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SOME RESULTS ON THE TOPOLOGY OF FOUR-MANIFOLDS WITH NONNEGATIVE CURVATURE

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1. Introduction

One of the most interesting problems in Riemannian geometry is the study of the topology of manifolds which admit a metric with nonnegative sectional curvatures. Although many results are known, such as pinching theorems, in general the problem is quite open. In the case that the curvature operator is nonnegative, the results of several authors lead to a topological classification of such manifolds. This classification can be found in [MN].

If the dimension of the manifold is three, the nonnegativity of the sectional curvatures implies the nonnegativity of the curvature operator because the Weyl tensor is identically zero. If the dimension is four, the work of Walschap [W] gives a thorough understanding of complete noncompact 4-manifolds with nonnegative sectional curvatures. However, in the compact case this problem has only been solved so far under additional assumptions (see [F], [HK], [S_1] [S_2], [Se], for instance) and the Hopf conjecture remains unsolved: does $S^2 \times S^2$ admit a positively curved Riemannian metric?

It follows from Theorem 3 in [CG], that if M is a compact 4-manifold with nonnegative sectional curvatures then the universal covering $\tilde{M}$ splits isometrically as $\overline{M}^{4-k} \times \mathbf{R}^k$, where $\overline{M}$ is compact. If $k = 1$, the topological classification of compact 3-manifolds with nonnegative Ricci curvature in [H], implies that $\overline{M}^3$ is homeomorphic to the sphere S^3. If $k = 2$, $\overline{M}^2$ is homeomorphic to S^2 by the classical Theorem of Gauss-Bonnet. Hence, if the fundamental group $\pi_1(M)$ is infinite, M is covered either by $\mathbf{R}^4$ or by $S^{4-k} \times \mathbf{R}^k$ for $k = 1, 2$. Therefore, we consider in this paper the case that $\tilde{M}$ is compact with some hypotheses that will imply that $\tilde{M}$ is definite. The description of the topology of $\tilde{M}$ will then follow from [Do] and [Fr].

First, recall that for an oriented 4-manifold, the bundle of exterior 2-forms Λ^2 splits as the Whitney sum $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ where $\Lambda^2_\pm$ are the eigenspaces of the Hodge $*$ operator. The Weyl conformal tensor W leaves $\Lambda^2_\pm$ invariant and $W^\pm$ will denote its restriction to $\Lambda^2_\pm$. It is known that a compact half-conformally flat manifold (either $W^+ = 0$ or $W^- = 0$) with nonnegative scalar curvature is definite or Kähler in the conjugate orientation (see [L] or [N]). In Section 3 we generalize this fact in Proposition 3.2. We obtain the same conclusion under the assumption that

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$||W^-||^2 \leq \frac{S^2}{24}$ for all points of M, where S is the scalar curvature. As in [N], we then apply this proposition to obtain a theorem for compact 4-manifolds with nonnegative Ricci curvature. We recall that a K3 surface is a complex surface with first Betti number $b_1(M) = 0$ and first Chern class $c_1 = 0$. Denoting the dimension of the subspace of harmonic 2-forms which are anti-self-dual by b_2^-, we have the following first result.

THEOREM 1. *Let M be an oriented, compact 4-manifold with nonnegative Ricci curvature. Suppose that $||W^-||^2 \leq \frac{S^2}{24}$ for all points of M. Then one of the following holds*:

(a) *$\tilde{M}$ is either homeomorphic to S^4 or to the connected sum $\mathbf{CP}^2\#...\#\mathbf{CP}^2$.*
(b) *$\tilde{M}$ splits isometrically as $S^3 \times \mathbf{R}$ or $S^2 \times \mathbf{R}^2$ or $S^2 \times S^2$.*
(c) *$\tilde{M}$ is a K3 surface with a Ricci flat Kähler (i.e., Calabi - Yau) metric.*
(d) *M is a Kähler manifold with the conjugate orientation and $b_2^-(M) = 1$.*

For the case of nonnegative sectional curvatures, we examine these relations between the norm of the components of the Weyl tensor and the scalar curvature S, proving the next result.

THEOREM 2. *Let M be an oriented, compact 4-manifold with nonnegative sectional curvatures. Let us suppose that $||W^+||^2 \leq \frac{S^2}{24}$ or $||W^+||^2 \geq \frac{S^2}{24}$ for all points of M.*

(a) *If at some point of M we have $||W^+||^2 < \frac{S^2}{24}$ or $||W^+||^2 > \frac{S^2}{24}$, then M is definite. It follows that if $\pi_1(M)$ is finite then the universal covering $\tilde{M}$ is homeomorphic to either S^4 or to the connected sum $\mathbf{CP}^2\#\ldots\#\mathbf{CP}^2$.*
(b) *If $||W^+||^2 = \frac{S^2}{24}$ for all points of M and $\pi_1(M)$ is finite, then one of the following holds*:

(i) *M is homeomorphic to the sphere S^4.*
(ii) *M is a locally a Riemannian product of two surfaces.*
(iii) *M is a Kähler manifold biholomorphic to the complex projective space $\mathbf{CP}^2$.*

Although the assumption of the above theorem seems quite restrictive, it is verified for all Kähler manifolds (with the natural orientation, $||W^+||^2 = \frac{S^2}{24}$). Moreover, if we do not assume orientability and M has a parallel two-form, applying Theorem 2 to the double cover of M we conclude that either M is locally a product of two surfaces or M itself is orientable and M is biholomorphic to $\mathbf{CP}^2$. Therefore, Theorem 2 generalizes the results of Seaman in [S_1].

Theorem 2 has other consequences too. As is pointed out in [HK], the known examples of compact, orientable 4-manifolds with nonnegative sectional curvatures (S^4, $\mathbf{CP}^2$ and local product of surfaces with nonnegative sectional curvatures), admit metrics with a lot of symmetry. On a Riemannian homogeneous space, the scalar curvature, $||W^-||$ and $||W^+||$ are constant. Supposing the constancy of the norm of only one component of the Weyl tensor W, Theorem 2 implies the following result:

COROLLARY 1. *Let M be an oriented, compact 4-manifold with nonnegative sectional curvatures. Let us suppose that the scalar curvature and $||W^-||$ are constant on M. Then M is either definite or is locally a Riemannian product of two surfaces.*

Comparing the norms of the components of the Weyl tensor we obtain the corollary below.

COROLLARY 2. *Let M be an oriented, compact 4-manifold with nonnegative sectional curvatures. Let us suppose that $||W^-||^2 \leq ||W^+||^2$.*

(a) *If at some point of M we have $||W^-||^2 < ||W^+||^2$, then M is definite.*
(b) *If $||W^-||^2 = ||W^+||^2$ for all points of M and $\pi_1(M)$ is finite, then M is either homeomorphic to S^4 or is locally a Riemannian product of two surfaces.*

The first Pontrjagin form of a 4-manifold is given by $p_1(M) - (||W^+||^2 - ||W^-||^2)\, dV$. It then follows from Corollary 2 that an oriented compact, positively curved Riemannian manifold with $p_1(M) = 0$ is homeomorphic to S^4. Notice that in $S^2 \times S^2$ the first Pontrjagin class is zero. Therefore, Corollary 2 answers the Hopf conjecture under the stronger assumption of first Pontrjagin form zero, instead of the first Pontrjagin class.

As a concluding remark, let us point out that all the conclusions of Theorem 2 and its corollaries hold assuming a weaker condition on the sectional curvatures. This conditions is $K_{ij} + K_{mm} \geq 0$, whenever X_i, X_j, X_m, X_n are orthonormal vectors of the tangent space where K_{ij} denotes the sectional curvature of the plane spanned by $\{X_i, X_j\}$ (see Remark 4.4).

2. Notations and preliminaries

Let M be an oriented Riemannian manifold of dimension 4, and let Λ^2 denote the bundle of exterior 2-forms and $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ the eigenspace splitting for the Hodge $*-$ operator.

The Riemann curvature tensor defines a symmetric operator $\mathfrak{R}\colon \Lambda^2 \to \Lambda^2$ given by

$$\mathfrak{R}(e_{ij}) = \frac{1}{2} \sum_{k,1} R_{ijlk} e_{kl}$$

where $\{e_i\}$ is a local orthonormal basis of 1-forms, e_{ij} denotes the 2-torm $e_i \wedge e_j$ and $R_{ijlk} = \langle R(e_i, e_j)e_l, e_k \rangle$. The operator $\mathfrak{R}$ can be decomposed as

$$\mathfrak{R} = \mathfrak{R}^+_+ + \mathfrak{R}^+_- + \mathfrak{R}^-_+ + \mathfrak{R}^-_-$$

with respect to the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$. This decomposition gives the irreducible components of $\mathfrak{R}$ (see [ST]). They are tr $\mathfrak{R}^+_+ =$ tr $\mathfrak{R}^-_- = \frac{S}{4}$, where S is the scalar curvature, the traceless Ricci tensor $\mathfrak{R}^+_-$ and the two components of the Weyl tensor W^+ and W^- given by $W^+ = \mathfrak{R}^+_+ - \frac{S}{12}$ and $W^- = \mathfrak{R}^-_- - \frac{S}{12}$.

Let F: $\Lambda^2(T_X M) \to \Lambda^2(T_X M)$ be the Weitzenböck operator given by (see [S₁])

$$\begin{aligned}\langle F(e_{ij}), e_{kl}\rangle &= \text{Ric}\,(e_i, e_k)\delta_{jl} + \text{Ric}\,(e_j, e_l)\delta_{ik} - \text{Ric}\,(e_i, e_l)\delta_{jk} \\ &\quad - \text{Ric}\,(e_j, e_k)\delta_{il} - 2\text{R}_{ijlk},\end{aligned}$$

where Ric denotes the Ricci curvature. This operator satisfies the well known Weitzenböck formula; e.g., $\Delta\omega = -\operatorname{div}\nabla\omega + F(\omega)$. Moreover F is a symmetric operator and Λ^2_+ and Λ^2_- are F-invariant (see [S₂], Proposition 1). Then $*$ F=F$*$ and we can find a normal form (as in [ST] for the curvature tensor R) for F at each point of M. Since this normal form will be used in the rest of the paper, we repeat the arguments used in [ST].

PROPOSITION 2.1. *Let M be an oriented four-manifold. Then for each x in M there exists a positively oriented orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of T_xM such that relative to the corresponding basis $\{e_{12}, e_{13}, e_{14}, e_{34}, e_{42}, e_{23}\}$, F takes the form*

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

where

$$A = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix}, \qquad B = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix}$$

Proof. Let $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\beta_1, \beta_2, \beta_3\}$ be orthonormal bases of eigenvectors of F^+ and F^- respectively, and r_i and s_i, $i = 1, 2, 3$, be the corresponding eigenvalues. Let us define the planes $P_i = \frac{\alpha_i + \beta_i}{\sqrt{2}}$ and $P_i^\perp = \frac{\alpha_i - \beta_i}{\sqrt{2}}$. Therefore $\{P_1, P_2, P_3, P_1^\perp, P_2^\perp, P_3^\perp\}$ is an orthonormal basis of $\Lambda^2(T_xM)$ and $F(P_i) = \delta_i P_i + \mu_i P_i^\perp$ and $F(P_i^\perp) = \delta_i P_i^\perp + \mu_i P_i$, where $\delta_i = \frac{r_i + s_i}{2}$ and $\mu_i = \frac{r_i - s_i}{2}$. Moreover since $*P_i = P_i^\perp$ we have $P_i \wedge P_i = 0 = P_i^\perp \wedge P_i^\perp$ which implies that P_i and $P_i^\perp$ are decomposable. We also have $P_1 \wedge P_2 = 0$ and hence $P_1 \cap P_2 \neq \{0\}$. Let e_1 in $P_1 \cap P_2$ be a unit vector and e_2 and e_3 such that $\{e_1, e_2\}$ and $\{e_1, e_3\}$ are oriented orthonormal bases for P_1 and P_2 respectively. Choose e_4 to complete a positive oriented orthonormal basis of T_xM. Then $P_1 = e_1 \wedge e_2$, $P_2 = e_1 \wedge e_3$ and $e_1 \wedge e_4$ is $\pm P_3$ or $P_3^\perp$. The matrix of F relative to $\{e_{12}, e_{13}, e_{14}, e_{34}, e_{42}, e_{23}\}$ is of the above type. □

It follows from the above proposition that the self-dual-2-forms

$$\alpha_1 = \frac{\sqrt{2}}{2}(e_{12} + e_{34}), \quad \alpha_2 = \frac{\sqrt{2}}{2}(e_{13} - e_{24}), \quad \alpha_3 = \frac{\sqrt{2}}{2}(e_{14} + e_{23})$$

are the eigenvectors of the symmetric operator $F^+ = F\colon \Lambda^2_+ \to \Lambda^2_+$ with corresponding eigenvalues given by $r_i = \delta_i + \mu_i$ and the anti-self-dual 2-forms

$$\beta_1 = \frac{\sqrt{2}}{2}(e_{12} - e_{34}), \quad \beta_2 = \frac{\sqrt{2}}{2}(e_{13} + e_{24}), \quad \beta_3 = \frac{\sqrt{2}}{2}(e_{14} - e_{23})$$

are the eigenvectors of the symmetric operator $F^- = F\colon \Lambda^2_- \to \Lambda^2_-$ with corresponding eigenvalues given by $s_i = \delta_i - \mu_i$.

PROPOSITION 2.2. *Let $\{e_1, e_2, e_3, e_4\}$ be the orthonormal basis of Proposition 2.1 Then the self-dual 2-forms α_1, α_2 and α_3 are the eigenvectors of $\mathfrak{R}^+_+$ and the anti-self-dual 2-forms β_1, β_2 and β_3 are the eigenvectors of $\mathfrak{R}^-_-$. Moreover if $\lambda_1, \lambda_2, \lambda_3$ and $\varphi_1, \varphi_2, \varphi_3$ are the corresponding eigenvalues of $\mathfrak{R}^+_+$ and $\mathfrak{R}^-_-$ respectively, we have*

$$\begin{aligned} r_1 &= 2(\lambda_2 + \lambda_3), \quad r_2 = 2(\lambda_1 + \lambda_3), \quad r_3 = 2(\lambda_1 + \lambda_2), \\ s_1 &= 2(\varphi_2 + \varphi_3), \quad s_2 = 2(\varphi_1 + \varphi_3), \quad s_3 = 2(\varphi_1 + \varphi_2), \end{aligned}$$

where r_i and s_i denote the eigenvalues of F^+ and F^- respectively.

Proof. We will show that $\langle \mathfrak{R}(\alpha_i), \alpha_j \rangle = 0$ and $\langle \mathfrak{R}(\beta_i), \beta_j \rangle = 0$ for $i \neq j$. For simplicity, we will show that $\langle \mathfrak{R}(\alpha_1), \alpha_2 \rangle = 0$ and the other ones are proved in similar manner. Since $\langle F(\alpha_1), \alpha_2 \rangle = 0$, we have

$$\langle F(e_{12}), e_{13} \rangle - \langle F(e_{12}), e_{24} \rangle + \langle F(e_{34}), e_{13} \rangle - \langle F(e_{34}), e_{24} \rangle = 0$$

From the definition of F we get

$$\begin{aligned} 0 &= \operatorname{Ric}(e_2, e_3) - 2R_{1231} + \operatorname{Ric}(e_1, e_4) + 2R_{1242} - \operatorname{Ric}(e_1, e_4) - 2R_{3431} \\ &\quad - \operatorname{Ric}(e_2, e_3) + 2R_{3442} = -2R_{1231} + 2R_{1242} - 2R_{3431} + 2R_{3442} \\ &= -4\langle \mathfrak{R}(\alpha_1), \alpha_2 \rangle. \end{aligned}$$

Now, the eigenvalues λ and φ are given by

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(K_{12} + K_{34}) - R_{1234}, & \varphi_1 &= \frac{1}{2}(K_{12} + K_{34}) + R_{1234}, \\ \lambda_2 &= \frac{1}{2}(K_{13} + K_{24}) + R_{1324}, & \varphi_2 &= \frac{1}{2}(K_{13} + K_{24}) - R_{1324}, \\ \lambda_3 &= \frac{1}{2}(K_{14} + K_{23}) - R_{1423}, & \varphi_3 &= \frac{1}{2}(K_{14} + K_{23}) + R_{1423}, \end{aligned} \tag{2.3}$$

where K_{ij} denotes the curvature of the plane $\{e_i, e_j\}$. On the other hand, from the definition of F we have

$$\begin{aligned} r_1 = \langle F(\alpha_1), \alpha_1 \rangle &= \frac{1}{2}(\operatorname{Ric}(e_1) + \operatorname{Ric}(e_2) + \operatorname{Ric}(e_3) + \operatorname{Ric}(e_4) - 2K_{12} - 2K_{34} \\ &\quad + 4R_{1234}) = K_{13} + K_{24} + K_{14} + K_{23} + 2R_{1234}. \end{aligned}$$

Using the first Bianchi identity, we conclude

$$r_1 = \langle F(\alpha_1), \alpha_1\rangle = K_{13} + K_{24} - 2R_{1324} + K_{14} + K_{23} - 2R_{1423} = 2(\lambda_2 + \lambda_3).$$

Similarly we obtain

$$\begin{aligned} s_1 &= \langle F(\beta_1), \beta_1\rangle = K_{13} + K_{24} + K_{14} + K_{23} - 2R_{1234} = 2(\varphi_2 + \varphi_3) \\ r_2 &= \langle F(\alpha_2), \alpha_2\rangle = K_{12} + K_{34} + K_{14} + K_{23} - 2R_{1324} = 2(\lambda_1 + \lambda_3) \\ s_2 &= \langle F(\beta_2), \beta_2\rangle = K_{12} + K_{34} + K_{14} + K_{23} + 2R_{1324} = 2(\varphi_1 + \varphi_3) \quad (2.4) \\ r_3 &= \langle F(\alpha_3), \alpha_3\rangle = K_{12} + K_{34} + K_{13} + K_{24} + 2R_{1423} = 2(\lambda_1 + \lambda_2) \\ s_3 &= \langle F(\beta_3), \beta_3\rangle = K_{12} + K_{34} + K_{13} + K_{24} - 2R_{1423} = 2(\varphi_1 + \varphi_2). \end{aligned}$$

□

From the proof of Proposition 2.1 we conclude that $r_i + 2\lambda_i = s_i + 2\varphi_i = \frac{S}{2}$, where S is the scalar curvature. Therefore we can state:

PROPOSITION 2.3. *The Weitzenböck operator is given in terms of the scalar curvature S by*

$$F^+ = \frac{S}{2} - 2\mathfrak{R}_+^+ = \frac{S}{3} - 2W^+ \quad \text{and} \quad F^- = \frac{S}{2} - 2\mathfrak{R}_-^- = \frac{S}{3} - 2W^-$$

3. Four manifolds with nonnegative Ricci curvature

In this section we prove Theorem 1, stated in the introduction. The result will follow from Proposition 3.2 below and Theorem 3 in [CG].

LEMMA 3.1. *Let $||W^\pm||$ denote the norm of the components $W^\pm$ of the Weyl tensor and S the scalar curvature. If $||W^+||^2 \leq \frac{S^2}{24}$ ($||W^-||^2 \leq \frac{S^2}{24}$), then F^+ (respectively F^-) is nonnegative if $S \geq 0$ and is nonpositive if $S \leq 0$. Moreover a strict inequality implies F^+ (respectively F^-) strictly positive or strictly negative.*

Proof. Let $W_i^\pm$, $i = 1, 2, 3$, be the eigenvalues of $W^\pm$. We claim that $(W_i^\pm)^2 \leq \frac{2}{3}||W^\pm||^2$. In fact, since trace of $W^\pm$ is zero, we have

$$(W_i^\pm)^2 = (W_j^\pm + W_k^\pm)^2 = ||W^\pm||^2 - (W_i^\pm)^2 + 2W_j^\pm W_k^\pm$$

It follows from the discriminant of the characteristic polynomial of $W^\pm$ that

$$(W_i^\pm)^2 - 4W_j^\pm W_k^\pm \geq 0$$

which substituted in the above equation implies the claim. Now if $||W^\pm||^2 \leq \frac{S^2}{24}$ we have

$$(W_i^\pm)^2 \leq \frac{2}{3}||W^\pm||^2 \leq \frac{S^2}{36} \text{ for } i = 1, 2, 3.$$

This together with Proposition 2.5 concludes the proof of the lemma. □

It is well known that a compact half-conformally flat manifold (either $W^+ = 0$ or $W^- = 0$) with nonnegative scalar curvature is either definite or Kähler with the conjugate orientation (see, for instance, [L], Proposition 1 or [N], Proposition 2.4). We generalize this result proving the following proposition.

PROPOSITION 3.2. *Let M be an oriented, locally irreducible, compact four dimensional manifold with nonnegative scalar curvature S and such that $||W^-||^2 \le \frac{S^2}{24}$ for all points of M. Then either M is definite or Kähler with the conjugate orientation. In the latter case either M is covered by a K3 surface with a Ricci flat metric or $b_2^-(M) = 1$.*

Proof. Let G be the restricted holonomy group of M. Since M is locally irreducible, so is G. Therefore the universal covering $\tilde{M}$ (with the covering metric) has the irreducible holonomy group G and $\tilde{M}$ is also irreducible. Recall that in [B], Berger proved that if for some $x \in \tilde{M}$, G acts irreducibly on $T_x\tilde{M}$, then either $\tilde{M}$ is locally symmetric or G is one of the following standard subgroups of SO(4): SO(4), U(2), or SU(2).

If $\tilde{M}$ is locally symmetric so is M. Then by Corollary 4 in [D], M is half-conformally-flat and the result follows. If G=SU(2), Berger also proved that $\tilde{M}$ is Ricci-flat and then has the Calabi-Yau metric (see [Y]).

We were left with two possibilities for G, SO(4) or U(2). Hence the restricted holonomy group of M is either SO(4) or U(2) and these are the only two possibilities for the holonomy group H of M. The assumption that $||W^-||^2 \le \frac{S^2}{24}$, implies, by Lemma 3.1, that F^- is nonnegative. Integrating by parts the Weitzenböck formula over M yields

$$(\Delta\omega, \omega) = (\nabla\omega, \nabla\omega) + \int_M \langle F(\omega), \omega\rangle dV \tag{3.3}$$

where (,) is the inner product on $\Lambda^2(M)$ given by

$$(\phi, \psi) = \int_M \langle\phi, \psi\rangle dV ,$$

where dV is the volume form of M and $\langle , \rangle$ is the naturally induced inner product on the space of 2-forms $\Lambda^2(T_xM)$. If $||W^-||^2 < \frac{S^2}{24}$ for some point p in M, F^- is positive on a neighborhood of p and then (3.3) implies that there are no harmonic anti-self-dual 2-forms and M is definite. Therefore if ω is an anti-self-dual harmonic 2-form, we have $||W^-||^2 = \frac{S^2}{24}$. Since F^- is nonnegative, we conclude that $(\nabla\omega, \nabla\omega) = 0$ and then ω is parallel; i.e., ω is left invariant by H. By the holonomy principle, if $H = SO(4)$, since the sphere S^4 has the same holonomy, the existence of ω would give rise to a parallel and hence harmonic 2-form on S^4, implying that the second Betti number $b_2(S^4) > 0$, which is a contradiction. If $H = U(2)$, since the complex projective space $\mathbf{CP}^2$ has the same holonomy and $b_2(\mathbf{CP})^2 = 1$ we conclude $b_2^-(M) = 1$, using again the holonomy principle. □

Proof of Theorem 1. As we observed in the introduction, if if $\pi_1(M)$ is infinite then $\tilde{M}$ is either $\mathbf{R}^4$ or splits isometrically as $S^{4-k} \times \mathbf{R}^4$ for $k = 1, 2$. If $\pi_1(M)$ is finite and the restricted holonomy group G of M is reducible, then $\tilde{M}$ splits isometrically in the product of two surfaces each of them homeomorphic to the sphere S^2. If G is irreducible, the result follows from Proposition 3.2.

4. Nonnegatively curved four manifolds

In this section we investigate some conditions on compact four manifolds with nonnegative sectional curvatures which imply the nonnegativity of F^+ or F^- or both.

LEMMA 4.1. *Let M be an oriented four dimensional manifold with nonnegative sectional curvatures. If there exists a point p in M such that the operator $\mathfrak{R}_+^+$ (or $\mathfrak{R}_-^-$) has one nonpositive eigenvalue, then $\mathfrak{R}_-^-$ (respectively $\mathfrak{R}_+^+$) is nonnegative at p.*

Proof. Using the notation of Section 2, let us suppsoe that after reordering the basis $\{\alpha_1, \alpha_2, \alpha_3\}$ we have $\lambda_1 \le \lambda_2 \le \lambda_3$. If $\lambda_1 \le 0$ then (2.3) implies that $R_{1234} \ge 0$ and hence $\varphi_1 \ge 0$. In order to show that $\varphi_2 \ge 0$ and $\varphi_3 \ge 0$, we consider the planes $P = \frac{\alpha_1+\beta_2}{\sqrt{2}}$ and $P^\perp = \frac{\alpha_1-\beta_2}{\sqrt{2}}$. The proof of Proposition 2.1 shows that there is an orthonormal basis $\{f_1, f_2, f_3, f_4\}$ of the tangent space such that $f_{12} = \frac{\alpha_1+\beta_2}{\sqrt{2}}$ and $f_{34} = \frac{\alpha_1-\beta_2}{\sqrt{2}}$. Hence the sectional curvatures $K(f_1, f_2)$ and $K(f_3, f_4)$ are given by

$$K(f_1, f_2) = \frac{1}{2}\lambda_1 + \frac{1}{2}\varphi_2 + \langle \mathfrak{R}(\alpha_1), \beta_2 \rangle$$

$$K(f_3, f_4) = \frac{1}{2}\lambda_1 + \frac{1}{2}\varphi_2 - \langle \mathfrak{R}(\alpha_1), \beta_2 \rangle$$

Since $0 \le K(f_1, f_2) + K(f_3, f_4) = \lambda_1 + \varphi_2$, if $\lambda_1 \le 0$ then $\varphi_2 \ge 0$. In a similar manner we show that $\varphi_3 \ge 0$. □

LEMMA 4.2. *If $\mathfrak{R}_+^+ \ge 0$ (or $\mathfrak{R}_-^- \ge 0$) then $||W^+||^2 \le \frac{S^2}{24}$ (respectively $||W^-||^2 \le \frac{S^2}{24}$).*

Proof. Since $\lambda_1 + \lambda_2 + \lambda_3 = \frac{S}{4}$, we have

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 + 2\lambda_1\lambda_3 + 2\lambda_2\lambda_3 = \frac{S^2}{16}.$$

But $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = ||W^+||^2 + \frac{S^2}{48}$ and therefore

$$2\lambda_1\lambda_2 + 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 = \frac{S^2}{24} - ||W^+||^2.$$

The nonnegativity of $\mathfrak{R}_+^+$ implies the result. □

Remark 4.3. Notice that the proof of Lemma 4.1 shows that if $\mathfrak{R}_+^+$ (or $\mathfrak{R}_-^-$) has one negative eigenvalue, then $\mathfrak{R}_-^-$ (respectively $\mathfrak{R}_+^+$) is positive at p. We want to observe that in Lemma 4.2 the positivity of $\mathfrak{R}_+^+$ (or $\mathfrak{R}_-^-$) implies $||W^+||^2 < \frac{s^2}{24}$ (respectively $||W^-||^2 < \frac{s^2}{24}$). This is turn implies in Lemma 3.1 that F^+ (respectively F^-) is positive.

Remark 4.4. The proof of Lemma 4.1 shows that the result still holds if instead of nonnegative sectional curvatures, we assume

(*) $K(X_i, X_j) + K(X_m, X_n) \geq 0$, whenever X_i, X_j, X_m, X_n are orthonormal vectors of the tangent space.

Since condition (*) implies the nonnegativity of the scalar curvature, Remark 4.3 also remains true. Now we prove Theorem 2 and its corollaries using the above lemmas and remarks. It then follows that all our results hold under the weaker assumption (*), as we claimed in the introduction.

Proof of Theorem 2. First, let us suppose that $||W^+||^2 \leq \frac{s^2}{24}$. Then F^+ is nonnegative; if there is a point p in M such that $||W^+||^2 < \frac{s^2}{24}$ we have that F^+ is positive at p. This implies that $b_2^+(M) = 0$. If $||W^+||^2 \geq \frac{s^2}{24}$, then it follows from Lemma 4.2 and Remark 4.3 that $\mathfrak{R}_+^+$ has at each point of M a nonpositive eigenvalue (otherwise $||W^+||^2 < \frac{s^2}{24}$). Then Lemma 4.1 implies $\mathfrak{R}_-^-$ is nonnegative and therefore $||W^-||^2 \leq \frac{s^2}{24}$ and F^- is nonnegative. If $||W^+||^2 > \frac{s^2}{24}$ for some point p, then $||W^-||^2 < \frac{s^2}{24}$ at this point. This implies that F^- is positive at p and then $b_2^-(M) = 0$. The classification of [Do] and [Fr] for definite, smooth, simply connected compact 4-manifolds finishes this part of the proof.

If $||W^+||^2 = \frac{s^2}{24}$ for all points of M, we claim that this fact implies that $||W^-||^2 \leq \frac{s^2}{24}$. If not, $\mathfrak{R}_-^-$ would have a negative eigenvalue at some point p, implying by Lemma 4.1 that $\mathfrak{R}_+^+$ is positive at p and therefore $||W^+||^2 < \frac{s^2}{24}$ at p, contradicting $||W^+||^2 = \frac{s^2}{24}$. If $\pi_1(M)$ is finite and the restricted holonomy group G is reducible, the universal cover $\tilde{M}$ splits isometrically and M is locally a product of two surfaces. If G is irreducible, we proceed as in the proof of Proposition 3.2, using the possibilities for G described in [B]. Notice that now we have $||W^-||^2 \leq \frac{s^2}{24}$ and then F is a nonnegative operator. If G is $SU(2)$, M has the Ricci-flat metric. Then the scalar curvature is identically zero and we conclude from Proposition 2.5 that the Weyl tensor is identically zero; i.e., M is conformally flat. The assumption that $\pi_1(M)$ is finite excludes this case, since we would conclude that M is covered by $\mathbf{R}^4$. Therefore G cannot be $SU(2)$. If $\tilde{M}$ is symmetric space then $\tilde{M}$ is an analytic Riemannian manifold ([He], p. 222, Proposition 5.5); this implies that if $\tilde{M}$ is irreducible then $\tilde{M}$ and M are locally irreducible. By Corollary 4 in [D], M is half-conformally flat. If the second Betti number $b_2 = 0$, M is conformally flat. Conformally flat, locally irreducible and locally symmetric manifolds have constant sectional curvatures and, since M is oriented, in this case we have M isometric to S^4. If $b_2 > 0$ then there exists an harmonic 2-form ω which is parallel, implying that M is Kähler manifold. A Kähler manifold with nonnegative scalar curvature that is locally irreducible and locally symmetric space is isometric to the complex projective space $\mathbf{CP}^2$. Therefore, the only remaining cases are the ones whose holonomy group H is either $SO(4)$ or $U(2)$. Again using the fact that F is a nonnegative operator we consider the following cases:

(i) If $b_2 > 0$, then there exists a harmonic 2-form ω which is parallel, implying again that M is Kähler manifold. Then the holonomy group is $U(2)$ and then the second Betti number $b_2 = 1$. Now, let J denote the complex structure and ρ the Ricci form given by $\rho(X, Y) = \mathrm{Ric}(X, JY)$. Since ρ is closed (see [KN], vol. 2, p. 154) and $b_2 = 1$, it follows from the Hodge Theorem that $\rho = c\omega + \phi$, where c is a real number and ϕ is an exact form orthogonal to ω. Now $(\rho, \omega) = c(\omega, \omega)$, where $(;)$ is as in (3.3). On the other hand, $(\rho, \omega) = \int_M S\, dV$ which implies that $c > 0$ (because S is nonnegative and not identically zero). Since ρ and $c\omega$ are homologous, by Yau's solution to the Calabi conjecture (see [Y]), M admits a Kähler metric with positive Ricci curvature. This in turn implies that M is simply connected, by a result of Kobayashi in [K]. Then $H_2(M, \mathbf{Z}) = \mathbf{Z}$ and, by a result of Whitehead [W], M is homotopy equivalent to $\mathbf{CP}^2$. A result of Yau [Y] implies that a Kähler manifold homotopy equivalent to $\mathbf{CP}^2$ is biholomorphic to $\mathbf{CP}^2$.

(ii) If $b_2 = 0$, let us consider the universal covering $\tilde{M}$. If $b_2(\tilde{M}) > 0$, it follows from part (i) of this proof that $\tilde{M}$ is biholomorphic to $\mathbf{CP}^2$ and therefore cannot cover M. Then, $b_2(\tilde{M}) = 0$ and by Freedman's solution to the Poincare conjecture $\tilde{M}$ is homeomorphic to the sphere. Since M is oriented, we conclude that M is homeomorphic to S^4.

Proof of Corollary 1. If $||W^+||^2 \le \frac{S^2}{24}$ and for some point in M we have $||W^+|| < \frac{S^2}{24}$, the result follows from Theorem 2. If $||W^+||^2 > \frac{S^2}{24}$ for some point p in M, it follows from Lemma 4.2 that $\mathfrak{R}_+^+$ has one negative eigenvalue. Now, from Lemma 4.1 and Remark 4.3 we conclude that $||W^-||^2 < \frac{S^2}{24}$ at p. Now the hypothesis imply that $||W^-||^2 < \frac{S^2}{24}$ for all points of M. Therefore $F^- > 0$ and then (3.3) shows that $b_2^-(M) = 0$ and M is positive definite.

Proof of the Corollary 2. First, suppose that $||W^+||^2 > \frac{S^2}{24}$ for some point p in M. It follows from Lemma 4.2 that $\mathfrak{R}_+^+$ has one negative eigenvalue. Now, from Lemma 4.1 and Remark 4.3 we conclude that $||W^-||^2 < \frac{S^2}{24}$ which implies that F^- is positive at p. For the points in M such that $||W^+||^2 \le \frac{S^2}{24}$, the hypothesis in (a) implies that $||W^-||^2 \le \frac{S^2}{24}$. Therefore F^- is a nonnegative operator that is positive on a neighborhood of p. Now (3.3) shows that $b_2^-(M) = 0$ and M is positive definite. If $||W^+||^2 \le \frac{S^2}{24}$ for all points of M and $||W^-||^2 < \frac{S^2}{24}$ for some point p, we conclude again that $b_2^-(M) = 0$.

Now suppose that $||W^-||^2 = ||W^+||^2$ for all points of M. Recall that the first Pontrjagin form $p_1(M)$ on an oriented 4-manifold is given by $p_1 = (||W^+||^2 - ||W^-||^2)dV$, where dV is the volume form of M (see for instance [AHS], p. 428). Therefore, we have $p_1(M) = 0$ and Lemmas 4.1 and 4.2 imply that $||W^-||^2 = ||W^+||^2 \leq \frac{S^2}{24}$. If $\pi_1(M)$ is finite, considering all the possibilities for restricted holonomy group G in Theorem 2, we conclude that the only possible cases now are S^4 or local Riemannian product of two surfaces, since $p_1(M) = 0$ implies that the signature of M is zero which excludes the possibility of being biholomorphic to $\mathbf{CP}^2$.

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