ILLINOIS JOURNAL OF MATHEMATICS
Volume 39, Number 3, Fall 1995

STABILITY OF SOME TYPES OF RADON-NIKODYM PROPERTIES

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I. Introduction

Let X be a Banach space, let (Ω, Σ, μ) be a measure space and let $1 \leq p \leq \infty$. $L^p(\mu, X)$ will denote the Banach space of all (classes of) X-valued μ-p-Bochner integrable with the usual norm. If X is scalar field, then we will write $L^p(\mu)$ for $L^p(\mu, X)$.

In this note, we consider some types of Radon Nikodym properties associated with subsets of countable discrete abelian group (type I-Λ-RNP and type II-Λ-RNP) which generalize the usual Radon Nikodym property and the Analytic Radon Nikodym property. These properties were introduced by Dowling [D1] and Edgar [E].

In [D1], it is shown that if Λ is a Riesz set then $L^1[0,1]$ has type I-Λ-RNP. It is then natural to ask if these two properties pass from X to $L^p(\mu, X)$. Dowling proved in [D1] that if Λ is a Riesz subset of $\mathbf{Z}$, then $L^1(\mathbf{T}, X)$ has type II-Λ-RNP whenever X does. In this paper, we will show that the same result holds regardless of the group G and the measure space (Ω, Σ, μ). We will give also some generalization for non Riesz sets.

We will discuss also when type I-Λ-RNP and type II-Λ-RNP pass from a Banach space X to $C_\Lambda(G, X)$ if Λ is a Rosenthal set. Other related results are obtained.

All unexplained terminologies can be found in [D] and [DU].

II. Preliminaries and definitions

Throughout this paper G will denote a compact metrizable abelian group, $\mathscr{B}(G)$ is the σ-algebra of the Borel subsets of G, and λ the normalized Haar measure on G. We will denote by Γ the dual group of G, i.e., the set of continuous homomorphisms $\gamma\colon G \to \mathbf{C}$ (Γ is a countable discrete abelian group).

Let X be a Banach space and $1 \leq p \leq \infty$, we will denote by $L^p(G, X)$ the usual Bochner function spaces for the measure space $(G, \mathscr{B}(G), \lambda)$, $M(G, X)$ the space of X-valued countably additive measure of bounded variation,

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Received April 5, 1993.

[1]This work will constitute a portion of the Ph.D. thesis of the first named author at the University of Missouri-Columbia.

1991 Mathematics Subject Classification. Primary 46E40, 46G10; Secondary 28B05, 28B20.

$C(G, X)$ the space of X-valued continuous functions and $M^\infty(G, X) = \{\mu \in M(G, X), |\mu| \leq C\lambda$ for some $C > 0\}$.

(i) If $f \in L^1(G, X)$, we denote by $\hat{f}$ the Fourier transform of f which is the map from Γ to X defined by $\hat{f}(\gamma) = \int_G \bar{\gamma} f \, d\lambda$.

(ii) If $\mu \in M(G, X)$, we denote by $\hat{\mu}$ the Fourier transform of μ which is the map from Γ to X defined by $\hat{\mu}(\gamma) = \int_G \bar{\gamma} \, d\mu$.

If $\Lambda \subset \Gamma$ is a set of characters, let

$$L^p_\Lambda(G, X) = \{f \in L^p(G, X), \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\}$$
$$C_\Lambda(G, X) = \{f \in C(G, X), \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\}$$
$$M_\Lambda(G, X) = \{\mu \in M(G, X), \hat{\mu}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\}$$
$$M^\infty_\Lambda(G, X) = \{\mu \in M^\infty(G, X), \hat{\mu}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\}$$

DEFINITION 1. (i) A subset Λ of Γ is a Riesz set if and only if $M_\Lambda(G, \mathbf{C}) = L^1_\Lambda(G, \mathbf{C})$.

(ii) A subset Λ of Γ is a Rosenthal set if and only if $C_\Lambda(G) = L^\infty_\Lambda(G)$.

Recent information about Riesz sets can be found in [G].

The following properties were introduced by Edgar [E], and Dowling [D1].

DEFINITION 2. (i) A Banach space X is said to have type I-Λ-Radon Nikodym Property (type-I-Λ-RNP) if and only if $M^\infty_\Lambda(G, X) = L^\infty_\Lambda(G, X)$.

(ii) A Banach space X is said to have type II-Λ-Radon Nikodym Property (type II-Λ-RNP) if and only if $M_{\Lambda, ac}(G, X) = L^1_\Lambda(G, X)$ where

$$M_{\Lambda, ac}(G, X) = \{\mu \in M_\Lambda(G, X),\ \mu \text{ is absolutely continuous with respect to } \lambda\}.$$

Remarks. (a) It is obvious that type II-Λ-RNP implies type I-Λ-RNP.

(b) Since $\mathscr{B}(G)$ is countably generated, one can see that these two properties are separably determined.

(c) If $G = \mathbf{T}$ then $\Gamma = \mathbf{Z}$. Then type I-$\mathbf{Z}$-RNP is equivalent to type II-$\mathbf{Z}$-RNP which is also equivalent to the usual RNP. Similarly, type I-$\mathbb{N}$-RNP is equivalent to type II-$\mathbf{N}$-RNP and is equivalent to the analytic Radon Nikodym Property (see [E]).

(d) If Λ is a Riesz subset, then $M_{\Lambda, ac}(G, X) = M_\Lambda(G, X)$.

We need the following previously known results.

THEOREM 1. [E] (Edgar). *Let G be a compact abelian metrizable group and let $\Lambda \subset \Gamma$. For a Banach space X the following conditions are equivalent.*

(i) *X has type I-Λ-RNP;*

(ii) *If*

$$T: L^1(G)/L^1_{\Lambda'}(G) \to X$$

is a bounded linear operator, where $\Lambda' = \{\gamma \in \Gamma : \bar{\gamma} \notin \Lambda\}$, *and*

$$Q: L^1(G) \to L^1(G)/L^1_{\Lambda'}(G)$$

is the natural quotient map, then TQ is a representable operator.

PROPOSITION 1. [D2] (Dowling). *Let G be a compact abelian metrizable group and let* Λ *be a Riesz subset of* Γ. *If* $L^1(G, X)$ *has type I-*Λ*-RNP then X has type II-*Λ*-RNP.*

DEFINITION 3. Let E and F be Banach spaces and suppose $T: E \to F$ is a bounded linear operator. T is said to be an *absolutely summing* operator if there is a constant $C > 0$ such that for any finite set $(x_m)_{1 \le m \le n}$ in E the following inequality holds:

$$\sum_{m=1}^{n} \|Tx_m\| \le C \sup\left\{ \sum_{m=1}^{n} |x^*(x_m)|;\ x^* \in E^*,\ \|x^*\| \le 1 \right\}.$$

The least constant C for the inequality above to hold will be denoted by $\|T\|_{as}$. It is easy to check that the class of all absolutely summing operators from E to F is a Banach space under the norm $\|T\|_{as}$. This Banach space will be denoted by $\Pi_1(E, F)$.

Equivalent formulations for the operator $T: E \to F$ to be absolutely summing can be found in [DU, p. 162]. If E happens to be $C(K)$ where K is a compact Hausdorff space, then T is absolutely summing if and only if its representing measure G (see [DU], p. 152) is of bounded variation and in this case $\|T\|_{as} = |G|(K)$ where $|G|(K)$ denotes the total variation of G. In this paper we take advantage of this fact by identifying the two Banach spaces $\Pi_1(C(K), F)$ and $M(K, F)$.

The following representation theorem due to Kalton [K] is the main ingredient of our approach.

THEOREM 2. [K] (Kalton). *Suppose that*

(i) *K is a compact metric space and λ is a Radon measure on K;*
(ii) *Ω is a Polish space and μ is a Radon finite measure on Ω;*
(iii) *X is a separable Banach space;*
(iv) *$T: L^1(\lambda) \to L^1(\mu, X)$ is a bounded linear operator.*

Then there is a map $\omega \to T_\omega(\Omega \to \Pi_1(C(K), X))$ *such that for every* $f \in C(K)$, *the map* $\omega \to T_\omega(f)$ *is Borel measurable from* Ω *to X and:*

α) *If* μ_ω *is the representing measure of* T_ω *then* $\int_\Omega |\mu_\omega|(B)\, d\mu(\omega) \le \|T\| \lambda(B)$ *for* $B \in \mathscr{B}(K)$.
β) *If* $f \in L^1(\lambda)$, *then for* μ a.e. *one has* $f \in L^1(|\mu_\omega|)$.
γ) $Tf(\omega) = T_\omega f$ μ a.e. *for every* $f \in L^1(\lambda)$.

The following proposition gives a characterization of representable operators.

PROPOSITION 2. *Under the assumptions of Theorem* 2, *the following two statements are equivalent*:

(i) *The operator T is representable;*

(ii) μ a.e. ω, μ_ω *has Bochner integrable density with respect to* λ.

Proof. Without loss of generality we can assume that $K = [0,1]$ and λ the Lebesgue measure.

Assume that $T\colon L^1(\lambda) \to L^1(\mu, X)$ is a representable operator. There exists $\psi \in L^\infty(\lambda, L^1(\mu, X))$ such that

$$Tf = \int f(t)\psi(t)\,d\lambda(t) \quad \text{for all } f \in L^1(\lambda).$$

Notice that for any measurable subset A of Ω, the map $I_A\colon L^1(\mu, X) \to X$ given by $I_A(h) = \int_A h(\omega)\,d\mu(\omega)$ is a bounded linear operator. Hence the operator $I_A \circ T$ is representable and its kernel is given by $K\colon \to X$ ($t\colon \to \int_A \psi(t)(\omega)\,d\mu(\omega)$). By Lemma 16 of [DS], there is a map $\Gamma\colon K \times \Omega \to X$ which is $\lambda \otimes \mu$-integrable and $\Gamma(t,.) = \psi(t)$ for λ a.e. $t \in K$. For any measurable subset A of Ω one has

$$\begin{aligned} I_A \circ T(f) &= \int_A T(f)(\omega)\,d\mu(\omega) \\ &= \int_K f(t) \int_A \Gamma(t,\omega)\,d\mu(\omega)\,d\lambda(t) \end{aligned}$$

and by Fubini's Theorem, this equals

$$\int_A \left(\int_K f(t)\Gamma(t,\omega)\,d\lambda(t) \right) d\mu(\omega).$$

Hence

$$Tf(\omega) = \int_K f(t)\Gamma(t,\omega)\,d\lambda(t) \text{ for } \mu \text{ a.e. } \omega.$$

One can apply Fubini's Theorem above since

$$\int_K |f(t)| \int_\Omega \|\Gamma(t,\omega)\|\,d\mu(\omega)\,d\lambda(t) \le \|\psi\|_\infty \|f\|_1$$

Now let $d\nu_\omega(\cdot) = \Gamma(\cdot, \omega)\, d\lambda(\cdot)$. We will show that $\nu_\mu = \mu_\omega$ for a.e. ω. For that, note that

$$\int_K f(t)\, d(\nu_\omega - \mu_\omega)(t) = 0 \text{ for } \mu \text{ a.e. } \omega.$$

Taking the exceptional set over all f_n's in a countable dense subset of $C(K)$, we can fix $\Omega_1 \subset \Omega$, $\mu(\Omega_1) = 0$ and for each $n \in \mathbf{N}$ and $\omega \notin \Omega_1$,

$$\int_K f_n(t)\, d(\nu_\omega - \mu_\omega)(t) = 0.$$

Since $(f_n)_n$ is dense in $C(K)$ then it follows that $\nu_\omega = \mu_\omega$ for every $\omega \in \Omega \setminus \Omega_1$.

Conversely, assume that there is a subset Ω_0 of Ω with $\mu(\Omega_0) = 0$ and for each $\omega \notin \Omega_0$, μ_ω has a Bochner integrable density with respect to λ i.e. there exists a map $\psi_\omega\colon K \to X$ Bochner integrable such that

$$T_\omega f = \int_K f(t)\psi_\omega(t)\, d\lambda(t) \quad \mu \text{ a.e.}$$

for all $f \in L^1(\lambda)$.

We need some lemmas.

LEMMA 1. *For each measurable subset A of K, the map*

$$\Omega \to X$$
$$\omega \to \int_A \psi_\omega(t)\, d\lambda(t) = \mu_\omega(A)$$

is norm measurable.

Proof. Fix $x^* \in X^*$, and consider the map $T^{x^*}\colon L^1(\lambda) \to L^1(\mu)$ given $T^{x^*}f(\omega) = x^*(Tf(\omega))$. Then the operator T^{x^*} is bounded and if we denote by $(\mu_\omega^{x^*})$ the representation given by Theorem 1 of [F], we have $\mu_\omega^{x^*}(A) = \langle \mu_\omega(A), x^* \rangle$, and, by (i) of the same theorem, $\omega \to \mu_\omega^{x^*}(A)$ is μ-measurable. Now an appeal to the Pettis measurability Theorem (see [DU]) shows that $\omega \to \mu_\omega(A)$ is norm measurable.

LEMMA 2. *The map $\Omega \to L^1(\lambda, X)$ which takes ω to $\psi_\omega(\cdot)$ is norm measurable.*

Proof. By Lemma 1, the map $\omega \to \int_A \psi_\omega(t)\, d\lambda(t)$ is measurable for each measurable subset A of K.

Let $(h_n)_n$ be a sequence of functions defined as follows:

$$h_n\colon \Omega \to L^1(\lambda, X)$$
$$\omega \to \sum_{i=2}^{2^n} 2^n \left(\int_{I_{n,i}} \psi_\omega(s)\, d\lambda(s) \right) \chi_{I_{n,i}}(\cdot).$$

where $(I_{n,i})_{1 \leq i \leq 2^n;\, n \in \mathbf{N}}$ denote the dyadic intervals of $[0,1]$.

The map h_n is clearly measurable for every $n \in \mathbf{N}$ and for every $\omega \in \Omega$, we have

$$\lim_{n \to \infty} \|h_n(\omega) - \psi_\omega(\cdot)\|_1 = 0.$$

Hence $\omega \to \psi_\omega(\cdot)$ is measurable.

To finish the proof of Proposition 1, notice that the map $h\colon \Omega \to L^1(\lambda, X)(\omega \to \psi_\omega(\cdot))$ belongs to $L^1(\mu, L^1(\lambda, X))$. In fact by the definition of ψ_ω we have $\|h(\omega)\| \leq |\mu_\omega|(K)$ and therefore $\int_\Omega \|h(\omega)\|\, d\mu(\omega) \leq \|T\|$ by α) of Theorem 1.

Now by Lemma 6 of [DS], there exists a map $H\colon K \times \Omega \to X$ which is $\lambda \otimes \mu$-integrable and such that $h(\omega) = H(.,\omega)$ for μ a.e. $\omega \in \Omega$.

Without loss of generality we can suppose that for every $t \in K$, the function $H(t,.) \in L^1(\mu, X)$ and the map $U\colon K \to L^1(\mu, X)$ defined by $U(t) = H(t,\cdot)$ is λ-integrable (see [DS], Lemma 16, p. 196). We claim that the map U represents T. To see that fix A a λ measurable subset of K and notice that $T(\chi_A)(\omega) = \int_A d\mu_\omega = \mu_\omega(A)$ for a.e. ω. On the other hand

$$\begin{aligned}\mu_\omega(A) &= \int_A \psi_\omega(t)\, d\lambda(t)\\ &= \int_A H(t,\omega)\, d\lambda(t)\\ &= \left(\int_A U(t)\, d\lambda(t) \right)(\omega).\end{aligned}$$

For the last equality see [DS, Theorem 17, p. 198].

This shows that $T(\chi_A) = \int_A U(t)\, d\lambda(t)$ for every λ-measurable subset A. Hence U represents T. □

III. Main results

Type I-II in Bochner functions spaces

Let (Ω, Σ, μ) be a finite measure space and E be a Köthe function space on (Ω, Σ, μ) (see [LT], p. 28). If X is a Banach space, then we will denote by

$E(X)$ the Banach space of all (classes of) measurable functions $f: \Omega \to X$ such that $\omega \to \|f(\omega)\|$ belongs to E. In particular, if $E = L^p(\mu)$, then $E(X) = L^p(\mu, X)$. In the sequel, we suppose that E is a linear subspace of $L^1(\mu)$ such that its unit ball is closed in $L^1(\mu)$.

DEFINITION 4 [LPP]. Let X and Y be two Banach spaces. We say that X semi-embeds into Y if there is a continuous bounded operator $T: X \to Y$ such that T is one to one and the image of the closed unit ball of X is closed in Y.

It is clear that if $1 \le p \le \infty$, then $L^p(\mu)$ semi-embeds in $L^1(\mu)$.
The following two lemmas are needed in the proof of Theorem 3.

LEMMA 3. *The space $E(X)$ semi-embeds into $L^1(\mu, X)$.*

Proof. Let $(f_n)_{n\in\mathbf{N}}$ be a sequence in the unit ball of $E(X)$ that converges to $f \in L^1(\mu, X)$. Let h_n and h be two functions defined as follows: $h_n(\omega) = \|f_n(\omega)\|$ and $h(\omega) = \|f(\omega)\|$. It is clear that h_n converges to h in $L^1(\mu)$. Since h_n belongs to the unit ball of E which is closed in $L^1(\mu)$, then h belongs to the unit ball of E. Hence f belongs to the unit ball of $E(X)$.

LEMMA 4. *Let Y and Z be Banach spaces such that Y is separable and semi-embeds into Z. If $(\Omega, \Sigma, \lambda)$ is a finite measure space, then the space $L^1(\lambda, Y)$ semi-embeds into $L^1(\lambda, Z)$.*

Proof. Let $J: Y \to Z$ be a semi-embedding. Assume that $\|J\| = 1$. We need the following fact.

FACT. *Let $(y_n)_n$ be a bounded sequence in Y such that $(Jy_n)_n$ converges to z in Z. Then there exists $y \in Y$ so that*
(a) $z = Jy$,
(b) $\|y\| \le \limsup_{n\to\infty} \|y_n\|$.

For this fact, notice that since J is a semi-embedding and $(y_n)_n$ is bounded, the existence of $y \in Y$ that satisfies $z = Jy$ is trivial. Now to prove (b), let us fix $k \in \mathbf{N}$ and let $r_k = \sup_{n\ge k} \|y_n\|$. Since $(y_n)_{n\ge l} \subset B_Y(0, r_k)$, we have $\|y\| \le r_k$ so we get

$$\|y\| \le \lim_{k\to\infty} r_k = \lim_{n\to\infty} \sup_{n\ge k} \|y_n\| = \limsup_{n\to\infty} \|y_n\|.$$

Now back to the proof of Lemma 4. Let $J^{\#}\colon L^1(\lambda, Y) \to L^1(\lambda, Z)$ defined by

$$J^{\#}(f)(\omega) = J(f(\omega)).$$

We will show that $J^{\#}$ is a semi-embedding. Let $(f_n)_n$ be a sequence in the unit ball of $L^1(\lambda, Y)$ so that $J^{\#}(f_n)$ converges to g in $L^1(\lambda, Z)$. There exists a subsequence $(f_{n_k})_k$ of $(f_n)_n$ so that

$$\lim_{k\to\infty} J^{\#}(f_{n_k})(t) = g(t) \quad \text{for a.e. } t \in \Omega$$

that is equivalent to

$$\lim_{k\to\infty} J\big(f_{n_k}(t)\big) = g(t) \quad \text{for a.e. } t \in \Omega.$$

On the other hand, since $(\|f_{n_k}(\cdot)\|)_k$ is bounded in $L^1(\lambda)$, one can find, using Komlòs result (see [D], p. 121), a further subsequence $(f'_k)_k$ and a function $F \in L^1(\lambda)$ so that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \|f'_k(t)\| = F(t) \quad \text{for a.e. } t \in \Omega.$$

Notice that F belongs to the unit ball of $L^1(\lambda)$ and

$$\lim_{m\to\infty} J\left(\frac{1}{m} \sum_{k=1}^{m} f'_k(t)\right) = g(t) \quad \text{for a.e. } t \in \Omega$$

and

$$\left(\frac{1}{m} \sum_{k=1}^{m} f'_k(t)\right)_k$$

is bounded for $t \notin \Omega_1$ where $\lambda(\Omega_1) = 0$. For $t \notin \Omega_1$, let us define $f(t)$ to be the unique element of Y such that $g(t) = J(f(t))$. For $t \in \Omega_1$, define $f(t) = 0$. By [CO, p. 276], the function f is λ measurable. Use now the previous fact to deduce that

$$\begin{aligned}\|f(t)\| &\le \limsup_{m\to\infty} \left\| \frac{1}{m} \sum_{k=1}^{m} f'_k(t) \right\| \\ &\le \limsup_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \|f'_k(t)\| \\ &= F(t)\end{aligned}$$

and since $F \in L^1(\lambda)$, we have $\|f(\cdot)\| \in L^1(\lambda)$ and therefore $f \in L^1(\lambda, Y)$ and $g = J^{\#}(f)$. □

Remark 1. In [D1], Dowling observed that the Bourgain-Rosenthal [BR] technique would imply that if a separable Banach space Y semi-embeds in a Banach space X that has type I-Λ-RNP, then Y has the same property. This fact is needed in the proof of Theorem 3.

Before stating Theorem 3, we claim that using Lemma 4, one can show the same stability result of semi-embedding for type II-Λ-RNP. This is done in the next proposition which is needed in the proof of Theorem 4.

PROPOSITION 3. *If a separable Banach space Y semi-embeds in a Banach space X that has type II-Λ-RNP, then Y has the same property.*

Proof. Let J be the semi-embedding from Y to X and let $m \in M_{\Lambda, ac}(G, Y)$ this implies that $Jm \in M_{\Lambda, ac}(G, X)$. Since X has type II-Λ-RNP, choose $g \in L^1_\Lambda(G, X)$ such that $Jm(A) = \int_A g\, d\lambda$ for every measurable subset A of G. For every finite measurable partition π of G consider the function

$$g_\pi = \sum_{A \in \pi} \frac{\int_A g\, d\lambda}{\lambda(A)} \chi_A$$

and let

$$f_\pi = \sum_{A \in \pi} \frac{m(A)}{\lambda(A)} \chi_A.$$

It is well known that the net g_π converges to g in mean. It is clear also that $\|f_\pi\|_1 \leq |m|(G)$. Now let $J^\#$ be the semi-embedding from $L^1(G, Y)$ to $L^1(G, X)$ as in Lemma 4. By the definition of $J^\#$, we have $g_\pi = J^\#(f_\pi)$. Apply now Lemma 4 to get a function $f \in L^1(G, Y)$ such that $J(f(t)) = g(t)$ for a.e. $t \in G$. Notice that if A is measurable subset of G, then

$$J\left(\int_A f\, d\lambda\right) = \int_A Jf\, d\lambda = \int_A g\, d\lambda = J(m(A)).$$

Since J is one to one, this shows that $\int_A f\, d\lambda = m(A)$. □

THEOREM 3. *Let Λ be a Riesz set of Γ and let (Ω, Σ, μ) be a finite measure space. If E is a separable Köthe function space that semi-embeds into $L^1(\mu)$, then the following assertions are equivalent*:

(i) *X has type II-Λ-RNP.*

(ii) *$E(X)$ has type II-Λ-RNP.*

In particular, for $1 \leq p < \infty$, then $L^p(\mu, X)$ has type II-Λ-RNP whenever X does.

That (ii) implies (i) is evident. For the proof the converse implication, we can assume without loss of generalities, that X is separable, Ω is a compact metric space.

Let $\Omega_1 = G \times \Omega$ with the product measure $\lambda \otimes \mu$. We first claim that $L^1(\Omega_1, \lambda \otimes \mu, X)$ has type I-Λ-RNP. To do that let us consider a bounded linear operator T from $L^1(G)$ to $L^1(\Omega_1, X)$ that factors through $L^1(G)/L^1_{\Lambda'}(G)$ via the quotient map Q, where $\Lambda' = \{\gamma \in \Gamma, \bar{\gamma} \notin \Lambda\}$. To prove our claim, it is enough to show that T is representable. To do that, consider $\Omega_1 \to M(G, X)(s \to \mu_s)$ the representation of T given by Theorem 2. By Proposition 2, it is enough to show that μ_s has Bochner integrable density with respect to λ for $\lambda \otimes \mu$ a.e. $s \in \Omega_1$. For that let $\gamma \notin \Lambda$ fixed. Since $\bar{\gamma} \in \Lambda', T(\bar{\gamma}) = 0$. On the other hand

$$T(\bar{\gamma})(s) = \int_G \bar{\gamma}\, d\mu_s = \hat{\mu}_s(\gamma)$$

for $s \notin \Omega$, where Ω_γ is of measure zero in Ω_1. Since Γ is countable, we can deduce that $\mu_s \in M_\Lambda(G, X)$ except on a set of measure zero. This fact together with the assumption that Λ is a Riesz set shows that there exists $\Omega_0 \subset \Omega_1$ with $(\lambda \otimes \mu)(\Omega_0) = 0$ such that for every $s \in \Omega_1 \setminus \Omega_0$, $\mu_s \in M_{\Lambda, ac}(G, X)$ and since X has type II-Λ-RNP, μ_s has Bochner integrable density with respect to λ.

To complete the proof of the theorem notice that by Lemma 3, the space $E(X)$ semi-embeds into $L^1(\mu, X)$ and by Lemma 4, $L^1(G, E(X))$ semi-embeds into $L^1(G, L^1(\mu, X))$ which can be identified with $L^1(\Omega_1, X)$ and therefore by Remark 1, $L^1(G, E(X))$ has type I-Λ-RNP. Now Proposition 1 implies that $E(X)$ has type II-Λ-RNP. □

In [E], Edgar asked the following question: If Λ is a Riesz set are the type I-Λ-RNP and type II-Λ-RNP equivalent properties?

As a corollary of the theorem, we get the following positive result:

COROLLARY 1. *Let Λ be a Riesz set of Γ and let X be a Banach space. Then $L^1(G, X)$ has type I-Λ-RNP if and only if it has type II-Λ-RNP.*

Remark 2. In [D1], Dowling showed that if Λ is a Riesz subset of $\mathbf{Z}$, then X has type II-Λ-RNP if and only if $L^1(\mathbf{T}, X)$ has the type I-Λ-RNP. Corollary 1 extends Dowling's result to any compact abelian metrizable group G.

If one wants to consider non-Riesz subsets, the following (weaker) result holds.

PROPOSITION 4. *Let $\Lambda \subset \Gamma$ (not necessarily a Riesz set) and let X be a Banach space. If E has type I-Λ-RNP and X has type II-Λ-RNP then $E(X)$ has type I-Λ-RNP.*

Proof. Same as before, we can assume that X is separable. Let T: $L^1(G) \to E(X)$ be a linear operator that factors through $L^1(G)/L^1_{\Lambda'}(G)$ via the quotient map and J: $E(X) \to L^1(\mu, X)$ the semi-embedding. We need to show that $S = J \circ T$ is representable.

Consider as in the proof of Theorem 3, $\Omega \to M(G, X)(\omega \to \mu_\omega)$ the representation of S. With the same argument as in Theorem 3, we have $\mu_\omega \in M_\Lambda(G, X)$ for a.e. $\omega \in \Omega$. We are done if we can show that $\mu_\omega \ll \lambda$ for a.e. $\omega \in \Omega$. For that let us fix $x^* \in X^*$, the map

$$S^{x^*}: L^1(G) \to L^1(\mu)$$
$$f \to \langle Sf(\cdot), x^* \rangle$$

is represented by $\omega \to \mu_\omega^{x^*}(A) = \langle \mu_\omega(A), x^* \rangle)$. It is also easy to see that S^{x^*} factors through $L^1(G)/L^1_{\Lambda'}(G)$ and E as follows:

$$\begin{array}{ccc} L^1(G) & \xrightarrow{S^{x^*}} & L^1(\mu) \\ Q\big\downarrow & & \big\uparrow j \\ L^1(G)/L^1_{\Lambda'}(G) & \xrightarrow{L} & E \end{array}$$

where j: E $\to$ L$^1(\mu)$ *is the semi-embedding and L is a bounded linear operator.*

Since E has type I-Λ-RNP, S^{x^*} is representable. Now we have $\mu_\omega^{x^*} \ll \lambda$ for a.e. ω by Proposition 1 of [F] and we conclude the proof using similar argument as in Lemma 1 of [RS]. □

Remark 3. Theorem 3 and Proposition 4 deal with when some type of Radon-Nikodym property passes from a Banach space X to some function spaces with values in X. This question has been investigated before. Turett and Uhl [TU] showed that the RNP passes from X to $L^p(X)$ for $1 < p < \infty$. Dowling [D4] showed that the analytic RNP passes from X to $L^p(X)$ for $1 \leq p < \infty$. It is clear that while Theorem 3 and Proposition 4 give a different proof to these results, they are also more general and provide a new approach to dealing with this kind of stability problem.

In [D3], Dowling showed directly that if Λ is subset of Γ and X is Banach space having the RNP and if $L^\infty_\Lambda(G, X)$ is separable then $L^\infty_\Lambda(G, X)$ has the RNP. The next theorem is in the same spirit. Before stating the theorem, let us give a quick proof of Dowling's result. To do that notice that $L^\infty_\Lambda(G, X)$ semi-embeds in $L^2(G, X)$ which has RNP and hence any separable Banach space that semi-embeds into it will inherit the RNP [BR].

For the next result, we need to introduce a new compact metrizable abelian group $\tilde{G}$ which is not necessarily the same as G. We will denote by $\tilde{\Gamma}$ its dual and $\tilde{\lambda}$ its normalized Haar measure.

THEOREM 4. *Let Λ be a Riesz-subset of Γ and X be a separable Banach space having type II-Λ-RNP (resp. type I-Λ-RNP). Consider $\tilde{\Lambda}$ a subset of $\tilde{\Gamma}$. If $L^{\infty}_{\tilde{\Lambda}}(\tilde{G}, X)$ is separable then it has type II-Λ-RNP (resp. type I-Λ-RNP).*

For the type I case, we do not need the assumption Λ being a Riesz set.

COROLLARY 4. *Assume that*
(1) *$\tilde{\Lambda}$ is a Rosenthal set of $\tilde{\Gamma}$ and*
(2) *X is a Banach space that has the Schur property.*
This $C_{\tilde{\Lambda}}(\tilde{G}, X)$ has type I-Λ-RNP if and only if X does. (Here Λ is not necessarily a Riesz-set.)

In particular, if X has the Schur property and $\tilde{\Lambda}$ is a Rosenthal set, then $C_{\tilde{\Lambda}}(\tilde{G}, X)$ has RNP (resp. ARNP) if and only if X has RNP (resp. ARNP).

Proof of Theorem 4. Assume first that X has type II-Λ-RNP. By Theorem 3, $L^1(\tilde{G}, X)$ has type II-Λ-RNP and since $L^{\infty}_{\tilde{\Lambda}}(\tilde{G}, X)$ is separable and semi-embeds in $L^1(\tilde{G}, X)$, the conclusion follows by Proposition 3.

For the type I-case, we will use similar argument as in the proof of Theorem 3.

Let T be an operator form $L^1(G)$ to $L^{\infty}_{\tilde{\Lambda}}(\tilde{G}, X)$ that factors through $L^1(G)/L^1_{\Lambda'}(G)$ via the quotient map Q and J: $L^{\infty}_{\tilde{\Lambda}}(\tilde{G}, X) \to L^1(\tilde{G}, X)$ the natural inclusion which is a semi-embedding.

Consider the representation of $J \circ T$ given by Theorem 2.

$$\begin{aligned} \tilde{G} &\to M(G, X) \\ \omega &\to \mu_{\omega}. \end{aligned}$$

Since $J \circ T$ factors through $L^1(G)/L^1_{\Lambda'}(G)$, we have $\mu_{\omega} \in M_{\Lambda}(G, X)$ for a.e. $\omega \in \tilde{G}$. We claim that for $\tilde{\lambda}$ a.e. ω, $|\mu_{\omega}| \leq \|T\| \lambda$. To see this, consider $(f_n)_n$ be a countable dense subset of $C(G)$. Since T takes its value in $L^{\infty}(\tilde{G}, X)$, we have $\|J \circ T(f_n)\|_{\infty} \leq \|T\| \|f_n\|_1$. Hence one can find a measurable subset A_n of $\tilde{G}$ such that $\tilde{\lambda}(A_n) = 0$ and

$$\|T(f_n)(\omega)\| = \left\| \int f_n \, d\mu_{\omega} \right\| \leq \|T\| \|f_n\|_1$$

for each $\omega \notin A_n$. Now let $A = \bigcup_{n \in \mathbf{N}} A_n$, $\tilde{\lambda}(A) = 0$ and for every $\omega \notin A$, $n \in \mathbf{N}$, we have $\|\int f_n \, d\mu_{\omega}\| \leq \|T\| \|f_n\|_1$ and since $(f_n)_n$ is also dense in $L^1(G)$, we have

$$\left\| \int f \, d\mu_{\omega} \right\| \leq \|T\| \|f\|_1 \quad \text{for every } f \in L^1(G),$$

in particular for every Borel subset of G, $\|\mu_{\omega}(B)\| \leq \|T\| \lambda(B)$ which shows the claim.

As a consequence, $\mu_\omega \in M^\infty_\Lambda(G, X)$ for a.e. $\omega \in \tilde{G}$. If X has type I-Λ-RNP, then μ_ω has Bochner density with respect to λ for a.e. ω and apply Proposition 2 to conclude that $J \circ T$ is representable. Since J is a semi-embedding and $L^\infty_\Lambda(\tilde{G}, X)$ is separable, then one can conclude that T is representable [BR]. □

Type II-Λ-RNP for the space of vector valued measures

In this section, we will show that a similar stability results holds for the space of vector valued measures when G is the circle group $\mathbf{T}$.

THEOREM 5. *Let $X = Y^*$ be a dual space and (Ω, Σ) a measure space. Assume that Λ is a Riesz subset of $\mathbf{Z}$ then $M(\Omega, X)$ has type II-Λ-RNP whenever X does. In particular $M(\Omega, X)$ has the ARNP whenever X does.*

For the proof, we will view the space $M(\Omega, X)$ as a function space using weak*-densities and liftings.

Proof. Suppose that X has type II-Λ-RNP and $\{a_m\}_{m \in \Lambda} \subset M(\Omega, X)$.
Let $(P_r)_{0 \le r < 1}$ be the usual Poisson Kernel. We have $\hat{P}_r(m) = r^{|m|}$. For $r_n = 1 - 1/n$, define

$$f_n(t) = \sum_{m \in \Lambda} r_n^{|m|} a_m e^{imt}.$$

It sufficesto consider the case where $(f_n)_{n \in \mathbf{N}}$ is bounded in $L^1(\mathbf{T}, M(\Omega, X))$. Note that $P_{r_n/r_{n+1}} * f_{n+1} = f_n$ and $\|P_{r_n/r_{n+1}}\|_1 = 1$. Therefore

$$\|f_n\|_{L^1(\mathbf{T}, M(\Omega, X))} \le \|f_{n+1}\|_{L^1(\mathbf{T}, M(\Omega, X))}$$

and so we have

$$\lim_{n \to \infty} \|f_n\|_{L^1(\mathbf{T}, M(\Omega, X))} = \sup_{n \in \mathbf{N}} \|f_n\|_{L^1(\mathbf{T}, M(\Omega, X))} < \infty.$$

Suppose the sequence $(f_n)_{n \in \mathbf{N}}$ is bounded in $L^1(\mathbf{T}, M(\Omega, X))$, one can easily deduce that the sequence $(a_m)_{m \in \Lambda}$ is bounded. Now define a measure $\mu \in M(\Omega)$ by

$$\mu = \sum_{m \in \Lambda} \frac{|a_m|}{2^m}$$

and let ρ be a lifting of $L^\infty(\mu)$ (see [IT]).

By a similar argument as above, if we define $\rho[f_n(\cdot)](\omega)$ by

$$\rho[f_n(t)](\omega) = \sum_{m\in\Lambda} r_n^{|m|}\rho(a_m)(\omega)\, e^{imt}$$

then

$$\lim_{n\to\infty} \|\rho[f_n(\cdot)](\omega)\|_{L^1(\mathbf{T},X)} = \sup_{n\in\mathbf{N}} \|\rho[f_n(\cdot)]\|_{L^1(\mathbf{T},X)}$$

for every $\omega \in \Omega$.

Now using the same method as in [D1], we can show that

$$\lim_{n,k\to\infty} \|f_n - f_k\|_{L^1(\mathbf{T},M(\Omega,X))} = 0$$

taking advantage of the well-known fact that $\|m\| = \int_\Omega \|\rho(m)(\omega)\|\, d\mu(\omega)$.

So $(f_n)_{n\in\mathbf{N}}$ is convergent in $L^1(\mathbf{T}, M(\Omega, X))$ which shows that $M(\Omega, X)$ has type II-Λ-RNP (see Theorem 6 of [D1]). □

Remark. If the space X is not a dual space, then the conclusion of Theorem 5 is no longer valid. In fact the space E constructed by Talagrand in [T] is a Banach lattice that does not contain c_0 and therefore has type II-Λ-RNP for every Riesz subset of $\mathbf{Z}$ but $M([0,1], E)$ contains a copy of c_0.

Let us finish by asking the following question.

Question. Does type I-Λ-RNP pass from X to $L^p(\mu, X)$ for a Riesz subsets Λ of G where $1 \le p < \infty$?

This question is equivalent to the still open problem of whether or not type II-Λ-RNP and type I-Λ-RNP are equivalent for a Banach space X and a Riesz subset Λ.

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