \xrightarrow{k_n} X(\Omega \times S) \xrightarrow{\sigma_n} X_{n-1}(l_2),$$

with $k_n = j_n \circ i_n$, $\sigma_n = q_n \circ \pi_n$, where i_n is the natural injection of $X_{n-1}(l_2)$ into $X_n(l_2)$, and q_n the expectation projection from $X_n(l_2)$ onto $X_{n-1}(l_2)$.

Then

$$\begin{aligned} k_n(x_{i,n-1} \otimes e_m) &= j_n((x_{2i-1,n} + x_{2i,n}) \otimes e_m) \\ &= A_{2i-1,n} \otimes G_{2i-1}^m + A_{2i,n} \otimes G_{2i}^m. \end{aligned}$$

Hence the sequence $(k_n(x_{i,n-1} \otimes e_m)_m$, $i = 1, \dots 2^n$, is jointly equimeasurable with $(A_{i,n-1}^{(n)} \otimes G_i^m)_m$, where $A_{i,n-1}^{(n)} = (A_{2i-1,n}^2 + A_{2i,n}^2)^{1/2}$. Again (with a measure-preserving transform on $\Omega \times S$), we deduce another C-representation of $X_{n-1}(l_2)$ given by

$$X_{n-1}(l_2) \xrightarrow{j_{n-1}^{(n)}} X(\Omega \times S) \xrightarrow{\pi_{n-1}^{(n)}} X_n(l_2)$$

with $j_{n-1}^{(n)}(x_{i,n-1} \otimes e_m) = A_{i,n-1}^{(n)} \otimes G_i^m$ ($m \geq 1$, $i = 1, \dots 2^{n-1}$). In the same way we recursively define $j_k^{(n)}, \pi_k^{(n)}$, for each $k \leq n$, giving a C-representation of $X_k(l_2)$ as a complemented subspace of $X(\Omega \times S)$.

We claim that for fixed k, i, m, the sequence $(j_k^{(n)}(x_{i,k} \otimes e_m))_{n \geq k} = (A_{i,k}^{(n)} \otimes G_i^m)_{n \geq k}$ is X-equiintegrable. For each n, $A_{i,k}^{(n)} \otimes G_i^m$ is equimeasurable with

$$u_{i,k}^{(n)} = \sum_{(i-1)2^{-k} \leq j2^{-n} \leq i2^{-k}} A_{j,n} \otimes G_j^n$$

which by construction is the wcd limit of a sequence $(V(f_m^n))_{m \geq 1}$, where the elements f_m^n are disjoint in the lattice $X(l_2)$ but verify $\|f_m^n\|_2(\omega) = x_{i,k}(\omega)$ a.e. In fact we may even suppose that $f_m^n \in X(H_m^n)$, where $H_m^n = \overline{\operatorname{span}}[e_l]_{l \in L_m^n}$ and $L_1^n < L_2^n < \cdots$ are successive disjoint intervals of $\mathbf{N}$. If the sequence $(u_{i,k}^{(n)})_n$ is not X-equiintegrable, it is possible to find sequences $n_1 < \cdots < n_l < n_{l+1} < \cdots$ and $m_1 < \cdots < m_l < m_{l+1} < \cdots$ such that the sequence $(V(f_{m_l}^{n_l}))_l$ is not X-equiintegrable, while $(f_{m_l}^{n_l})_l$ is disjoint in the lattice $X(l_2)$. But then

$$\left\| \sum_l \alpha_l f_{m_l}^{n_l} \right\|_2 (\omega) = \left(\sum_l \alpha_l^2 \right)^{1/2} x_{i,k}(\omega) \quad \text{a.e.}$$

hence $(f_{m_l}^{n_l})_l$ is equivalent in X to the l_2-basis; so is $(V(f_{m_l}^{n_l}))_l$, which is then X-equiintegrable by a preceding remark, a contradiction which proves our claim.

The sequence $(j_k^{(n)}(x_{i,k} \otimes e_m))_{n \geq k}$ defines an element $\tilde{j}_k(x_{i,k} \otimes e_m)$ of the ultrapower $\hat{X} = X(\Omega \times S)^{\mathbf{N}}/\mathscr{U}$, in fact of $\hat{X}^{\text{eq}}$. Similarly the sequence $(A_{i,k}^{(n)})_{n \geq k}$ (which is a fortiori X-equiintegrable) defines an element $\tilde{A}_{i,k}$ of $\tilde{X} = X(\Omega)^{\mathbf{N}}/\mathscr{U}$, in fact of $\tilde{X}^{\text{eq}}$. We identify as usual $\tilde{X}^{\text{eq}} = X(\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{\mu})$. We have a natural isometric embedding Θ: $X(\tilde{\Omega} \times S) \to \hat{X}^{\text{eq}}$: if $\tilde{f} \in \tilde{X}^{\text{eq}}$ is defined by the sequence $(f_n)_n$ and $g \in L_\infty(S)$, then $\Theta(\tilde{f} \otimes g)$ is simply the element of $\hat{X}^{\mathrm{eq}}$ defined by the sequence $(f_n \otimes g)_n$. We have clearly $\Theta(\tilde{A}_{i,k} \otimes G_i^m) = \tilde{j}_k(x_{i,k} \otimes e_m)$.

Now define $\tilde{\pi}_k$: $\hat{X} \to X_k(l_2)$ by $\tilde{\pi}_k(\tilde{h}) = w - \lim_{n,\mathscr{U}} \pi_k^{(n)}(h_n)$ when $\tilde{h}$ is defined by the sequence $(h_n)_n$. We now have a C-factorization of the identity of $X_k(l_2)$ through $\hat{X}^{\mathrm{eq}}$:

$$X_k(l_2) \xrightarrow{\tilde{j}_k} \hat{X}^{\mathrm{eq}} \xrightarrow{\tilde{\pi}_k} X_k(l_2)$$

In fact we can replace $\hat{X}^{\mathrm{eq}}$ by $X(\tilde{\Omega} \times S)$, replacing $\tilde{j}_k$ by $\Theta^{-1} \circ \tilde{j}_k$ and $\tilde{\pi}_k$ by $\tilde{\pi}_k \circ \Theta$. But from

$$A_{i,k}^{(n)} = \left(A_{2i-1,k+1}^{(n)2} + A_{2i,k+1}^{(n)2}\right)^{1/2}, \quad \forall n \geq k+1$$

we get

$$\tilde{A}_{i,k} = \left(\tilde{A}_{2i-1,k+1}^2 + \tilde{A}_{2i,k}^2\right)^{1/2}.$$

Since these $\tilde{A}_{i,k}$ live in a separable sublattice of $X(\tilde{\Omega})$, we can find a new family $(\overline{A}_{i,k})_{k \geq 1; i=1,\ldots 2^k}$ in $X(\Omega)$, jointly equimeasurable with $(\tilde{A}_{i,k})_{i,k}$, and sharing the same properties. So (writing $A_{i,k}$ in place of $\overline{A}_{i,k}$) we are back to our starting point, i.e., we have a family $(A_{i,k})_{i,k}$ such that $(A_{i,k} \otimes G_i^m)_{m \geq 1, 1 \leq i \leq 2^k}$ generates a complemented subspace of $X(\Omega \times S)$ isomorphic to $X_k(l_2)$, but we gained the compatibility condition

$$\forall k, \forall i, 1 \leq i \leq 2^k, \quad A_{i,k}^2 = A_{2i-1,k+1}^2 + A_{2i,k+1}^2.$$

We define the operator S on positive dyadic functions by

$$f = \sum_{i=1}^{2^k} \lambda_i^2 x_{i,k} \Rightarrow S(f) = \sum_{i=1}^{2^k} \lambda_i^2 A_{i,k}^2$$

Due to the compatibility condition, $S(f)$ does not depend on the way of writing f as a simple dyadic function. We have

$$\begin{aligned}\left\| S\left(\sum_i \lambda_i^2 x_{i,k}\right)\right\|_{X_{1/2}} &= \left\|\left(\sum_i \lambda_i^2 A_{i,k}^2\right)^{1/2}\right\|_X^2 \sim \left\|\left(\sum_i \lambda_i^2 A_{i,k}^2\right)^{1/2} \otimes G\right\|_X^2 \\ &= \left\|\sum_i \lambda_i A_{i,k} \otimes G_i^k\right\|_X^2 \sim \left\|\sum_i \lambda_i x_{i,k}\right\|_X^2 = \left\|\sum_i \lambda_i^2 x_{i,k}\right\|_{X_{1/2}},\end{aligned}$$

thus $\|Sf\|_{X_{1/2}} \sim \|f\|_{X_{1/2}}$ for every positive dyadic function f. Note that $|Sf - Sg| \leq S(f \vee g) - S(f \wedge g) = S(|f-g|)$, by positivity of S; thus S is lipschitzian on the cone of positive dyadic functions, and it extends by density to a positively linear operator $X_{1/2} \to X_{1/2}$, with again $\|Sf\|_{X_{1/2}} \sim \|f\|_{X_{1/2}}$ for all f.

(B) *Construction of T.* For each $k \geq 1$, we have a projection

$$P_k\colon X(\Omega \times S) \to E_k = \overline{\operatorname{span}}[A_{i,k} \otimes G_i^m]_{m \geq 1, i=1,\dots 2^k}.$$

From now on, we suppose that (Σ, σ) is generated by the variables $(G_i^m)_{i,m}$. We consider the action of the group $G_{n,p} = O(n)^p$ on the space $X(\Omega \times S)$, defined by

$$\begin{aligned}(U_1, \dots U_p)f\big(\omega, (G_1^m)_m, (G_2^m)_m \dots (G_p^m)_m, (G_{p+1}^m)_m, \dots\big)\\ = f\big(\omega, U_1^*(G_1^m)_m, U_2^*(G_2^m)_m, \dots U_p^*(G_p^m)_m, (G_{p+1}^m)_m, \dots\big)\end{aligned}$$

where, as in §2, $O(n)$ acts on $\mathbf{R}^{\mathbf{N}}$ by $U(x_1, \dots, x_n, x_{n+1} \dots) = (U(x_1, \dots x_n), x_{n+1} \dots)$. The subspace E_k is invariant under the action of $G_{n,p}$ (in fact each "fiber" $E_{k,i} = \overline{\operatorname{span}}[A_{i,k} \otimes G_i^m]_{m \geq 1}$ is invariant). The reasoning of Proposition 6 gives us a projection $R_k\colon X(\Omega \times S) \to E_k$ which is invariant under the action of all the groups $G_{n,p}$; the method of Proposition 6 shows that R_k takes necessarily the form

$$R_k f = 2^k \sum_{\substack{1 \leq i \leq 2^k\\ m \geq 1}} \langle f, B_{i,k} \otimes G_i^m \rangle A_{i,k} \otimes G_i^m$$

with $B_{i,k} \otimes G_i^m \in X'$ and $\langle A_{i,k}, B_{i,k} \rangle = 2^{-k}$.

Since $(B_{i,k} \otimes G_i^m)_{i,m}$ is biorthogonal to $(B_{i,k} \otimes G_i^m)_{i,m}$, which is equivalent to $(x_{i,k} \otimes e_m)_{i,m}$ and spans a complemented subspace of $X(\Omega \times S)$, and since $\langle B_{i,k}, A_{i,k} \rangle = 2^{-k} = \langle x_{i,k}, x_{i,k} \rangle$, we see that

$$\Big\| \sum_{i,m} \alpha_{i,m} B_{i,k} \otimes G_i^m \Big\|_{X'} \sim \Big\| \sum_{i,m} \alpha_{i,m} x_{i,k} \otimes e_m \Big\|_{X'(l_2)}.$$

Again there is a priori no relation between the systems $(B_{i,k})_{i=1,\dots 2^k}$ for different values of k, so we modify them to have a compatibility condition. The method we used for the $(A_{i,k})$ does not work here because X' has no more non trivial concavity. However we set

$$B_{i,k-1}^{(k)} = \left(B_{2i-1,k}^2 + B_{2i,k}^2\right)^{1/2}$$

and note that $(B_{i,k}^{(k)} \otimes G_i^m)_{m \geq 1, 1 \leq i \leq 2^{k-1}}$ span an isomorph of $X'_{k-1}(l_2)$ in X' (with no loss on the equivalence constant), and that

$$\left\langle B_{i,k-1}^{(k)}, A_{i,k-1}\right\rangle = \left\langle \left(B_{2i-1,k}^2 + B_{2i,k}^2\right)^{1/2}, \left(A_{2i-1,k}^2 + A_{2i,k}^2\right)^{1/2}\right\rangle$$
$$\geq \langle B_{2i-1,k}, A_{2i-1,k}\rangle + \langle B_{2i,k}, A_{2i,k}\rangle = 2^{-(k-1)}$$

Similarly we recursively define $B_{i,p}^{(k)}$, $p \leq k$, which satisfies

$$\left\| \sum_{\substack{1 \leq i \leq 2^p \\ m \geq 1}} \alpha_{i,m} B_{i,p}^{(k)} \otimes G_i^m \right\|_{X'} \sim \left\| \sum_{\substack{1 \leq i \leq 2^p \\ m \geq 1}} \alpha_{i,m} x_{i,p} \otimes e_m \right\|_{X'}$$

(with equivalence constants independent from k, p) and

$$\langle B_{i,p}^{(k)}, A_{i,p}\rangle \geq 2^{-p}.$$

The sequence $(B_{i,p}^{(k)2})$ is bounded in the lattice $X'_{1/2}$, which is a dual lattice (since it is r-convex, $r > 1$, and a maximal r.i. space, $X'_{1/2} = Z^*$ with $Z = (X'_{1/2})'$). We define $\hat{B}_{i,p}^2 = w^* - \lim_{k,\mathscr{U}} B_{i,p}^{(k)2}$. We have $\hat{B}_{i,p} \in X'$ and in
fact $\hat{B}_{i,p} \otimes G \in X'$, with $\hat{B}_{i,p}^2 \otimes G^2 = w^* - \lim_{k,\mathscr{U}} B_{i,p}^{(k)2} \otimes G^2$. Thus

$$\left\| \sum_{i,m} \alpha_{i,m} \hat{B}_{i,p} \otimes G_i^m \right\|_{X'} = \left\| \left(\sum_{i,m} \alpha_{i,m}^2 \hat{B}_{i,p}^2 \right)^{1/2} \otimes G \right\|_{X'}$$
$$= \left\| \sum_{i,m} \alpha_{i,m}^2 \hat{B}_{i,p}^2 \otimes G^2 \right\|_{X'_{1/2}}^{1/2}$$
$$\leq \lim_{k,\mathscr{U}} \left\| \sum_{i,m} \alpha_{i,m}^2 B_{i,p}^{(k)2} \otimes G^2 \right\|_{X'_{1/2}}^{1/2}$$
$$= \lim_{k,\mathscr{U}} \left\| \sum_{i,m} \alpha_{i,m} B_{i,p}^{(k)} \otimes G_i^m \right\|_{X'}$$
$$\sim \left\| \sum_{i,m} \alpha_{i,m} x_{i,p} \otimes e_m \right\|_{X'(l_2)}$$

Set $C_{i,p} = w^* - \lim_{k,\mathscr{U}} B_{i,p}^{(k)}$, where now the w^*-limit is relative to $\sigma(X', X)$.

We have $C_{i,p} \le \hat{B}_{i,p}$, as is well known; hence, for nonnegative reals $\alpha_{i,m}$,

$$\begin{aligned}\Big\langle \sum_{i,m} \alpha_{i,m}\hat{B}_{i,p} \otimes G_{i,m}, \sum_{j,l} \beta_{j,l}A_{j,p} \otimes G_j^l \Big\rangle &= \sum_{i,m} \alpha_{i,m}\beta_{i,m}\langle \hat{B}_{i,p}, A_{i,p}\rangle \\ &\ge \sum_{i,m} \alpha_{i,m}\beta_{i,m}\langle C_{i,p}, A_{i,p}\rangle \\ &= \lim_{k,\mathscr{U}} \sum_{i,m} \alpha_{i,m}\beta_{i,m}\langle B_{i,p}^{(k)}, A_{I,p}\rangle \ge 2^{-p} \sum_{i,m} \alpha_{i,m}\beta_{i,m} \\ &= \Big\langle \sum_{i,m} \alpha_{i,m}x_{i,p} \otimes e_m, \sum_{j,l} \beta_{j,l}x_{j,p} \otimes e_l \Big\rangle\end{aligned}$$

And since

$$\Big\| \sum_{j,l} \beta_{j,l}A_{j,p} \otimes G_j^l \Big\|_X \sim \Big\| \sum_{j,l} \beta_{j,l}x_{j,p} \otimes e_l \Big\|_{X(l_2)},$$

we have

$$\Big\| \sum_{i,m} \alpha_{i,m}\hat{B}_{i,p} \otimes G_{i,m} \Big\|_{X'} \gtrsim \Big\| \sum_{i,m} \alpha_{i,m}x_{i,p} \otimes e_m \Big\|_{X'(l_2)}.$$

Since we have, by construction, $B_{i,p}^{(k)2} = B_{2i-1,p+1}^{(k)2} + B_{2i,p+1}^{(k)2}$, for $k \ge p+1$, we obtain, by passing to the limit, the compatibility condition

$$\hat{B}_{i,p}^2 = \hat{B}_{2i-1,p+1}^2 + \hat{B}_{2i,p+1}^2.$$

We can now define Tf for a nonnegative dyadic function $f = \sum_{i=1}^{2^p} \alpha_i^2 x_{i,p}$ by $Tf = \sum_{i=1}^{2^p} \alpha_i^2 \hat{B}_{i,p}^2$, and we have

$$\|(Tf) \otimes G^2\|_{X'_{1/2}} \sim \|f\|_{X'_{1/2}}$$

and

$$\big\langle (Tf)^{1/2}, (Sg)^{1/2} \big\rangle \ge \langle f^{1/2}, g^{1/2}\rangle$$

for every nonnegative dyadic f and g.

Let Z be the space $\{f \in L_0 | f \otimes G \in X'_{1/2}\}$. We can extend (by density) the operator T to an operator $E_+ \to Z$, where E is the closure of dyadic functions in $X'_{1/2}$ which is also the closure of L_∞ in $X'_{1/2}$ (since $X'_{1/2} \ne L_\infty$, i.e. $X \ne L_1$). If $f \in X'_{1/2}$ and $f_n \in E_+$, $f_n \uparrow f$ a.e., we have

$$\forall n, \ \|Tf_n\|_Z \le C\|f_n\|_{X'_{1/2}} \le C\|f\|_{X'_{1/2}}$$

hence $F = \sup_n Tf_n$ belongs to Z'', with norm $\|F\|_{Z''} \leq C\|f\|_{X'_{1/2}}$. This element F does not depend on the sequence $f_n \uparrow f$, since if $0 \leq g_m \uparrow f$ we have: $g_m \wedge f_n \uparrow f_n$ a.e. and in the norm of E, hence $T(g_m \wedge f_n) \to Tf_n$ as $m \to \infty$ and $\sup_m Tg_m \geq \sup_{m,n} T(g_m \wedge f_n) \geq \sup_n Tf_n$. We set $Tf = F$. The operator T is then positively additive and order continuous, and, choosing (f_n) such that $\|f_n\| \to \|f\|$, we have

$$\|Tf\|_Z \geq \sup_n \|Tf_n\|_Z \sim \sup_n \|f_n\|_{X'_{1/2}} = \|f\|_{X'_{1/2}}.$$

This ends the proof of Theorem 16. □

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