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FINITENESS OF $\bigcup_e \operatorname{Ass} F^e(M)$ AND ITS CONNECTIONS TO TIGHT CLOSURE

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1. Introduction

Throughout this paper, all rings are commutative with identity and Noetherian; p will always denote a prime integer, and q will be some power p^e. A local ring is defined as a Noetherian ring with a unique maximal ideal. Let R be a ring of prime characteristic p, let $N \subset M$ be finitely generated R-modules. In [HH], M. Hochster and C. Huneke introduced the notion of *the tight closure of* N *in* M as follows:

Let S be R viewed as an R-algebra via the iterated Frobenius endomorphism $r \mapsto r^q$ and define the *Peskine-Szpiro functor* F^e from R-modules to S-modules by $F^e(M) = S \otimes_R M$. Since the category of S-modules *is* the category of R-modules, we may view F^e as a functor from the category of R-modules to itself.

The R-module structure on $F^e(M)$ is such that $r'(r \otimes m) = (rr') \otimes m$ and we also have $r' \otimes (rm) = (r'r^q) \otimes m$. If $I \subset R$ is an ideal then $F^e(R/I) = R/I^{[q]}$, and generally if we apply F^e to a map $R^a \to R^b$ given by a matrix (c_{ij}) by identifying $F^e(R^a) \cong R^a$ and $F^e(R^b) \cong R^b$ (this identification is not canonical, it depends on a choice of generators for the free modules), we obtain a map $F^e(c_{ij})$: $R^a \to R^b$ given by the matrix (c_{ij}^q). There is a natural map $M \to F^e(M)$ given by $m \mapsto m \otimes 1$, and we denote the image of m under this map by m^q.

If $N \subset M$ are R-modules, we have an exact sequence

$$F^e(N) \to F^e(M) \to F^e(M/N) \to 0$$

and we write $N_M^{[q]}$ for

$$\operatorname{Ker}(F^e(M) \to F^e(M/N)) \cong \operatorname{Im}(F^e(N) \to F^e(M)).$$

Let R^0 be the set of all elements in R not in any minimal prime of R. Let $N \subset M$ be R-modules. The *tight closure of* N *in* M, N_M^*, is defined as the set of all elements $m \in M$ such that $cm^q \in N_M^{[q]}$ for some $c \in R^0$ and all large q. If $N_M^* = N$ we say that N *is tightly closed in* M.

We also define $G^e(M) = F^e(M)/0^*_{F^e(M)}$. Notice that $G^e(M/N) \cong F^e(M)/N_M^{[q]*}$.

We refer the reader to [HH] for a description of the basic properties of tight closure.

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2. Commutativity of localization with tight closure and the set $\bigcup_e \operatorname{Ass} F^e(M)$

With the notation above, let $S \subset R$ be a multiplicative system.

We always have $S^{-1}(N_M^*) \subset (S^{-1}N)^*_{S^{-1}M}$ and we would like to know whether $S^{-1}(N_M^*) = (S^{-1}N)^*_{S^{-1}M}$. This question still remains open in this generality. However, in a special case, to be discussed below, an affirmative answer has been found.

DEFINITION 1. *Let R be a ring of prime characteristic p, and let*

$$G_\bullet = 0 \to G_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} G_0 \to 0$$

be a complex of finitely generated projective R-modules.

(1) *The complex $G_\bullet$ is said to have phantom homology at the ith spot if* $\operatorname{Im} d_{i+1}$ *is in the tight closure of* $\operatorname{Ker} d_i$ *in G_i.*
(2) *The complex $G_\bullet$ is said to be stably phantom acyclic if the complex $F^e(G_\bullet)$ has phantom homology for all $i \geq 1$ and all $e > 0$.*
(3) *An R-module M is said to have finite phantom projective dimension if there exists a finite stably phantom acyclic complex of projective R-modules whose zeroth homology is isomorphic to M.*

In [AHH] it is shown that if $N \subset M$ are R-modules and M/N has finite phantom projective dimension then $S^{-1}(N_M^*) = (S^{-1}N)^*_{S^{-1}M}$ for any multiplicative system $S \subset R$, and the proof of this statement uses the fact that under the hypotheses above $\bigcup_e \operatorname{Ass} F^e(M/N)$ is finite. This, together with the following two theorems below, give a motivation for studying the problem of whether in general $\bigcup_e \operatorname{Ass} F^e(M/N)$ (or $\bigcup_e \operatorname{Ass} G^e(M/N)$) is finite or has finitely many maximal elements.

THEOREM [AHH] 2. *Let R be a ring of prime characteristic p, let $N \subset M$ be finitely generated R-modules and let $S \subset R$ be multiplicative system.*

(*a*) *Every element of $(S^{-1}R)^0$ is a product of a unit in $(S^{-1}R)^0$ and an element in the image of R^0.*
(*b*) *Let $u \in M$ and $s \in S$. Then $u/w \in (S^{-1}N)^*_{S^{-1}M}$ if and only if there exists a $d \in R^0$ such that $s_e d u^q \in N_M^{[q]}$ for some $s_e \in S$ for all large q. If R has a locally stable weak test element* (*resp. a completely stable weak test element*) *then d may be chosen to be a locally stable weak test element* (*resp. a completely stable weak test element*).
(*c*) *If S is disjoint from $\bigcup_e \operatorname{Ass} F^e(M/N)$ then $S^{-1}(N_M^*) = (S^{-1}N)^*_{S^{-1}M}$.*
(*d*) *If S is disjoint from $\bigcup_e \operatorname{Ass} G^e(M/N)$ then $S^{-1}(N_M^*) = (S^{-1}N)^*_{S^{-1}M}$.*

Further motivation for the study of $\bigcup_e \operatorname{Ass} F^e(M)$ is given by Theorem 5 whose proof relies on the following two lemmas.

LEMMA 3. *Let R be a semi-local ring orprime characteristic p with a completely stable q'-weak test element c, and let J be its Jacobson radical. Denote by $\widehat{}$ the completion with respect to J. If tight closure fails to commute with localization at a multiplicative system $W \subset R$ for some pair of R-modules $N \subset M$, then it fails to commute for a pair of $\hat{R}$-modules.*

Proof. Pick a $u \in M$ with $u \notin N_M^*$ while $u \in (W^{-1}N)^*_{W^{-1}M}$. For all $q > q'$, $cu^q \notin N_M^{[q]}$, and since $\widehat{R}$ is faithfully flat, tensoring with $\widehat{R}$ we obtain

$$c\widehat{u}^q \notin \widehat{N}^{[q]}_{\widehat{M}}.$$

On the other hand, since $u \in (W^{-1}N)^*_{W^{-1}M}$, by Lemma 2b we have $w_e c u^q \in N_M^{[q]}$ for some $w_e \in W$ and for all $q > q'$. Tensoring the exact sequence

$$0 \to \operatorname{Ann}_{M/N_M^{[q]}} w_e \to M/N_M^{[q]} \xrightarrow{w_e} M/N_M^{[q]}$$

with the faithfully flat extension $\widehat{R}$ we obtain $\widehat{(\operatorname{Ann}_{M/N_M^{[q]}} w_e)} = \operatorname{Ann}_{\widehat{M}/\widehat{N}^{[q]}_{\widehat{M}}} w_e$ hence $c\widehat{u}^q \in \operatorname{Ann}_{\widehat{M}/\widehat{N}^{[q]}_{\widehat{M}}} w_e$ and by Lemma 2b, $u \in (W^{-1}\widehat{N})^*_{W^{-1}\widehat{M}}$. $\square$

LEMMA 4. *Let $R = R_1 \times \cdots \times R_n$ be a ring of prime characteristic p, and let $N \subset M$ be R-modules. Let $N_i \subset M_i (1 \leq i \leq n)$ be R-modules, and let $M = M_1 \times \cdots \times M_n$, $N = N_1 \times \cdots \times N_n$ be the corresponding decompositions of N and M. Then*

$$N_M^* = N^*_{1\,M_1} \times \ldots \times N^*_{n\,M_n}.$$

Proof. Notice that $R^0 = R_1^0 \times \cdots \times R_n^0$, $F_R^e(M) \cong F^e_{R_1}(M_1) \times \cdots \times F^e_{R_n}(M_n)$ and $N_M^{[q]} = N_1^{[q]}{}_{M_1} \times \ldots \times N_n^{[q]}{}_{M_n}$ where each $N_i^{[q]}{}_{M_i}$ is computed over R_i. Now, $u = (u_1, \ldots, u_n) \in N_M^* \Leftrightarrow$ there exists a $c = (c_1, \ldots, c_n) \in R^0$ such that $cu^q \in N^{[q]}$ $\Leftrightarrow c_i u_i^q \in N_i^{[q]}{}_{M_i}$ for all $1 \leq i \leq n \Leftrightarrow u_i \in N_i^*{}_{M_i}$ for all $1 \leq i \leq n$. $\square$

THEOREM 5. *Let R have a q'-weak test element, and let $N \subset M$ be R-modules. Assume that either $S = \bigcup_e \operatorname{Ass} F^e(M/N)$ or $S' = \bigcup_e \operatorname{Ass} G^e(M/N)$ has finitely many maximal elements. If localization does not commute with tight closure for the pair $N \subset M$, then we can find a counter-example when R is complete local and we are localizing at a prime ideal $P \subset R$ with* $\dim(R/P) = 1$.

Proof. By Lemma 3.5a in [AHH], we may assume that we have a counterexample in which localization at a prime ideal P fails to commute with tight closure. Let W be the complement of $P \cup \left(\bigcup S\right)$ (respectively $P \cup \left(\bigcup S'\right)$) in R. By the previous theorem, we may localize R at W without affecting any relevant issues, hence we may assume that R is semi-local.

Let J be the Jacobson radical of R, and let $\widehat{\ }$ denote the completion at J. By Lemma 3 we can find a counter-example over $\widehat{R}$, hence we may substitute R with $\widehat{R}$, and we may assume that R is a product of complete local rings $R = (R_1, m_1) \times \cdots \times (R_n, m_n)$ and we also get a decomposition $M = M_1 \times \cdots \times M_n$ $N = N_1 \times \cdots \times N_n$ where $N_i \subset M_i$ are R_i modules.

By Lemma 4 we have

$$N_M^* = N_{1\,M_1}^* \times \cdots \times N_{n\,M_n}^*$$

and one of the pairs, say $N_1 \subset M_1$, must give a counterexample over a local ring (R_1, m_1); hence we may assume R is local.

If $P = m$, we obviously cannot have a counterexample, so assume that P is not maximal, and let

$$P = P_0 \subset P_1 \subset \cdots \subset P_l = m$$

be a saturated chain of primes. Let $0 \leq i < l$ be the maximal number such that localization at P_i does not commute with tight closure for the pair $N \subset M$ and replace R, M, N and P with $R_{P_{i+1}}$, $M_{P_{i+1}}$, $N_{P_{i+1}}$ and P_i. $\square$

THEOREM 6. *Assume that for any local ring (R, m) of prime characteristic p and every finitely generated R-module $\overline{M}$ the set $\bigcup_e \operatorname{Ass} G^e(\overline{M})$ has finitely many maximal elements. If, in addition, for every R-module $\overline{M}$ there exists a positive integer $B > 0$ such that m^{qB} kills $\mathrm{H}_m^0(F^e(\overline{M}))$ (or $\mathrm{H}_m^0(G^e(\overline{M}))$) then tight closure commutes with localization.*

Proof. Pick a counterexample consisting of a local ring R, R-modules $N \subset M$ and a multiplicative system $S \subset R$. By the previous theorem we may assume that R is a complete local ring, and we are localizing at a prime $P \subset R$ with $\dim R/P = 1$. We may also assume that we have chosen our counterexample with $\dim R$ minimal. Since tight closure can be computed modulo the minimal primes or R, we may further assume that R is a domain, and hence module finite and torsion free over a regular ring, and we may assume R has weak test elements (see Section 6 in [HH].)

We may replace the pair $N \subset M$ with the pair $0 \subset \overline{M} = M/N_M^*$, hence we may assume that $N = 0$ and 0 is tightly closed in M.

Pick some $u \in M$ with $u \in 0_{M_P}^*$ while $u/1 \neq 0$ in M_P. For all $f \in m - P$ we have $u/1 \in 0_{M_f}^*$, otherwise we get a counterexample over a ring of smaller dimension. Hence for all $q \gg 0$ there exists a positive integer $N(q)$ such that $f^{N(q)}u^q = 0$ in $F^e(M)$, and since the ideal generated by all elements in $m - P$ is m, we have $u^q \in \mathrm{H}_m^0(F^e(M))$ (and hence $u^q \in \mathrm{H}_m^0(G^e(M))$.) Pick a test element $c \in R^0$.

If m^{qB} kills $\mathrm{H}_m^0(F^e(\overline{M}))$, then for all $f \in m - P$ we have $f^{Bq}cu^q = 0$ and $f^B u \in 0_M^*$ but as $0_M^* = 0$ we have $f^B u = 0$ in M, contradicting the choice of $u/1 \neq 0$ in M_P.

If m^{qB} kills $H^0_m(G^e(\overline{M}))$, then $f^{Bq}cu^q \in 0^*_{F^e(M)}$ for all $q \gg 0$, and by Lemma 8.16 in [HH] we have $f^B u \in 0^*_M = 0$ arriving again at a contradiction. $\square$

3. The set $\bigcup_e \mathrm{Ass}(F^e(M))$ over a hypersurface

In the rest of this section we will study the set $\bigcup_e \mathrm{Ass}(F^e(M))$, over a hypersurface $R = A[x_1, \dots, x_n]/F$ where A is a domain and with $M = R/(x_1, \dots, x_n)$. It has been shown that in some interesting cases the set $\bigcup_e \mathrm{Ass}(F^e(M))$ is finite [Kat], but there is a surprisingly simple counterexample for the finiteness of this set in the general case.

We fix the ring A to be a domain of characteristic $p > 0$, and q will always denote p^e for some positive integer e. For any $F \in A[x_1, \dots, x_n]$ let $R_F = A[x_1, \dots, x_n]/F$ and let $M_F = R_F/(x_1, \dots, x_n)R_F$.

DEFINITION 7. *A sequence $\{M_n\}_n$ of A-modules has finite torsion if there exists a non-zero $a \in A$ such that $(M_n)_a$ is a torsion free A_a module for all n.*

LEMMA 8. *Let A be a domain and let $B \supset A$ be a module finite extension domain. Let $\{M_i\}_i$ be a sequence of B modules such that $\{M_i\}_i$ has finite torsion over B. Then $\{M_i\}_i$ has finite torsion over A.*

Proof. Pick some nonzero $b \in B$ such that the modules $(M_b)_i$ are torsion free over B and choose $a \in A$ to be a nonzero multiple of b in A. Clearly, $(M_a)_i$ are torsion free over A. $\square$

LEMMA 9. *Let A be a domain which is also a k-algebra, let $R = A[x_1, \dots, x_n]$ and let $I \subset R$ be an ideal generated by elements in $k[x_1, \dots, x_n]$. Then any non-zero $\alpha \in A$ is a nonzero divisor on R/I.*

Proof. Since $k[x_1, \dots, x_n]/I$ is flat over k, $R = k[x_1, \dots, x_n]/I \otimes_k A$ is flat over A. $\square$

Let A be a domain and let $R = A[x, y]$. If $F \in R$ is a homogeneous polynomial, in view of Lemma 8, we may replace A with a localization at one element of a module finite extension of A to obtain a splitting of F into linear factors

$$F(x, y) = \sum_{k=1}^{s} (a_k x + b_k y)^{r_k}$$

where $a_k, b_k \in A$ for $1 \le k \le s$.

LEMMA 10. *If the number of different linear factors in F is at most 3, then the modules $\{F^e(M_F)\}_e$ have finite A-torsion.*

Proof. We can make a change of variables so that the different linear factors of F are among x, y and $(x-y)$.

When $F = x^{s_1}$ or $F = x^{s_1}y^{s_2}$ or $F = x^{s_1}y^{s_2}(x-y)^{s_3}$ the modules $\{F^e(M_F)\}_e$ have no A–torsion by Lemma 9. □

In view of this lemma, the first interesting case is when F is a product of four different linear factors, and indeed our next aim is to produce a F which is a product of four linear factors for which the modules $\{F^e(M_F)\}_e$ do *not* have finite A-torsion.

But first we need the following lemma:

LEMMA 11. *Let A be a domain and let $R = A[x, y]$. Then*

$$\left(x^{q-1}, y^{q-1}\right) :_R (x-y)$$

is generated by y^{q-1} and $\gamma = x^{q-2} + x^{q-3}y + \ldots + xy^{q-3} + y^{q-2}$.

Proof. Assume that $a(x-y) = bx^{q-1} + cy^{q-1}$ for some $a, b, c \in R$. Working modulo $x-y$ we have $\bar{b}x^{q-1} + \bar{c}x^{q-1} \equiv 0$; hence $\bar{b} + \bar{c} \equiv 0$ and we can write $c = -b + d(x-y)$ for some $d \in R$. We can write $a(x-y) = bx^{q-1} + (-b + d(x-y))y^{q-1} \Rightarrow (a - dy^{q-1})(x-y) = bx^{q-1} - by^{q-1} \Rightarrow a - dy^{q-1} = b\gamma \Rightarrow a \in \left(y^{q-1}, \gamma\right)$ □

THEOREM 12. *Let $A = k[t]$, $R = A[x, y]$. Let $F = xy(x-y)(x-ty) \in R$. The modules $\{F^e(M_F)\}_e$ do* **not** *have finite A-torsion.*

Proof. We will first show that for all $q = p^e$,

$$\tau G \in \left(x^q, y^q, xy(x-y)(x-ty)\right)$$

where $G = xy(x-y)y^{q-2}$ and $\tau = 1 + t + \ldots + t^{q-2}$, while $G \notin (x^q, y^q, xy(x-y)(x-ty))$.

Let $\gamma = x^{q-2} + x^{q-3}y + \ldots + xy^{q-3} + y^{q-2}$. We have $\tau y^{q-2} \in (\gamma, x-ty)$ therefore $\tau(x-y)y^{q-2} \in ((x-y)\gamma, (x-y)(x-ty))$ but by the previous lemma, $(x-y)\gamma \in (x^{q-1}, y^{q-1})$; hence $\tau(x-y)y^{q-2} \in (x^{q-1}, y^{q-1}, (x-y)(x-ty))$ and $\tau xy(x-y)y^{q-2} \in (x^q, y^q, F)$.

If $G \in (x^q, y^q, xy(x-y)(x-ty))$, since $G \equiv x^2y^{q-1} \pmod{(x^q, y^q)}$ we can write $x^2y^{q-1} = ax^q + by^q + cF$ for some $a, b, c \in R$ where the x, y degree of a, b is 1. Writing $y^{q-1}(x^2 - by) = ax^q + cF$ we see that $x \mid b$ and $y \mid a$. Let $a = a'y$ and $b = b'x$. Note that now $a', b' \in A$. Dividing throughout by xy we get

$$y^{q-2}(x - b'y) = a'x^{q-1} + c(x-y)(x-ty).$$

Modulo $x-y$ this gives $y^{q-1}(1-b') = a'y^{q-1} \Rightarrow 1 - b' = a'$, while modulo $x - ty$ this gives $y^{q-1}(t-b') = y^{q-1}t^{q-1}a' \Rightarrow t - b' = t^{q-1}a'$. Combining this we have $a'(t^{q-1} - 1) = t - 1 \Rightarrow a'\tau = 1$, which is impossible.

To finish the proof, we note that if for some $d \in k[t]$, $dxy(x-y)y^{q-2} \in (x^q, y^q, F)$ then for some $a, b, c \in R$ we have

$$x\left(dy(x-y)y^{q-2} - ax^{q-1} - cy(x-y)(x-ty)\right) = by^q$$

and since x, y is a regular sequence, $x \mid b$ and $y \mid a$ and we may write

$$d(x-y)y^{q-2} - a'x^{q-1} - c(x-y)(x-ty) = b'y^{q-1}$$

where $b = xb'$ and $a = ya'$. Grouping together the terms divisible by $(x-y)$ we get

$$(x-y)(dy^{q-2} - c(x-ty)) \in (x^{q-1}, y^{q-1})$$

and using Lemma 11 we deduce that

$$dy^{q-2} \in (x-ty, y^{q-1}, \gamma).$$

Working modulo $x-ty$ we see that $dy^{q-2} \in (y^{q-1}, \tau y^{q-2})$ so τ must divide d, and to kill all A-torsion we need to invert all $\tau = \tau(q)$, and these polynomials have infinitely many irreducible factors. $\square$

REMARK 13. Notice that the counterexample above shows that the set $\bigcup_e \operatorname{Ass}_R F^e_{R_F}(M)$ has infinitely many maximal elements. While $\bigcup_e \operatorname{Ass} F^e_{R_F}(M)$ may have infinitely many maximal elements, the question of whether the set $\bigcup_e \operatorname{Ass} G^e(M)$ is finite, or has finitely many maximal elements remains open.

With R_F and M_F as in the previous theorem, we can show that $G^e_{R_F}(M) \cong R_F/(x,y)^q R_F$: we can compute $0^*_{F^e(M)}$ working modulo each minimal prime of R_F (see Lemma 2.10 in [AHH]). Killing the minimal primes or R_F we obtain polynomial rings; hence

$$(x, y^q)^*_{R_F/xR_F} = (x, y^q),\ (x^q, y)^*_{R_F/yR_F} = (x^q, y)$$

$$(x-y, x^q, y^q)^*_{R_F/(x-y)R_F} = (x-y, x^q, y^q)$$

$$(x-ty, x^q, y^q)^*_{R_F/(x-ty)R_F} = (x^q, y^q, x-ty).$$

Lifting these ideals back to R_F we find that $0^*_{F^e_{R_F}(M_F)}$ is the image of $(x, y^q) \cap (x^q, y) \cap (x-y, x^q, y^q) \cap (x^q, y^q, x-ty)$ in $F^e_{R_F}(M_F)$. Each monomial $x^i y^j$ is in this intersection for all non negative integers i, j with $i + j = q$, while if the image of $H = \sum_{i+j<q} h_{ij}(t)x^i y^j$ is in $0^*_{F^e_{R_F}(M_F)}$, then H must be divisible by x, y, $(x-y)$ and $(x-ty)$, and since these are relatively prime, H must be divisible by F. Therefore, $0^*_{F^e_{R_F}(M_F)}$ is the image of $(x,y)^q$ in $F^e_{R_F}(M_F)$ and $G^e_{R_F}(M_F) = R_F/(x,y)^q R_F$ and $\operatorname{Ass} G^e_{R_F}(M_F) = \{(x,y)\}$.

In fact, this argument is valid for any choice of F which can be decomposed into a product of linear factors, and in view of Lemma 8, this holds for any homogeneous polynomial F.

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