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FINITE CRITERIA FOR WEAK F-REGULARITY

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1. Introduction

This paper will investigate the following question: given any Cohen-Macaulay ring R of prime characteristic p, can one construct in a natural way an ideal $I \subset R$ such that R is weakly F-regular if and only if I is tightly closed in R? We shall refer to such an ideal as a "critical" ideal.

This question is closely related to the TF-boundedness of the ring R, as defined by L. Williams in [W], where she used this notion to investigate the connection between weak and strong F-regularity. The first part of this paper will define this property (which will henceforth be referred to as "F-boundedness"). Theorem 5 will show the existence of critical ideals in F-bounded rings. The rest of the paper will extend these results for tight closure in characteristic zero where the notion of "fiberwise tightly closed" ideals is defined and later used in Theorems 11, 12 and 15 to produce a critical ideal in the sense that if this ideal is fiberwise tightly closed then a certain set of ideals are tightly closed. In Theorem 17 we also identify certain affine $\mathbb{N}$-graded rings where the notions of weak F-regular type and strong F-regularity are equivalent.

Throughout this paper, all rings are commutative with identity and Noetherian; p will always denote a prime integer, and q will be some power p^e. A local ring is defined as a Noetherian ring with a unique maximal ideal. Let R be a ring of prime characteristic p and let $N \subset M$ be finitely generated R-modules. In [HH2] M. Hochster and C. Huneke introduced the notion of *the tight closure of N in M* as follows:

Let S be R viewed as an R-algebra via the iterated Frobenius endomorphism $r \mapsto r^q$ and define the *Peskine-Szpiro functor* F^e from R-modules to S-modules by $F^e(M) = S \otimes_R M$. Since the category of S-modules *is* the category of R-modules, we may view F^e as a functor from the category of R-modules to itself.

Thus the R-module structure on $F^e(M)$ is such that $r'(r \otimes m) = (rr') \otimes m$ and we also have $r' \otimes (rm) = (r'r^q) \otimes m$. If $I \subset R$ is an ideal then $F^e(R/I) = R/I^{[q]}$, and, generally, if we apply F^e to a map $R^a \to R^b$ given by a matrix (c_{ij}) by identifying $F^e(R^a) \cong R^a$ and $F^e(R^b) \cong R^b$ (this identification is not canonical, it depends on a choice of generators for the free modules), we obtain a map $F^e(c_{ij})$: $R^a \to R^b$ given

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by the matrix (c_{ij}^q). There is a natural map $M \to F^e(M)$ given by $m \mapsto 1 \otimes m$, and we denote the image of m under this map as m^q. Notice that when $M = R$ and N is an ideal of R, the image of $F^e(N)$ in $F^e(R) = R$ is $N^{[q]}$ where $N^{[q]}$ is the ideal generated by $\{n^q | n \in N\}$.

If $N \subset M$ are R-modules, we have an exact sequence

$$F^e(N) \to F^e(M) \to F^e(M/N) \to 0$$

and we write $N_M^{[q]}$ for

$$\operatorname{Ker}(F^e(M) \to F^e(M/N)) \cong \operatorname{Im}(F^e(N) \to F^e(M)).$$

Let R^0 be the set of all elements in R not in any minimal prime of R. Let $N \subset M$ be R-modules. The *tight closure of N in M*, N_M^*, is defined as the set of all elements $m \in M$ such that $cm^q \in N_M^{[q]}$ for some $c \in R^0$ and all large q. If $N_M^* = N$ we say that *N is tightly closed in M*.

We refer the reader to [HH2] for a description of the basic properties of tight closure.

2. F-boundedness

Definition 1. Let R be a Cohen-Macaulay ring of characteristic $p > 0$ with canonical module $\omega \subset R$. Fix some system of parameters $\underline{x} = x_1, \dots, x_n$ and define

$$\rho_{q,t}\colon \frac{\omega^{[q]}}{\underline{x}^q \omega^{[q]}} \xrightarrow{(x_1 \cdot \ldots \cdot x_n)^{q(t-1)}} \frac{\omega^{[q]}}{\underline{x}^{qt} \omega^{[q]}}$$

We define R to be F-bounded relative to $\underline{x}$ and $c \in R^0$ if for all large q there exists an integer $t_0 \geq 2$ independent of q such that $c \ker(\rho_{q,t}) \subseteq \ker(\rho_{q,t_0})$ for all $t \geq t_0$.

If R is F-bounded relative to $\underline{x}$ and $c = 1$, we say that R is F-bounded relative to $\underline{x}$. If R is F-bounded with respect to one system of parameters, then it is F-bounded with respect to all systems of parameters ([W], Theorem 4.1) and in that case we will simply refer to R as being F-bounded. Notice that in [W], L. Williams refers to "F-boundedness" as "TF-boundedness".

Notice that if $\dim(R) = 1$ and x is a parameter, x^q is a non zerodivisor on $\omega^{[q]}$ and R is then clearly F-bounded. Also, if R is Gorenstein, $\omega \cong R$, and if $x_1, \dots, x_n$ is a regular sequence on R, so is $x_1^q, \dots, x_n^q$, and R is clearly F-bounded.

The following theorems, proved in [W], describe known cases of Cohen-Macaulay rings which are F-bounded.

THEOREM 2 [W]. *Let A be a regular domain of prime characteristic p, and let $R \cong A[H]/\Sigma$ be a discrete, square free Hodge algebra. Then R is F-bounded with respect to any system of parameters $\underline{x}$ and some $c \in R^0$.*

THEOREM 3 [W]. *Let R be a ring of prime characteristic p with canonical module ω. Then if R is a pure subring of a regular ring, R is F-bounded with respect to all systems of parameters.*

THEOREM 4 [W]. *Let (R, m) be a local normal domain of prime characteristic p with canonical module ω. Then if either* $\dim(R) \leq 2$ *or if* $\dim(R) = 3$ *and R is* F*-rational on its punctured spectrum, then R is* F*-bounded with respect to all systems of parameters.*

3. F-boundedness and critical ideals

Throughout the rest of this chapter whenever R is a Cohen-Macaulay ring, ω will denote an ideal of R isomorphic with its canonical module.

The connection between F-boundedness and tight closure is given in the following theorem:

THEOREM 5. *Let (R, m, k) be a Cohen-Macaulay local domain of characteristic $p > 0$ with canonical module $\omega \subset R$. Let $n = \dim(R)$ and let $\underline{x} = x_1, \dots, x_n$ be a system of parameters. Assume R is F-bounded relative to $\underline{x}$ and some $c \in R^0$ with t_0 as in the definition above. Then:*

(1) *If $\underline{x}^{t_0}\omega$ is tightly closed in R, so is $\underline{x}^t\omega$ for every $t \geq t_0$.*
(2) *The ideal $\underline{x}^{t_0}\omega$ is a critical ideal; i.e., R is weakly F-regular if and only if $\underline{x}^{t_0}\omega$ is tightly closed in R.*

Proof. (1) If $\underline{x}^t\omega$ is not tightly closed, since $(\underline{x}^t\omega)^* \subset (\underline{x}^{t_0}\omega)^* = \underline{x}^{t_0}\omega \subset \omega$, we can chose an element $r \in (\underline{x}^t\omega)^*$ representing a non-zero element of the socle of $\omega/\underline{x}^t\omega$. We can write $r = (x_1 \cdot \dots x_n)^{t-1}s$ where s represents an element in the socle of $\omega/\underline{x}\omega$. Then $cr^q = c(x_1 \cdot \dots x_n)^{q(t-1)}s^q \in \underline{x}^{qt}\omega^{[q]}$ for all large q. By the F-boundedness of R we have $(x_1 \cdot \dots x_n)^{q(t_0-1)}cs^q \in \underline{x}^{qt_0}\omega^{[q]}$ hence $(x_1 \cdot \dots x_n)^{t_0-1}s \in (\underline{x}^{t_0}\omega)^* = \underline{x}^{t_0}\omega$. But $\underline{x}$ is a regular sequence on ω, hence $s \in \underline{x}\omega$ and $r = (x_1 \cdot \dots x_n)^{t-1}s \in \underline{x}^t\omega$, a contradiction.

(2) Pick some $y \in \omega \setminus \underline{x}^{t_0}\omega$ and let $I_n = \underline{x}^{nt_0}\omega :_R yR$. Since $\underline{x}^{nt_0}\omega$ are tightly closed for all $n \geq 1$, so are I_n. Therefore, using Proposition 8.9 in [HH2] it suffices to show that $\{I_n\}_{n\geq 1}$ are a decreasing sequence of m-primary irreducible ideals, cofinal with the powers of m. Using the Artin-Rees Lemma, we can find a $c > 0$ such that for all large n, $yR \cap \underline{x}^{nt_0}\omega \subset yR \cap m^{nt_0}\omega = m^{nt_0-c}(yR \cap m^c\omega) \subset m^{nt_0-c}yR$ hence $I_n \subset m^{nt_0-c}$ and $\{I_n\}_{n\geq 1}$ are cofinal with the powers of m. Clearly, each I_n is m-primary as it contains $\underline{x}^{nt_0}$. To see that the ideals I_n are irreducible, notice that the Artin ring R/I_n injects into the R/I_n-module $\omega/\underline{x}^{nt_0}\omega$ which has a one-dimensional socle. $\square$

COROLLARY 6. *Let R be a Cohen-Macaulay domain of characteristic $p > 0$ with canonical module $\omega \subset R$. Let $d = \dim(R)$ and let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters, with $\mathrm{Rad}(\underline{x}) = m$, $m \subset R$ a maximal ideal. Assume R is* F*-bounded relative to $\underline{x}$ with t_0 as in Definition* 1 *and $\underline{x}^{t_0}\omega$ is tightly closed in R. Then every m-primary ideal of R is tightly closed, and all ideals contracted from R_m are tightly closed.*

Proof. The first assertion is immediate from the fact that if I is m-primary, localization at m commutes with tight closure (Proposition 1.14 in [HH2]), and after localization at m we are in the local case described in the previous theorem.

If $I = (IR_m) \cap R$ is not tightly closed, pick some $f \in I^* - I$. We have $\bigcap_{t>0}(I + m^t) = I$, for if $u \in \bigcap_{t>0}(I + m^t)$, then in R_m, $u/1 \in \bigcap_{t>0}(I + m^t)R_m = \bigcap_{t>0}(IR_m + (mR_m)^t) = IR_m$, and since I is contracted from R_m, $u \in I$. Pick some $l > 0$ such that $f \notin Q = I + m^l R_m$, but as $I \subset Q$, $f \in Q^*$. Let $P = Q \cap R$. Since P is m-primary, we have $f \in (PR_m)^* = P^* R_m = PR_m = Q$ $\square$

4. Tight closure in algebras over a field of characteristic zero

Definition 7. Let R be a finitely generated algebra over a field K of characteristic 0, let $N \subset M$ be finitely generated R modules and let $u \in M$. A quintuple (A, R_A, M_A, N_A, u_A) will be called *descent data* for (K, R, M, N, u) if it satisfies the following conditions:

1. A is a finitely generated $\mathbb{Z}$-subalgebra of K.
2. R_A is a finitely generated A-subalgebra of R such that the inclusion $R_A \subset R$ induces an isomorphism of $R_K = R_A \otimes_A K$ with R.
3. R_A is A-free.
4. M_A,N_A are finitely generated A-submodules of M and N such that $N_A \subset M_A$ and all of the A-modules M_A,N_A,M_A/N_A are A-free.
5. The inclusion $M_A \subset M$ induces an isomorphism $M \cong M_K = M_A \otimes_A K$ as R-modules.
6. The element $u \in M$ is in M_A and $u_A = u$.

Given R,K,M,N,u as in the previous definition, descent data exists ([HH4] Section 2.) Furthermore, given a field K, a finitely generated K-algebra, a finite set of finitely generated R-modules, finitely many elements in those modules, finitely many maps between the modules which may be specified to take some of the elements above into certain elements, and given finitely many commutative diagrams involving these modules and maps, one can descend this setup, in the sense defined in Section 2 of [HH4].

Our next step towards the definition of tight closure for algebras over fields of characteristic zero, is to define the *tight closure of N_A in M_A over A* where $N_A \subset M_A$

are finitely generated R_A-modules, R_A is a finitely generated A algebra and $A \subset R$. With this setup, we will use the following notation: μ will denote a maximal ideal of A, $\kappa = \kappa(\mu)$ will denote the (finite) field A/μ, $p = p(\mu)$ its characteristic, and $q = q(\mu)$ will be an integer of the form $p(\mu)^e$ for some non-negative integer e. A property will hold "for almost all μ" if it holds for all μ in an non-empty Zariski open set of MaxSpec(A). We will also call the algebra $\kappa \to R_\kappa$ *the fiber of* $A \to R_A$ *over* μ where $\kappa = \kappa(\mu)$. A property will hold "for almost all fibers" if it holds in $R_\kappa(\mu)$ for almost all μ.

Definition 8. Let A be a domain, finitely generated over $\mathbb{Z}$, let R_A be a finitely generated A algebra with $A \subset R_A$, and let $N_A \subset M_A$ be finitely generated R_A modules. We say that $u_A \in M_A$ is in *the tight closure of* N_A *in* M_A *over* R_A *relative to* A, denoted by ${N_A^{*/A}}_{M_A}$ if for almost all $\mu \in$ MaxSpec(A), $u_\kappa \in (N_\kappa^*)_{M_\kappa}$ where $_\kappa$ denotes images after tensoring with κ over A.

We now have all the ingredients for the following:

Definition 9. Let R be a finitely generated algebra over a field K of characteristic 0, let $N \subset M$ be finitely generated R-modules. We say that $u \in M$ is in the K*-tight closure of* N, denoted by N_M^{*K}, if there exists descent data (A,R_A,M_A,N_A,u_A) for (K,R,M,N,u) such that $u_A \in {N_A^{*/A}}_{M_A}$.

In the rest of this paper we will adopt the following notation: if S is an A-algebra or A-module, and B is a A-algebra, we will write S_B for $S \otimes_A B$. If $s \in S$, s_B will be the image $s \otimes 1$ of s in S_B.

Let A be a Noetherian domain of characteristic zero with fraction field K, and let R_A be a finitely generated A-algebra. It is not known whether tight closure in R is a geometric notion; i.e., if $I_K \subset R_K$ is a tightly closed ideal in R_K, is I_κ tightly closed in R_κ for almost all κ?

5. Fiberwise tightly closed submodules

We can make this property into a definition as follows:

Definition 10. Let R be a finitely generated algebra over a field K of characteristic 0, and let $N \subset M$ be finitely generated R-modules. We say that N is *fiberwise* tightly closed in M if there exists descent data such that N_κ is tightly closed in M_κ for almost all κ.

This definition can be shown to be independent of the choice of descent data: in comparing descent over A with descent over B, we may compare both with descent over a $\mathbb{Z}$-subalgebra C of K such that $A \subset C$ and $B \subset C$; hence we may assume without loss of generality that $A \subset B$, and by localizing at one element of A^0 and of B^0, we may also assume that B is A-free and smooth over A. Since A is a Hilbert ring, and B is a finitely generated A-algebra, every maximal ideal of B lies over a maximal ideal of A, and there is at least one maximal ideal of B lying over every maximal ideal of A. If $\mu' \in \operatorname{MaxSpec}(B)$ lies over $\mu \in \operatorname{MaxSpec}(A)$ and $\kappa' = B/\mu'$, $\kappa = A/\mu$, then N_κ is tightly closed in M_κ if and only if N'_κ is tightly closed in M'_κ. (See the proof of Theorem 2.5.3 in [HH4] which involves a similar argument. See also Theorem 6.24, Proposition 6.23 and Theorem 7.29 in [HH3].)

Clearly, if N is tightly closed fiberwise in M then N is tightly closed in M, and the two notions may be equivalent.

With the notation above, assume R_K is a Cohen-Macaulay normal domain. Then it can be shown that R_κ is a Cohen-Macaulay normal domain for almost all κ ([HH4], Theorems 2.3.15 and 2.3.17.) We can construct a canonical module ω_κ for R_κ for almost all κ as follows: map a polynomial ring $S_A = A[X_1, \ldots, X_n]$ onto R_A and after localizing at one element of A^0 we may assume that A is Gorenstein and, therefore,

$$\omega_A = \operatorname{Ext}_{S_A}^{\dim(S_A)-\dim(R_A)}(R_A, S_A)$$

will be a canonical module for R_A. Passing to a closed fiber κ and applying Theorem 2.3.5c,e in [HH4] we see that for almost all fibers κ,

$$\omega_\kappa = \operatorname{Ext}_{S_\kappa}^{\dim(S_\kappa)-\dim(R_\kappa)}(R_\kappa, S_\kappa)$$

In addition, for almost all fibers κ, S_κ will remain a Gorenstein ring (Theorem 2.3.15 in [HH4]) and will map onto R_κ ([HH4], Theorem 2.3.5) and we see that for almost all κ, ω_κ will be a canonical module for R_κ. Also, if $x_1, \ldots, x_n \in R_A$ and their images in R_K form a system of parameters then their images in R_κ form a system of parameters for almost all κ (Theorem 2.3.9b in [HH4].)

We next want to raise the following question:

Question. Can we find a system of parameters for R_K, $\underline{x} = x_1, \ldots, x_n \in R_A$ such that the kernels of the maps

$$\rho_{q,t}\colon \frac{\omega_\kappa^{[q]}}{\underline{x}^q \omega_\kappa^{[q]}} \xrightarrow{(x_1 \cdots x_n)^{q(t-1)}} \frac{\omega_\kappa^{[q]}}{\underline{x}^{qt} \omega_\kappa^{[q]}}$$

stabilize for all large q at $t \geq t_0$, for some $t_0 \geq 2$ *independent of* κ?

THEOREM 11. *If the answer is affirmative, with $\underline{x}$ and t_0 as above, then if $\underline{x}^{t_0}\omega$ is fiberwise tightly closed in R_K with* $\operatorname{Rad}(\underline{x}) = m$ *then all m-primary ideals and all ideals contracted from $(R_K)_m$ are tightly closed.*

Proof. Choose descent data for R and choose $\underline{x} = x_1, \ldots, x_n \in R_A$ and t_0 as in the previous paragraph.

If $\underline{x}^{t_0}\omega_K$ is fiberwise tightly closed in R_K, then $\underline{x}_\kappa^{t_0}\omega_\kappa$ will be tightly closed in R_κ for almost all κ. Also, by Theorem 2.3.9 in [HH4] if $I \subset R_K$ is m-primary, $\mathrm{Rad}(I_\kappa) = \mathrm{Rad}(\underline{x}_\kappa)$ after localizing at one element of A^0. Now both assertions follow from Corollary 6 applied to the fiber R_κ. □

We can give an affirmative answer to the question above for one dimensional Cohen-Macaulay K-algebras, where the result is almost trivial but we state it for the sake of completeness, and for two dimensional K-algebras which are normal domains:

THEOREM. 12. *Let R be a one dimensional finitely generated local K-algebra, where K is a field of characteristic* 0. *We can find descent data A for R and $x \in R_A$ a parameter for R, such that the kernels of the maps*

$$\rho_{q,t}\colon \frac{\omega_\kappa^{[q]}}{x^q\omega_\kappa^{[q]}} \xrightarrow{x^{q(t-1)}} \frac{\omega_\kappa^{[q]}}{x^{qt}\omega_\kappa^{[q]}}$$

stabilize for all large q at $t \geq t_0 = 2$, for almost all κ.

Proof. Choose descent data such that, R_κ will be Cohen-Macaulay for almost all fibers. Then x will be a non zero divisor on ω_κ for each such κ. □

Recall that every height one primary ideal Q in a normal ring R has the form $Q = P^{(m)} = P^m R_P \cap R$ for some prime ideal $P \subset R$. If R is a normal domain, and $I \subset R$ is an ideal of pure height one, then I has a primary decomposition $I = P_1^{(\alpha_1)} \cap \cdots \cap P_r^{(\alpha_r)}$ where P_i are prime ideals of height one, and we define the m^{th} *symbolic power* of I as $I^{(m)} = P_1^{(m\alpha_1)} \cap \cdots \cap P_r^{(m\alpha_r)}$. If R is a normal domain and $\omega \subset R$ is a canonical module, since ω has pure height one, we may define $\omega^{(m)}$. A priori this depends on the choice of $\omega \subset R$, but the isomorphism class of $\omega^{(m)}$ depends only on the isomorphism class of ω in the divisor class group of R, hence the isomorphism class of $\omega^{(m)}$ is independent of the embedding $\omega \subset R$.

For the next result, we shall need the following lemma from [W]

LEMMA 13. *Let R be a Noetherian ring of dimension n and prime characteristic $p > 0$. Let $x_1, \ldots, x_n$ be a system of parameters for R, and suppose that for all q we have an exact sequence of finitely generated R^q modules,*

$$0 \to A_q \to F_q \to T_q \to 0$$

such that $x_1^q, \ldots, x_n^q$ form a regular sequence on F_q and $x_1^q T_q = 0$. If there exists a $t_0 \geq 2$ such that for all $t \geq t_0$ and all q, the kernels of the maps

$$\frac{T_q}{(x_2^q, \ldots, x_n^q)T_q} \xrightarrow{(x_2 \cdot \ldots x_d)^{q(t-1)}} \frac{T_q}{(x_2^{qt}, \ldots, x_n^{qt})T_q}$$

are stable then for all $t \geq t_0$ and all q the kernels of the maps

$$\frac{A_q}{(x_1^q, \dots, x_n^q)A_q} \xrightarrow{(x_1 \cdot \dots x_d)^{q(t-1)}} \frac{A_q}{(x_1^q, \dots, x_n^q)A_q}$$

are stable.

(This lemma is stated in [W] with the additional hypothesis that R is local, but its proof does not depend on it.)

LEMMA 14. *Let R be a 2-dimensional finitely generated K-algebra, where K is a field of characteristic* 0, *and assume that R is a normal domain. We can find descent data A for R such that if $x \in \omega_A$ is a parameter for R, there exist $y \in R_A$ and $\gamma \in \omega_A$ such that for almost all fibers y_κ is not in any minimal prime of x_κ and $y_\kappa^m \omega_\kappa^{(m)} \subset \gamma_\kappa^m R_\kappa$ for all positive integers m where ω_A is a canonical module for R_A as in the previous discussion.*

Proof. Since K is a field of characteristic zero, R is geometrically normal, and we may replace R with the faithfully flat extension $R \to R \otimes_K \bar{K}$ where $\bar{K}$ is an algebraic closure of K. Hence, we may assume with no loss of generality that K is algebraically closed.

Let $P^1, \dots, P^n$ be the minimal primes of x in R, and let $S = R - \cup_{i=1}^n P^i$. Since $S^{-1}R$ is semi-local and one dimensional, $S^{-1}\omega$ is principal, therefore, there exists a $u \in S$ such that $u\omega \subset \gamma R$ for some $\gamma \in \omega$. We may assume we have chosen A such that u_A is not in any minimal prime of x_A, and by Theorem 2.3.9b in [HH4], we may conclude that this is preserved in almost all fibers. Also, $u_\kappa \omega_\kappa \subset \gamma_\kappa R_\kappa$ therefore $u_\kappa^m \omega_\kappa^m \subset \gamma_\kappa^m R_\kappa$ for all $m \geq 0$ in almost all fibers.

Let $T = gr_\omega R$, and let J be the S-torsion of T, i.e., $J = \ker(T \to S^{-1}T)$. We can pick a $v \in S$ such that $vJ = 0$. We may descend this setup, while the fact that K is algebraically closed implies that the ideals $(A^0)^{-1}P_A^i$ are absolute primes and, therefore, each P_κ^i is a prime ideal for almost all κ (See 2.3.18 in [HH4]).

Since T_A/J_A has no $(R_A - \cup_{i=1}^n P_A^i)$-torsion, the associated primes for T_κ/J_κ are contained in the P_κ^i's. (See 2.3.9f in [HH4]) Hence, T_κ/J_κ has no $(R_\kappa - \cup_{i=1}^n P_\kappa^i)$-torsion; i.e., the $R_\kappa - \cup_{i=1}^n P_\kappa^i$-torsion of T_κ is contained in J_κ.

We will now show that $v_\kappa^m \omega_\kappa^{(m)} \subset \omega_\kappa^m$ for almost all κ. Write $S_\kappa = R_\kappa - \cup_{i=1}^n P_\kappa^i$ and $I_\kappa = \ker(T_\kappa \to S_\kappa^{-1}T_\kappa)$ (notice that I_κ does not denote $I_A \otimes_A \kappa$ here!) Assume that $t \in \omega_\kappa^{(l)}$ for some l. Then $t \in \omega_\kappa^h$ for some $h \geq 0$, and let $d = l - h$. If $d = 0$ we are done, so assume $d > 0$. We have $S_\kappa^{-1}\omega_\kappa^{(l)} = S_\kappa^{-1}\omega^l$, so we can find an $s \in S_\kappa$ such that $st \in \omega_\kappa^l$, and since $l \geq h + 1$, we have $st \in \omega_\kappa^{h+1}$. Therefore the image of t in T_κ has S_κ-torsion, i.e., the image of t in T_κ is in I_κ. So by the previous paragraph v_κ kills it; i.e., $v_\kappa t \in \omega_\kappa^{h+1}$. Now we proceed by induction on d. If $d = 1$ we have $v_\kappa^{l-1} v_\kappa t \in \omega_\kappa^l$, otherwise assume that $v_\kappa^{d-1} t \in \omega_\kappa^{h+d-1} = \omega_\kappa^{l-1}$, and since $st \in \omega_\kappa^l$, $v_\kappa^{d-1}(st)\omega_\kappa^l$, therefore $v_\kappa^{d-1} t \in I_\kappa$ and we have $v_\kappa^d t \in \omega_\kappa^l$ and since $d \leq l$ we have $v_\kappa^l t \in \omega_\kappa^l$.

Now take $y = uv$, and it will have the desired properties. $\square$

THEOREM 15. *Let R be a 2-dimensional finitely generated K-algebra, where K is a field of characteristic 0, and assume that R is a normal domain. We can find descent data A for R such that if ω_A is a canonical module for R_A as in the previous discussion, then for some system of parameters $x, y \in R_A$ for R, the kernels of the maps*

$$\rho_{q,t}\colon \frac{\omega_\kappa^{[q]}}{(x^q, y^q)\omega_\kappa^{[q]}} \xrightarrow{(x\cdot y)^{q(t-1)}} \frac{\omega_\kappa^{[q]}}{(x^{qt}, y^{qt})\omega_\kappa^{[q]}}$$

stabilize for all large q at $t \geq t_0 = 2$, for almost all κ.

Proof. Pick a parameter $x \in \omega_A$ for R. By Corollary 2.3.12 in [HH4], the image of x in R_κ will be a parameter for almost all κ. Also, by the previous discussion, R_κ will be a normal domain for almost all κ. Applying Lemma 13 to the short exact sequence

$$0 \to \omega_\kappa^{[q]} \xrightarrow{x_\kappa^q} x_\kappa^q R_\kappa \to x_\kappa^q R_\kappa / x_\kappa^q \omega_\kappa^{[q]} \to 0$$

we see that it suffices to show that for some choice of $y \in R_A$ such that the images of x, y in R_κ are parameters for almost all κ, the kernels of the maps

$$\phi_\kappa\colon T_q / y_\kappa^q T_q \xrightarrow{y_\kappa^{q(t-1)}} T_q / y_\kappa^{qt} T_q$$

stabilize for all q and almost all κ at $t \geq 2$, where $T_q = x_\kappa^q R_\kappa / x_\kappa^q \omega_\kappa^{[q]} \cong R/\omega_\kappa^{[q]}$.

Using the previous lemma, choose $y \in R_A$, not in any minimal prime of xR_A such that y_κ is not in any minimal prime or $x_\kappa R_\kappa$ and $y^q \omega_\kappa^{(q)} \subset \omega_\kappa^q$ for almost all κ.

Working in each fiber R_κ satisfying the conditions in the previous paragraph, if the maps ϕ_κ do not stabilize for all q at $t \geq 2$ then they will not stabilize after localizing at a maximal ideal $m \subset R_\kappa$ containing x_κ and y_κ. Further completion of $(R_\kappa)_m$ at its maximal ideal will not affect the stabilization of the maps either. Let $S = \widehat{(R_\kappa)_m}$.

We may map $\kappa[[X, Y]] \to S$ by sending $X \mapsto x^q$ and $Y \mapsto y$, and since x, y were chosen to form a system of parameters in S, this map will be injective and module finite. Since x^q kills T_q, this map makes T_q a finitely generated module over the principal ideal domain $\kappa[[Y]]$, and we may write $T_q = F_q \oplus N_q$ where F_q is $\kappa[[X_2]]$-free and N_q is torsion. The restriction of ϕ_κ on $F_q/y^q F_q$ is clearly injective while y_κ^q was chosen to kill $N_q \cong \omega_\kappa^{(q)}/\omega_\kappa^{[q]}$; hence the maps must stabilize and we have a contradiction. Hence the maps ϕ_κ stabilize in R_κ for all q at $t \geq 2$ for almost all κ. $\square$

6. Strong F-regularity in graded affine algebras of characteristic 0

If R_κ is a reduced Noetherian ring of prime characteristic p we can define the ring $R_\kappa^{1/q}$ to be the ring obtained from R_κ by adjoining all qth roots of elements of R_κ to R_κ.

Let R_κ be a reduced Noetherian ring of prime characteristic p such that $R_\kappa^{1/p}$ is module finite over R_κ. We call R_κ *strongly F-regular* if for every $c \in R_\kappa^0$ there exists a q such that the R_κ-linear map $R_\kappa \to R_\kappa^{1/q}$ mapping 1 to $c^{1/q}$ splits as a map of R_κ modules.

We generalize this notion to affine algebras R over fields of characteristic zero, by calling such R strongly F-regular if R_κ is F-regular for almost all fibers. Notice that the $R_\kappa^{1/p}$ will be automatically module finite over R_κ, as κ is a finite field, hence perfect.

It is known that the strongly F-regular locus is open in prime characteristic p (Theorem 3.3 in [HH1]), and in finitely generated $\mathbb{N}$-graded κ-algebras where κ is an infinite field we further have:

LEMMA 16. *Let κ be an infinite field of prime characteristic p and let R_κ be a finitely generated $\mathbb{N}$-graded κ-algebra with $(R_\kappa)_0 = \kappa$. Let m be the ideal of R_κ generated by the forms of positive degree. The radical defining ideal for the non-strongly F-regular locus is homogeneous. Therefore, R_κ is strongly F-regular if and only if $(R_\kappa)_m$ is strongly F-regular.*

Proof. For every $\alpha \in \kappa$ let θ_α be the automorphism of R_κ sending a form $f \in R_\kappa$ of degree d to $\alpha^d f$.

Let $V(I)$ be the non-strong F-regular locus in R_κ and pick $f \in I$. By 3.2 in [HH1], for every $d \in R^0$ there exist $q \gg 0$ some positive integer t and an R_κ-linear map $\gamma\colon R_\kappa^{1/q} \to R_\kappa$ which sends $d^{1/q}$ to f^t. Hence $\theta_\alpha \circ \gamma$ maps $d^{1/q}$ to $\theta_\alpha(f)^t$, and again by 3.2 in [HH1], $\theta_\alpha(f) \in I$.

If $f = f_0 + f_1 + \cdots + f_d \in I$ where each f_i is a form of degree i, then pick $d+1$ distinct $\alpha_j \in \kappa$ $(1 \le j \le d+1)$. Then for all $1 \le j \le d+1$ we have $\sum_{i=0}^d \alpha_j^i f_i \in I$ and we may solve for the f_i's using the fact that the matrix of coefficients has determinant $\prod_{1\le r<s\le d+1}(\alpha_r - \alpha_s) \neq 0$. Therefore, $f \in I$. □

THEOREM 17. *Let R be a reduced $\mathbb{N}$-graded finitely generated K-algebra, where K is a field of characteristic 0. Assume we can find descent data A for R such that if ω_A is a canonical module for R_A as in the previous discussion, for some homogeneous system of parameters $\underline{x} = (x_1, \ldots, x_n)$ in R_A for R, the kernels of the maps*

$$\rho_{q,t}\colon \frac{\omega_\kappa^{[q]}}{(\underline{x^q})\omega_\kappa^{[q]}} \xrightarrow{(x_1\cdots x_n)^{q(t-1)}} \frac{\omega_\kappa^{[q]}}{(\underline{x^{qt}})\omega_\kappa^{[q]}}$$

stabilize for all large q at $t \ge t_0$, for almost all κ, where t_0 is independent of κ. If

$\underline{x}^{t_0}\omega$ is fiberwise tightly closed in R then R is strongly F-regular. In particular, the notions of weak F-regular type and strong F-regularity are equivalent in R.

Proof. Assume first that κ is infinite. Using the previous lemma we may deduce that R_κ is F-regular if and only if it is strongly F-regular at the origin. Since weakly F-regular, F-bounded and F-finite rings of prime characteristic p are strongly F-regular (Theorem 3.3 in [W]), we conclude that R_κ is strongly F-regular by applying Theorem 11.

If κ is finite, pick an infinite perfect extension field $\kappa \subset \lambda$. By the previous discussion, $\lambda \otimes_\kappa R_\kappa$ is strongly F-regular, and if $1 \otimes d^{1/q} \mapsto 1$ is a splitting of $\lambda \otimes_\kappa R_\kappa \to \lambda \otimes_\kappa R_\kappa^{1/q}$, since

$$(\lambda \otimes_\kappa R_\kappa)^{1/q} \cong \lambda^{1/q} \otimes_{\kappa^{1/q}} R_\kappa^{1/q} \cong \lambda \otimes_\kappa R^{1/q}$$

we obtain a splitting $1 \otimes d^{1/q} \mapsto 1$ of $R_\kappa \to R_\kappa^{1/q}$. $\square$

Added in proof. The author has recently learned that B. MacCrimmon showed in his Ph.D. thesis, submitted to The University of Michigan that *all* local rings with an isolated non-Gorenstein point are F-bounded.

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