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JACOBI FIELDS, RICCATI EQUATION AND RIEMANNIAN FOLIATIONS

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1. Introduction

Many properties of the geometry of a Riemannian manifold (M, g) may be studied using Jacobi vector fields [C-E], [DC]. Moreover, the consideration of Jacobi vector fields along a geodesic, and normal to that geodesic leads at once to a Riccati equation which is very useful in the study of the geometry in a normal neighborhood of a point or a tubular neighborhood of a submanifold. This Riccati type differential equation for the shape operator of small concentric geodesic spheres or parallel tubular hypersurfaces describes the evolution of that operator along geodesics normal to spheres or hypersurfaces (see [G], [V] for a detailed treatment and further references). These families of spheres or hypersurfaces form locally Riemannian foliations on (M, g) with bundle-like metric g since, by the Gauss Lemma, the geodesics orthogonal to one leaf of such a foliation are orthogonal to all leaves of the foliation.

These consideratons motivate the following generalization. Let (M, g) be a Riemannian manifold, and $\mathcal{F}$ a Riemannian foliation on M with bundle-like metric g. As pointed out already by Reinhart [R1], a characteristic property of geodesics transversal to the leaves is that orthogonality to a leaf at one point implies orthogonality to the leaves at all points of the geodesic. So, it is immediately clear that for Riemannian foliations by hypersurfaces, one may use the above mentioned Riccati equation to study the geometric properties of the foliation. In this paper we will show how this technique may also be used for Riemannian foliations of higher codimension. For this purpose we first adapt the general notion of Jacobi vector field to this situation. This leads to the concept of $\mathcal{F}$-Jacobi vector fields. Secondly, we show that in this more general situation also the evolution of the shape operator of the leaves along an orthogonal geodesic satisfies a Riccati equation. In the case of the codimension one foliations mentioned before, this reduces to the usual Riccati equation.

Our main purpose is to discuss this theory already introduced [Ki], [Ki-T], and to review and extend some immediate applications, thereby illustrating the uses of this Riccati equation. We point out that this equation, in one form or another, was already present in several papers discussing geometrical aspects of foliations.

In the last section we consider transversal Jacobi operators, and introduce the

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notions of transversally $\mathfrak{C}$- and $\mathfrak{P}$-foliations. These are used to derive a new characterization of the transversally symmetric Riemannian foliations discussed in [T-V].

2. $\mathcal{F}$-Jacobi fields

We begin with the basic data given by a foliation $\mathcal{F}$ on (M, g). The tangent bundle of $\mathcal{F}$ is denoted by $L \subset TM$, with orthogonal projection $\pi^L = \pi^\perp \colon TM \to L$. The orthogonal projection to the normal bundle $L^\perp \simeq Q = TM/L$ is denoted by $\pi^Q = \pi \colon TM \to Q$. The Levi Civita connection ∇^M on TM gives rise to a metric connection ∇^L on L defined by

$$\nabla^L_E V = \pi^\perp \nabla^M_E V \tag{2.1}$$

for $E \in \Gamma TM$, $V \in \Gamma L$. For a unit speed geodesic γ orthogonal to the leaves of $\mathcal{F}$, we consider $V \in \Gamma L_\gamma$, i.e., vector fields along γ which are tangential to $\mathcal{F}$, and define a concept introduced in [Ki].

DEFINITION 2.2. $V \in \Gamma L_\gamma$ is an *$\mathcal{F}$-Jacobi vector field* along γ if

$$\pi^\perp\{\ddot{V} + R^M(\dot\gamma, V)\dot\gamma\} = 0. \tag{2.3}$$

Here $\ddot{V}$ denotes as usual $\left(\frac{\nabla^M}{dt}\right)^2 V = \nabla^M_{\dot\gamma}\nabla^M_{\dot\gamma} V$ along γ. The curvature $R^M(E_1, E_2) = \nabla^M_{[E_1,E_2]} - [\nabla^M_{E_1}, \nabla^M_{E_2}]$ of ∇^M gives rise to the operator $\bar{R}_{\dot\gamma} \colon L_\gamma \to L_\gamma$ by

$$\bar{R}_{\dot\gamma} V = \pi^\perp R^M(\dot\gamma, V)\dot\gamma. \tag{2.4}$$

Let A be the integrability tensor defined by [B], [ON]

$$A_{E_1} E_2 = \pi \nabla^M_{\pi E_1} \pi^\perp E_2 + \pi^\perp \nabla^M_{\pi E_1} \pi E_2 \quad \text{for} \quad E_1, E_2 \in \Gamma TM. \tag{2.5}$$

In particular, for $X \in \Gamma L^\perp$, we have $A_X \colon L \to L^\perp$, $A_X \colon L^\perp \to L$ and $A^2_X | L \colon L \to L$. Then (2.3) can be equivalently expressed by

$$\left(\frac{\nabla^L}{dt}\right)^2 V + \bar{R}_{\dot\gamma} V + (A_{\dot\gamma})^2 V = 0. \tag{2.6}$$

This follows immediately from $A_{\dot\gamma} V = \pi \nabla^M_{\dot\gamma} V$ and

$$\begin{aligned} A_{\dot\gamma} A_{\dot\gamma} V &= A_{\dot\gamma} \pi \nabla^M_{\dot\gamma} V = \pi^\perp \nabla^M_{\dot\gamma} \pi \nabla^M_{\dot\gamma} V = \pi^\perp \nabla^M_{\dot\gamma}(\nabla^M_{\dot\gamma} V - \pi^\perp \nabla^M_{\dot\gamma} V) \\ &= \pi^\perp \nabla^M_{\dot\gamma} \nabla^M_{\dot\gamma} V - \nabla^L_{\dot\gamma} \nabla^L_{\dot\gamma} V. \end{aligned}$$

We prove now a relation between ordinary Jacobi vector fields and $\mathcal{F}$-Jacobi fields of a special type along γ. For this purpose consider first the *shape operator* $S_{\dot\gamma} \colon L_\gamma \to L_\gamma$ of $\mathcal{F}$, defined by

$$S_{\dot\gamma} U = \pi^\perp \nabla^M_U \dot\gamma, \quad \text{for} \quad U \in \Gamma L_\gamma. \tag{2.7}$$

Here $\dot\gamma$ is extended to a normal vector field along an integral curve α of U in a leaf of $\mathcal{F}$. For the initial point $\gamma(0) = m$ we write $S_{\dot\gamma(0)} = S_m \colon L_m \to L_m$.

THEOREM 2.8. *Let γ be a unit speed geodesic orthogonal to $\mathcal{F}$. Then the following holds.*

(*i*) *An ordinary Jacobi vector field V along γ is tangential to $\mathcal{F}$ if and only if it satisfies the initial conditions at $m = \gamma(0)$:*

$$V(0) = v \in L_m, \left(\frac{\nabla^M}{dt} V\right)(0) = S_m v + A_{\dot\gamma(0)} v; \tag{2.9}$$

(*ii*) *An $\mathcal{F}$-Jacobi vector field $V \in \Gamma L_\gamma$ is a tangential ordinary Jacobi vector field if and only if it satisfies the initial conditions at $m = \gamma(0)$:*

$$V(0) = v \in L_m, \left(\frac{\nabla^L}{dt} V\right)(0) = S_m v. \tag{2.10}$$

The main point of the arguments in the following proof is the fact that given γ, the choice of $v \in L_m$, $m = \gamma(0)$ determines a unique tangential Jacobi vector field V along γ satisfying (2.9).

Proof. (i) Let $V \in \Gamma L_\gamma$ be an ordinary Jacobi vector field with $V(0) = v \in L_m$. Then

$$\frac{\nabla^M}{dt} V = \pi^\perp \nabla^M_{\dot\gamma} V + \pi \nabla^M_{\dot\gamma} V = \pi^\perp \nabla^M_{\dot\gamma} V + A_{\dot\gamma} V.$$

Further, since V is Jacobi along γ and $V \perp \dot\gamma$, we have $\nabla^M_{\dot\gamma} V = \nabla^M_V \dot\gamma$ (see for example [C-E, p. 14]). Hence by (2.7)

$$\frac{\nabla^M}{dt} V = S_{\dot\gamma} V + A_{\dot\gamma} V,$$

and (2.9) follows. Conversely, let $\bar V \in \Gamma TM_\gamma$ be an ordinary Jacobi vector field satisfying (2.9). The initial condition $\bar V(0) = v \in L_m$ defines a tangential Jacobi vector field V along γ by variations of γ through orthogonal geodesics (if $\alpha(s)$ is a curve in the leaf through m with $\alpha(0) = m$, $\dot\alpha(0) = v$, the geodesic γ_s with $\gamma_s(0) = \alpha(s)$ has the initial velocity given by the unique horizontal lift of $\dot\gamma(0) \in L_m^\perp$ to $L^\perp_{\alpha(s)}$). Since $\bar V$, V are both Jacobi fields, and satisfy the same initial conditions (2.9), it follows that $\bar V = V$ and $\bar V$ is necessarily tangential.

(ii) Let $V \in \Gamma L_\gamma$ be an ordinary Jacobi vector field with $V(0) = v \in L_m$. Then (2.9) holds. The ordinary Jacobi equation implies the $\mathcal{F}$-Jacobi equation. Moreover (2.9) implies (2.10). Conversely, let $\bar V \in \Gamma L_\gamma$ be the $\mathcal{F}$-Jacobi vector field satisfying (2.10). Then $\bar V$ coincides with the Jacobi vector field $V \in \Gamma L_\gamma$ satisfying (2.9). □

3. Riccati equation

For the following discussion let $\{e_i\}_{i=1,\dots,p}$ be an orthonormal basis of L_m at $m = \gamma(0)$, where p is the dimension of the leaves of $\mathcal{F}$. By parallel translation along γ with respect to the metric connection ∇^L we obtain an orthonormal frame field $\{E_i\}$ of L along γ. Further, let $Y_i (i = 1, \dots, p)$ be the $\mathcal{F}$-Jacobi vector fields along γ satisfying the initial conditions

$$Y_i(0) = e_i, \quad \left(\frac{\nabla^L}{dt} Y_i\right)(0) = S_m e_i. \tag{3.1}$$

This gives rise to a linear operator D: $L_\gamma \to L_\gamma$ given by

$$Y_i = DE_i. \tag{3.2}$$

Clearly

$$\frac{\nabla^L}{dt} Y_i = \left(\frac{\nabla^L}{dt} D\right) E_i,$$

and

$$\left(\frac{\nabla^L}{dt}\right)^2 Y_i = \left(\left(\frac{\nabla^L}{dt}\right)^2 D\right) E_i.$$

Then from (2.6) we obtain

$$\left(\frac{\nabla^L}{dt}\right)^2 D + (\bar{R}_{\dot\gamma} + A^2_{\dot\gamma}) D = 0. \tag{3.3}$$

Thus D is the *$\mathcal{F}$-Jacobi tensor (endomorphism) field* along γ satisfying the initial conditions

$$D(0) = I, \quad \left(\frac{\nabla^L}{dt} D\right)(0) = S_m. \tag{3.4}$$

THEOREM 3.5. *Let γ be a unit speed geodesic orthogonal to $\mathcal{F}$, and D: $L_\gamma \to L_\gamma$ the endomorphism field defined by* (3.1), (3.2). *Then*

$$S_{\dot\gamma} = \frac{\nabla^L}{dt} D \cdot D^{-1}. \tag{3.6}$$

Proof. By construction

$$S_{\dot\gamma} Y_i = \pi^\perp \nabla^M_{Y_i} \dot\gamma.$$

Then, since $\dot\gamma$ and Y_i are Jacobi vector fields along γ, we have

$$S_{\dot\gamma} Y_i = \frac{\nabla^L}{dt} Y_i,$$

or

$$S_{\dot\gamma} D E_i = \frac{\nabla^L}{dt} D E_i,$$

i.e., $S_{\dot\gamma} D = \frac{\nabla^L}{dt} D$. As the Jacobi fields Y_i are independent, D is invertible and Theorem 3.5 follows. □

The next fact was established in [Ki-T] by a direct calculation.

THEOREM 3.7. *Let γ be a unit speed geodesic orthogonal to $\mathcal{F}$. Then the shape operator $S_{\dot\gamma}\colon L_\gamma \to L_\gamma$ of $\mathcal{F}$ along γ satisfies the Riccati equation*

$$\frac{\nabla^L}{dt} S_{\dot\gamma} + S_{\dot\gamma}^2 + \bar R_{\dot\gamma} + A_{\dot\gamma}^2 = 0. \tag{3.8}$$

Proof. By (3.6) we have

$$S_{\dot\gamma} D = \frac{\nabla^L}{dt} D. \tag{3.9}$$

Differentiating covariantly along γ, we get

$$\left(\frac{\nabla^L}{dt} S_{\dot\gamma}\right) D + S_{\dot\gamma}\left(\frac{\nabla^L}{dt} D\right) = \left(\frac{\nabla^L}{dt}\right)^2 D.$$

Using (3.3) and (3.9), this implies

$$\left(\frac{\nabla^L}{dt} S_{\dot\gamma}\right) D + S_{\dot\gamma}(S_{\dot\gamma} D) + (\bar R_{\dot\gamma} + A_{\dot\gamma}^2) D = 0.$$

Since D is invertible, the desired result follows. □

Remark. This Riccati equation can also be obtained from one of O'Neill's formulas. It suffices to evaluate (9.28c) on p. 241 of [B] for $X = Y = \dot\gamma$. Observing that $T_U \dot\gamma = S_{\dot\gamma} U$, the Riccati equation readily follows. Here T is the tensor (see [B], [ON]) defined by

$$T_{E_1} E_2 = \pi \nabla^M_{\pi^\perp E_1} \pi^\perp E_2 + \pi^\perp \nabla^M_{\pi^\perp E_1} \pi E_2 \quad \text{for} \quad E_1, E_2 \in \Gamma T M.$$

Returning to an $\mathcal{F}$-Jacobi vector field V along γ as described in Theorem 2.7, we observe that

$$\dot V \equiv \frac{\nabla^M}{dt} V = \pi^\perp \frac{\nabla^M}{dt} V + \pi \frac{\nabla^M}{dt} V = S_{\dot\gamma} V + A_{\dot\gamma} V.$$

It follows that

$$\ddot{V} \equiv \left(\frac{\nabla^M}{dt}\right)^2 V = (\dot{S}_{\dot\gamma} + \dot{A}_{\dot\gamma})V + (S_{\dot\gamma} + A_{\dot\gamma})\dot{V}$$
$$= [(\dot{S}_{\dot\gamma} + \dot{A}_{\dot\gamma}) + (S_{\dot\gamma} + A_{\dot\gamma})^2]V.$$

Since

$$\pi^{\perp}\ddot{V} + \bar{R}_{\dot\gamma}V = 0,$$

and all this holds for a frame field of L_γ, it follows that on L

$$\pi^{\perp}(\dot{S}_{\dot\gamma} + \dot{A}_{\dot\gamma}) + \pi^{\perp}(S_{\dot\gamma} + A_{\dot\gamma})^2 + \bar{R}_{\dot\gamma} = 0. \tag{3.10}$$

We have established the following fact.

PROPOSITION 3.11. *Let γ be a unit speed geodesic orthogonal to $\mathcal{F}$. Then* (3.10) *holds on L.*

We wish to show that (3.10) conversely implies (3.8). For this it suffices to show that on L

$$\pi^{\perp}\dot{A}_{\dot\gamma} + \pi^{\perp}(S_{\dot\gamma}A_{\dot\gamma} + A_{\dot\gamma}S_{\dot\gamma}) = 0.$$

Let $U, V \in L$. It suffices to show

$$g((\nabla^M_{\dot\gamma}A_{\dot\gamma})U + (S_{\dot\gamma}A_{\dot\gamma} + A_{\dot\gamma}S_{\dot\gamma})U, V) = 0,$$

or equivalently

$$g((\nabla^M_{\dot\gamma}A)_{\dot\gamma}U, V) + g(A_{\dot\gamma}U, S_{\dot\gamma}V) - g(S_{\dot\gamma}U, A_{\dot\gamma}V) = 0.$$

The first term vanishes by formula (9.32) on p. 242 of [B]. The second term vanishes since $A_{\dot\gamma}U \in L^{\perp}$, while $S_{\dot\gamma}V \in L$. The third term vanishes similarly. This completes the proof of the equivalence of Theorem 3.7 and Proposition 3.11.

4. The Wronskian

We now introduce a notion naturally associated to the defining equation for $\mathcal{F}$-Jacobi vector fields, and which has proved to be very useful for the codimension one case (for example, see [V]).

DEFINITION 4.1. If D, E are fields of endomorphisms of L along γ, the Wronskian is the field of endomorphisms along γ given by

$$W(D, E) = \left(\frac{\nabla^L}{dt}\right){}^tD \circ E - {}^tD \circ \frac{\nabla^L}{dt}E. \tag{4.2}$$

If $D, E \in \Gamma \operatorname{End} L_\gamma$ are both solutions of the differential equation along γ

$$\left(\frac{\nabla^L}{dt}\right)^2 F + \bar{R}_{\dot\gamma} F + A_{\dot\gamma}^2 F = 0, \tag{4.3}$$

then we have the following fact.

PROPOSITION 4.4. *Let γ be a unit speed geodesic orthogonal to $\mathcal{F}$ and D, E endomorphism fields of L along γ satisfying* (4.3). *Then $W(D, E)$ is* (*covariantly*) *constant along γ.*

Proof. We have to show that

$$\frac{\nabla^L}{dt} W(D, E) = 0.$$

We note that while $A_{\dot\gamma}$ has skew-symmetric aspects, the operator $A_{\dot\gamma}^2 \colon L \to L$ is symmetric, and so is $\bar{R}_{\dot\gamma}$. It follows that

$$\begin{aligned} \frac{\nabla^L}{dt} W(D, E) &= \left(\frac{\nabla^L}{dt}\right)^2 {}^tD \circ E + \left(\frac{\nabla^L}{dt}\right) {}^tD \circ \frac{\nabla^L}{dt} E \\ &\quad - \left(\frac{\nabla^L}{dt}\right) {}^tD \circ \frac{\nabla^L}{dt} E - {}^tD \circ \left(\frac{\nabla^L}{dt}\right)^2 E \\ &= -{}^tD \circ {}^t(\bar{R}_{\dot\gamma} + A_{\dot\gamma}^2) \circ E + {}^tD \circ (\bar{R}_{\dot\gamma} + A_{\dot\gamma}^2) \circ E \\ &= 0, \end{aligned}$$

since ${}^t(\bar{R}_{\dot\gamma} + A_{\dot\gamma}^2) = \bar{R}_{\dot\gamma} + A_{\dot\gamma}^2$. □

4.5 *Remarks.* (1) Let D be an endomorphism field of L along γ, satisfying (4.3) and with the initial conditions at $m = \gamma(0)$:

$$D(0) = I, \quad \left(\frac{\nabla^L}{dt} D\right)(0) = S_m.$$

Then $W(D, D) = 0$. This is equivalent to the symmetry of the shape operator.

(2) For applications of an analog of Proposition 4.4 to submanifold theory see [V].

5. Applications

(a) *Riemannian foliations $\mathcal{F}$ with bundle-like g and involutive normal bundle $L^\perp$.*

The last property is expressed by $A = 0$, and thus in particular, $A_{\dot\gamma} = 0$. Equations (3.3), (3.8) yield

$$\left(\frac{\nabla^L}{dt}\right)^2 D + \bar{R}_{\dot\gamma} D = 0 \tag{5.1}$$

for the endomorphism field D of L along γ defined by (3.2), and

$$\left(\frac{\nabla^L}{dt}\right)S_{\dot\gamma} + S_{\dot\gamma}^2 + \bar{R}_{\dot\gamma} = 0 \tag{5.2}$$

for the field of shape operators $S_{\dot\gamma}$ along γ.

If (M, g) is of constant curvature c, then $\bar{R}_{\dot\gamma} = c \cdot I\colon L_\gamma \to L_\gamma$. Equations (5.1), (5.2) then read

$$\left(\frac{\nabla^L}{dt}\right)^2 D + cD = 0, \tag{5.3}$$

$$\left(\frac{\nabla^L}{dt}\right)S_{\dot\gamma} + S_{\dot\gamma}^2 + cI = 0. \tag{5.4}$$

Note that $\mathcal{F}$ can only be totally geodesic if $c = 0$. Even if A is not assumed to vanish, the Riccati equation shows that $\mathcal{F}$ can only be totally geodesic provided $c \geqq 0$ (taking traces yields then $c = \frac{1}{p}|A_{\dot\gamma}|^2$, where p is the dimension of the leaves of $\mathcal{F}$). Returning to the case of foliations with involutive normal bundle, we observe that for $c > 0$ the explicit solution of equation (5.3) along the geodesic $\gamma(r)$ is given by

$$D(r) = \cos\sqrt{c}r \cdot I + \frac{\sin\sqrt{c}\,r}{\sqrt{c}} \cdot S(0).$$

Then $S = \dot{D}D^{-1}$ turns out to be diagonal, and of the form

$$S(r) = \begin{pmatrix} \dfrac{\kappa_1 - \sqrt{c}\cdot\tan\sqrt{c}\,r}{1 + \frac{\tan\sqrt{c}\,r}{\sqrt{c}}\cdot\kappa_1} & & & 0 \\ & \bullet & & \\ & & \bullet & \\ 0 & & & \bullet \end{pmatrix},$$

where $\kappa_1, \ldots, \kappa_p$ are the principal curvatures of $S(0)$.

Harmonic foliations are characterized by $\mathrm{Tr}S = 0$, i.e. the leaves are minimal. Taking the trace in (3.8) yields

$$|S|^2 + \sum_{i=1}^{p} g(R^M(\dot\gamma, E_i)\dot\gamma, E_i) = 0. \tag{5.5}$$

This implies the following fact [K-T; 2.27], which also holds if $\mathrm{Tr}S = $ constant.

PROPOSITION 5.6. *Let $\mathcal{F}$ be a Riemannian foliation with bundle-like metric on (M, g). Assume $L^\perp$ to be involutive. If the sectional curvature $K^M \geqq 0$, then the harmonicity of $\mathcal{F}$ implies that $\mathcal{F}$ is totally geodesic.*

Proof. (5.5) implies $|S|^2 = 0$, hence $S = 0$. □

Remark. Riemannian foliations with involutive normal bundle and totally geodesic leaves are locally Riemannian products. The proof is an application of DeRham's holonomy theorem, together with the fact that in this case the decomposition $TM \cong L \oplus L^\perp$ is preserved under parallel transport.

The condition $A = 0$ holds in particular for the case of foliations of codimension $q = 1$. An example is the following conclusion.

PROPOSITION 5.7. *If $\mathcal{F}$ is of codimension one and harmonic on (M, g) with non-negative Ricci curvature, then $\mathcal{F}$ is totally geodesic.*

Global arguments for this conclusion on a closed M were given in [Os] and [K-T] (see also [T, Theorem 7.50]). While the argument to follow is local in nature, and thus applies equally well off the singular set of a Riemannian foliation, the global arguments in [Os] and [K-T] imply the result as well as the Riemannian property of $\mathcal{F}$, while in the present context $\mathcal{F}$ is assumed to be Riemannian to begin with.

Proof. For $q = 1$ equation (5.5) implies

$$|S|^2 + \mathrm{Ric}(\dot\gamma, \dot\gamma) = 0. \tag{5.8}$$

Thus for non-negative Ricci curvature $|S|^2 = 0$, hence $S = 0$. □

As (5.8) moreover shows, the positivity of the Ricci operator at even a single point of M is incompatible with the existence of a foliation satisfying our assumptions.

(b) *Riemannian foliations $\mathcal{F}$ with bundle-like metric on (M, g) with strictly negative sectional curvatures for all 2-planes at a single point of (M, g).*

PROPOSITION 5.9. *With these assumptions $\mathcal{F}$ cannot be totally geodesic.*

Proof. Assume $S = 0$. It follows from (3.8) that $B = \bar R_{\dot\gamma} + A_{\dot\gamma}^2 = 0$. But

$$g(BE_i, E_i) = g(\bar R_{\dot\gamma} E_i, E_i) - |A_{\dot\gamma} E_i|^2.$$

Let now $\gamma(t)$ be the point at which $K^M < 0$ on all 2-planes. Then at this point

$$g(\bar R_{\dot\gamma} E_i, E_i) = g(R^M(\dot\gamma, E_i)\dot\gamma, E_i) = K^M(\dot\gamma, E_i) < 0.$$

This contradicts

$$K^M(\dot\gamma, E_i) - |\ A_{\dot\gamma}\ \ E_i|^2 = 0. \qquad \square$$

As in [Ki-T], it is convenient to consider the *partial Ricci curvature form* Ric^L defined by

$$\mathrm{Ric}^L(x, y) = \sum_{i=1}^{p} g(R^M(x, e_i)y, e_i)$$

for $x, y \in L_m^\perp$, and an orthonormal basis $\{e_i\}_{i=1,\dots,p}$ of L_m. The quadratic form associated to the bilinear symmetric form Ric^L on $L^\perp$ is then given by

$$\mathrm{Ric}^L(x, x) = \sum_{i=1}^{p} K^M(x, e_i).$$

Taking $\mathrm{Tr}\, B$ over L in the preceding argument, the conclusion is as follows.

PROPOSITION 5.10. *If $\mathcal{F}$ is a Riemannian foliation with bundle-like metric on (M, g), and $\mathrm{Ric}^L < 0$ at least at one point of M, then $\mathcal{F}$ cannot be totally geodesic.*

The same formulas show also that when $\mathrm{Tr}\,\mathrm{Ric}^L \le 0$ and $\mathcal{F}$ is totally geodesic, then $A = 0$, as is well-known.

In the case of codimension $q = 1$ we have $\mathrm{Ric}^L = \mathrm{Ric}^M$, and the condition above concerns the ordinary Ricci operator at a point of M.

(c) *Mean curvature conditions.*

Consider $\mathrm{Tr}\, S$. Note that S depends on the choice of a normal vector. Thus along a geodesic γ orthogonal to $\mathcal{F}$, the mean curvature function $h = \mathrm{Tr} S_{\dot\gamma}$ is well defined. Let

$$w = \frac{1}{p} h,$$

and consider the operator $S_{\dot\gamma} - w \cdot I \colon L \to L$. Then

$$\begin{aligned} |S_{\dot\gamma} - w \cdot I|^2 &= \sum_{i=1}^{p} g((S_{\dot\gamma} - w \cdot I)E_i,\ (S_{\dot\gamma} - w \cdot I)E_i) \\ &= |S_{\dot\gamma}|^2 - 2wh + pw^2 = |S_{\dot\gamma}|^2 - pw^2, \end{aligned}$$

or

$$|S_{\dot\gamma}|^2 = pw^2 + |S_{\dot\gamma} - w \cdot I|^2.$$

Taking traces over L in the Riccati equation (3.8) implies

$$p(\dot w + w^2) + |S_{\dot\gamma} - w \cdot I|^2 + \mathrm{Ric}^L(\dot\gamma, \dot\gamma) - |A_{\dot\gamma}|^2 = 0. \tag{5.11}$$

Here we have used

$$\mathrm{Tr}(A_{\dot\gamma}^2) = \sum_{i=1}^{p} g(A_{\dot\gamma}^2 E_i, E_i) = -\sum_{i=1}^{p} g(A_{\dot\gamma} E_i, A_{\dot\gamma} E_i) = -\sum_{i=1}^{p} |A_{\dot\gamma} E_i|^2 = -|A_{\dot\gamma}|^2,$$

while

$$\operatorname{Tr}(S_{\dot\gamma}^2) = \sum_{i=1}^{p} g(S_{\dot\gamma}^2 E_i, E_i) = \sum_{i=1}^{p} g(S_{\dot\gamma} E_i, S_{\dot\gamma} E_i) = \sum_{i=1}^{p} |S_{\dot\gamma} E_i|^2 = |S_{\dot\gamma}|^2.$$

To illustrate the method used in [Ki-T], we prove the following fact as an application of (5.11) under the additional completeness condition for the manifold (M, g). We refer to [Ki-T] for further applications.

PROPOSITION 5.12. *Let $\mathcal{F}$ be a foliation with bundle-like metric g on a complete Riemannian manifold (M, g). If for each geodesic γ orthogonal to $\mathcal{F}$ we have* $\operatorname{Ric}^L(\dot\gamma, \dot\gamma) \geqq |A_{\dot\gamma}|^2$, *then $\mathcal{F}$ is totally geodesic.*

Proof. (5.11) implies

$$\dot w + w^2 + c + r = 0,$$

with $c = \frac{1}{p}|S_{\dot\gamma} - w \cdot I|^2$, $r = \frac{1}{p}(\operatorname{Ric}^L(\dot\gamma, \dot\gamma) - |A_{\dot\gamma}|^2)$. Since $c \geqq 0$, $r \geqq 0$ it is clear that the solution decreases not less rapidly than the solution of $\dot w + w^2 = 0$ with the same initial condition. But that solution goes to $-\infty$ in finite time, contrary to the completeness assumption. This implies $w = 0$. Thus $c + r = 0$. Since $c \geqq 0$ and $r \geqq 0$, this implies $c = 0$. This means that $S_{\dot\gamma} = w \cdot I$, or, since $w = 0$, $S_{\dot\gamma} = 0$. □

For $q = 1$ the hypothesis reads $\operatorname{Ric}^M \geq 0$. Thus under the completeness assumption one obtains a better result than in Proposition 5.7. But note that these arguments assume $\mathcal{F}$ to be Riemannian to begin with.

As pointed out in [Ki-T], the inequality in the preceding proposition is in fact sharp. This was proved using an example in [H]. The hypotheses of Proposition 5.12 are in particular realized for a foliation of codimension $q = 1$ on (M, g) with $\operatorname{Ric}^M \geq 0$.

Note that the case where the normal bundle is involutive is also mentioned in [W].

Finally we turn to the case of a foliation of codimension one on a space of constant curvature. Consider the case where all the leaves have the *same constant mean curvature h*. Taking traces in (3.8) yields

$$|S_{\dot\gamma}|^2 + \operatorname{Tr}\bar R_{\dot\gamma} = 0.$$

But if M^n is of constant curvature c, $\operatorname{Tr}\bar R_\gamma = (n-1)c$. Thus we have the following cases to distinguish:

(i) $c > 0$, in which case no such $\mathcal{F}$ exists;
(ii) $c = 0$, in which case $\mathcal{F}$ is necessarily totally geodesic, and is induced by a hyperplane foliation on the universal covering in the complete case;
(iii) $c < 0$, in which case $|S|^2 = -c(n-1)$.

An example is a foliation of hyperbolic space by horospheres. As shown in [B-G-S], for 3-dimensional hyperbolic space, these are the only such examples with $h \geq 1$.

6. Transversal Jacobi fields

Jacobi vector fields along a geodesic γ on a Riemannian manifold are defined with the help of the Jacobi operator $R_{\dot\gamma} = R(\dot\gamma, -)\dot\gamma$. The study of the eigenvalues and the eigenspaces of these operators led in [B-V] to the consideration of the $\mathfrak{C}$-spaces and $\mathfrak{P}$-spaces which are natural generalizations of locally symmetric spaces. In this last section we will introduce an analog treatment with respect to the transversal geometry of a Riemannian foliation.

Let $L^\perp$ be the normal bundle of a Riemannian foliation on (M, g) with bundle-like metric g. Let ∇ denote the Levi Civita connection in $L^\perp$ and R^∇ its curvatur tensor. For a unit speed geodesic γ orthogonal to the leaves we consider the normal vector fields $\Gamma L_\gamma^\perp$ along γ.

DEFINITION 6.1. The *transversal Jacobi operator* along γ is the operator $R_\gamma^\nabla : X \mapsto R^\nabla(\dot\gamma, X)\dot\gamma$ on $\Gamma L_\gamma^\perp$. A vector field $Y \in \Gamma L_\gamma^\perp$ is a *transversal Jacobi vector field* along γ, if

$$\left(\frac{\nabla}{dt}\right)^2 Y + R^\nabla(\dot\gamma, Y)\dot\gamma = 0. \tag{6.2}$$

The transversal Jacobi operator field R_γ^∇ determines a field of symmetric endomorphisms of $\Gamma L_\gamma^\perp$. In what follows we concentrate on the eigenvalues and eigenspaces of these endomorphisms. In analogy with the theory developed in [B-V] we first introduce the following two new concepts.

DEFINITION 6.3. A Riemannian foliation $\mathcal{F}$ on (M, g) is a *transversally $\mathfrak{C}$-foliation*, if the eigenvalues of R_γ^∇ are constant along γ for each geodesic γ orthogonal to the leaves of the foliation. $\mathcal{F}$ is a *transversally $\mathfrak{P}$-foliation*, if the eigenspaces of the transversal Jacobi operators can be spanned by parallel fields of eigenvectors along γ for each geodesic γ orthogonal to the leaves of $\mathcal{F}$.

These conditions mean that $\mathcal{F}$ is locally modeled on a $\mathfrak{C}$-space or $\mathfrak{P}$-space, respectively. We use these concepts to give a new characterization of *transversally symmetric foliations*. This class of Riemannian foliations, locally modeled on a Riemannian symmetric space, can according to [T-V, Theorem 1] be analytically defined using the following result.

PROPOSITION 6.4. *$\mathcal{F}$ is transversally symmetric if and only if*

$$(\nabla_X R^\nabla)(X, Y, X, Y) = 0 \quad \textit{for all} \quad X, Y \in \Gamma L^\perp. \tag{6.5}$$

We refer to [T-V], [G-G-V], [GD] for a collection of results and examples. Note that trivially any codimension one Riemannian foliation is transversally symmetric

[T-V]. It follows also from [B-V] that any Riemannian foliation of codimension two is a transversally $\mathfrak{P}$-foliation.

To give examples of transversally $\mathfrak{C}$- and $\mathfrak{P}$-foliations it suffices to consider bundles over $\mathfrak{C}$- or $\mathfrak{P}$-spaces, as e.g. warped products $B \times_f F$ over a $\mathfrak{C}$- or $\mathfrak{P}$-space B.

We prove now the following result.

THEOREM 6.6. *$\mathcal{F}$ is transversally symmetric if and only if it is a transversally $\mathfrak{C}$- as well as a transversally $\mathfrak{P}$-foliation.*

Proof. First let $\mathcal{F}$ be transversally symmetric. Let m be a point of (M, g), and γ a unit speed geodesic orthogonal to $\mathcal{F}$ satisfying $\gamma(0) = m$ and $\dot\gamma(0) = u$, $|u| = 1$. Next let $e \in L_m^\perp$ be an eigenvector of R_u^∇ corresponding to the eigenvalue λ; i.e.,

$$R^\nabla(u, e)u = \lambda e. \tag{6.7}$$

Further let $E \in \Gamma L_\gamma^\perp$ be the vector field along γ obtained by parallel translation of e along γ with respect to ∇. Then λE is parallel. Moreover, (6.5) implies that $R^\nabla(\dot\gamma, E)\dot\gamma$ is parallel. Finally, since both vector fields have the same initial value at m, they concide; i.e.,

$$R^\nabla(\dot\gamma, E)\dot\gamma = \lambda E. \tag{6.8}$$

This implies that $\mathcal{F}$ is a transversally $\mathfrak{C}$- and a transversally $\mathfrak{P}$-foliation.

Conversely, let $\mathcal{F}$ be a transversally $\mathfrak{C}$- and transversally $\mathfrak{P}$-foliation. Then there exists a ∇-parallel frame field $\{E_i\}_{i=1,\dots,q}$, formed by eigenvector fields $E_i \in \Gamma L_\gamma^\perp$. Hence we have for the corresponding eigenvalues λ_i

$$R^\nabla(\dot\gamma, E_i)\dot\gamma = \lambda_i E_i, \quad i = 1, \dots, q. \tag{6.9}$$

By assumption the λ_i are constant along γ. This implies at once (6.5), and hence $\mathcal{F}$ is transversally symmetric. □

This theorem shows that for $q = 2$ the transversally symmetric foliations coincide with the transversally $\mathfrak{C}$-foliations. This is based on the fact that a connected 2-dimensional $\mathfrak{C}$-space is a space of constant curvature [B-V].

For $q = 3$ we refer to the classification of $\mathfrak{C}$-spaces and $\mathfrak{P}$-spaces given in [B-V].

Instead of considering the operators R_u^∇ for transversal unit vectors u one may also consider the operators $\nabla_u R_u^\nabla = (\nabla_u R^\nabla)(u, -)u$. These operators are symmetric and the consideration of their eigenvalues leads to the following new characterization of transversally symmetric foliations.

PROPOSITION 6.10. *A Riemannian foliation $\mathcal{F}$ on (M, g) is transversally symmetric if and only if at each point $m \in M$ the eigenvalues of $\nabla_u R_u^\nabla$ are independent of the choice of the transversal unit vector $u \in L_m^\perp$.*

Proof. First, let $\mathcal{F}$ be transversally symmetric. Then (6.5) implies $\nabla_u R_u^\nabla = 0$ for all $u \in L_m^\perp$, and the result holds trivially.

Conversely, the independence of the eigenvalues of $\nabla_u R_u^\nabla$ from u implies the independence of $\mathrm{Tr}(\nabla_u R_u^\nabla)^k$, $k = 1, \ldots, q-1$, from the choice of the unit vector $u \in L_m^\perp$. Following a recent result of [S] (see also [Gi]), this implies $\nabla_u R_u^\nabla = 0$, and then the result follows from Proposition 6.4. □

Remarks. 1. It would be worthwhile to study the geometry of these two new classes of Riemannian foliations, and to describe some interesting examples. We hope to return to this question on another occasion.

2. It is clear that (6.2) leads to a Riccati type differential equation. This equation decribes the evolution along an orthogonal geodesic of the shape operator of the geodesic spheres on the local model space for the transversal geometry of $\mathcal{F}$.

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