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ANALYSIS OF SOME FUNCTION SPACES ASSOCIATED TO HANKEL OPERATORS

STEVEN G. KRANTZ, SONG-YING LI AND RICHARD ROCHBERG

1. Introduction

Generally speaking, it may be said that a significant part of the analysis of (holomorphic) Bergman spaces on a bounded domain $D \subseteq \mathbb{C}^n$ can be understood by the device of restricting to D the corresponding holomorphic Hardy spaces on a suitable domain in $\mathbb{C}^{n+1}$.

These ideas have been used in several papers, including Forelli [FOR], Rudin [RUD], Coifman, Rochberg and Weiss [CRW], Beatrous and Burbea [BEB], Ligocka [LIG], and references therein. The main point of the present paper is to give several examples which develop this point of view. Our aim will be to extend these ideas to the *real* Hardy spaces so that we may obtain related results on the "real Bergman spaces" $L^p_b(D)$ by using known results on real Hardy spaces.

Historically, the "real variable" theory of Hardy spaces has proved important in the development of harmonic analysis. A secondary purpose of the present paper is to suggest one possible way to think about real variable Bergman spaces, and to prove some basic results about them.

The paper is organized as follows: In Section 2, we prove several preliminary results on a particular restriction operator R and a corresponding extension operator E. As an application we have a factorization theorem for the Bergman space $A^1(D)$ and a characterization of the boundedness of the small Hankel operator h_f by using the results in [KL1]. In Section 3, we begin to study the 'real Bergman' space, which is closely related to the space obtained by restricting a real Hardy space. We next combine the machinery and results in [KL2-3] to obtain some new results as well as some of the known results from [BL1], [LIH], and [LUL] on the boundedness and compactness of Hankel operators with non-holomorphic symbols on Bergman spaces. We conclude, in Section 4, with some remarks that look ahead to future work.

It is worth noting that the Bergman spaces that we consider in this paper are defined with a weighted measure that will be specified below. These Bergman spaces are clearly equivalent to the classical ones that are defined with respect to Euclidean volume measure. The connection between the two different Bergman projections is less obvious, and is explored in the paper [JAN]. A similar set of remarks applies to the Hankel operators being considered here.

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2. Preliminaries

Let D be a bounded domain in $\mathbb{C}^n$ with C^1 boundary, and let $r(z)$ be a C^1 function on $\overline{D}$, $r(z) > 0$ on D, $r(z) = 0$ on ∂D, $\nabla r \neq 0$ on ∂D. If D is C^2, then we may choose the function r so that $|\nabla r(z)| = 1$ on ∂D. Now we define an extension domain D_e in $\mathbb{C}^{n+1}$, based on D, as follows:

$$D_e = \left\{(z, z_{n+1}) \in \mathbb{C}^{n+1} : z \in D,\ |z_{n+1}|^2 < r(z)\right\}. \tag{2.1}$$

We define two mappings as follows:

(a) Let ϕ be the Euclidean orthogonal projection from $\mathbb{C}^{n+1}$ to $\mathbb{C}^n$; i.e., $\phi(z, z_{n+1}) = z$ for $z \in \mathbb{C}^n$ and $z_{n+1} \in \mathbb{C}$. Then we define a linear operator E from $L^1(D)$ to functions on ∂D_e as follows:

$$E(f)(z, z_{n+1}) = f \circ \phi(z, z_{n+1}) \tag{2.2}$$

(b) Let R be a linear operator defined on $L^1(\partial D_e)$ as follows:

$$R(g)(z) = \frac{1}{2\pi} \int_0^{2\pi} g\left(z, \sqrt{r(z)}e^{i\theta}\right) d\theta \tag{2.3}$$

We shall prove the following simple and fundamental proposition.

PROPOSITION 2.1. *The following two statements hold*:
(i) *For all* $g \in L^1(\partial D_e)$, *we have*

$$\int_{\partial D_e} g\, d\sigma = \int_D R(g)(z)\, \omega(z)\, dv(z). \tag{2.4}$$

(ii) *For any* $f \in L^1(D)$, *we have*

$$\int_{\partial D_e} E(f) d\sigma = \int_{\partial D_e} (f \circ \phi) d\sigma = \int_D f(w)\, \omega(w)\, dv(w). \tag{2.5}$$

Here $\omega(z) = \pi\sqrt{4r(z) + |\nabla r(z)|^2}$.

Proof. Let us first prove (i). We may write ∂D_e as the image of a map X: $D \times [0, 2\pi) \to \partial D_e$ defined by $X(z, \theta) = (z, \sqrt{r(z)}e^{i\theta})$. Thus we have the following formula for the surface measure on ∂D_e:

$$d\sigma(z, z_{n+1}) = \frac{1}{2}\sqrt{4r(z) + |\nabla r(z)|^2}\, dv(z) d\theta = \frac{1}{2\pi}\omega(z) dv(z)\, d\theta.$$

As a result,

$$\int_{\partial D_e} g\, d\sigma = \frac{1}{2\pi} \int_D \int_0^{2\pi} g\left(z, \sqrt{r(z)}e^{i\theta}\right) \omega(z)\, dv(z)\, d\theta$$

$$= \int_D \frac{1}{2\pi} \int_0^{2\pi} g\left(z, \sqrt{r(z)}e^{i\theta}\right) d\theta\, \omega(z)\, dv(z)$$
$$= \int_D R(g)(z)\, \omega(z)\, dv(z).$$

So (2.4) holds, and the proof of part (i) is complete. Next we prove (ii).

When D is the unit ball in $\mathbb{C}^n$, the above formula in (ii) is due to Forelli [FOR]. Applying the conclusion of part (i), we have

$$\int_{\partial D_e} E(f)\, d\sigma = \int_D R(E(f))(z)\omega(z)\, dv(z).$$

Let us calculate $R(E(f))(z)$. By definition, we have

$$R(E(f))(z) = \frac{1}{2\pi} \int_0^{2\pi} E(f)\left(z, \sqrt{r(z)}e^{i\theta}\right) d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z) d\theta = f(z).$$

Therefore (2.5) holds, and so does (ii), and the proof of the proposition is complete. $\square$

Let $\mathcal{H}^p(D_e)$ denote the usual holomorphic Hardy space on D_e with norm

$$\|f\|_{\mathcal{H}^p}^p = \sup_t \left\{ \int_{\partial D_e(t)} |f(z, z_{n+1})|^p d\sigma(z, z_{n+1}) \right\}, \tag{2.6}$$

where $D_e(t) = \{(z, z_{n+1})\colon\ |z_{n+1}|^2 - r(z) + t < 0\}$ for $t > 0$ small. A calculation shows that, if $f \in \mathcal{H}^p(D_e)$, then

$$\|f\|_{\mathcal{H}^p} = \left(\int_{\partial D_e} |f|^p d\sigma \right)^{1/p}, \tag{2.7}$$

where we use the symbol f to denote the boundary trace of f on ∂D_ϵ. Let $f \in \mathcal{H}^2(D_e)$. We may write

$$f(z, z_{n+1}) = \sum_{k=0}^{\infty} f_k(z) z_{n+1}^k.$$

Thus

$$\int_{\partial D_e} |f(z, z_{n+1})|^2 d\sigma(z, z_{n+1})$$
$$= \sum_{k=0}^{\infty} \int_{\partial D_e} |f_k(z)|^2 |z_{n+1}|^{2k} d\sigma(z, z_{n+1})$$
$$= \frac{1}{2\pi} \sum_{k=0}^{\infty} \int_D |f_k(z)|^2 \int_0^{2\pi} \left(\sqrt{r(z)}\right)^{2k} d\theta\, \omega(z)\, dv(z)$$
$$= \sum_{k=0}^{\infty} \int_D |f_k(z)|^2 r(z)^k\, \omega(z)\, dv(z).$$

For each nonnegative integer k, we let $dv_k = r(z)^k \omega(z) dv$, and let $L_k^p(D)$ be the weighted Lebesgue space with respect to the measure dv_k. Let $A_k^p(D)$ be the weighted Bergman space with respect to the measure dv_k. Let us consider the mapping P_k: $L_k^2(D) \to A_k^2(D)$, the Bergman projection with Bergman kernel $K_k(z, w)$. Let S: $L^2(\partial D_e) \to \mathcal{H}^2(\partial D_e)$ be the Szegö projection with the Szegö kernel $S((z, z_{n+1}), (w, w_{n+1}))$. The following formula is due to Ligocka [LIG]:

$$S((z, z_{n+1}), (w, w_{n+1})) = \sum_{k=0}^{\infty} K_k(z, w) z_{n+1}^k \overline{w}_{n+1}^k.$$

From this formula, we have that

$$S((z, 0), (w, 0)) = K_0(z, w),$$

where $K_0(z, w)$ is the weighted Bergman kernel with $dv_0 = \pi \omega(z) dv(z)$ for D. Since $\omega(z) \approx 1$, we may see that, for each $0 < p < \infty$, the relation between the Bergman space $A^p(D)$ and the Hardy space $\mathcal{H}^p(D_e)$ is

$$A^p(D) = \{f(z, 0): f \in \mathcal{H}^p(D_e)\}.$$

In fact the relation $\subseteq$ is clear and the relation $\supseteq$ follows from the subharmonicity of $|f|^p$ on vertical slices.

The following simple proposition now holds:

PROPOSITION 2.2. *Let $1 < p < \infty$. If the Szegö projection S: $L^p(\partial D_e) \to \mathcal{H}^p(D_e)$ is bounded, then the Bergman projection P_k: $L_k^p(D) \to A_k^p(D)$ is bounded, $k = 0, 1, \ldots$*

Proof. This is a standard result; we include the proof for completeness.

Without loss of generality, we shall treat the case $k = 0$; that for $k > 0$ follows similarly. Let $f \in L_k^p(D)$; it is clear that $E(f) \in L^p(\partial D_e)$. Let $\overline{K}_z(w) = K(z, w)$. Now

$$\begin{aligned} P_0(f)(z) &= \int_D K_0(z, w) f(w) \pi \omega dv(w) \\ &= \int_{\partial D_e} E(\overline{K}) E(f) d\sigma \\ &= \int_{\partial D_e} S((z, 0), (w, w_{n+1})) E(f)(w, w_{n+1}) d\sigma(w, w_{n+1}) \\ &= S(E(f))(z, 0) \\ &= R(S(E(f))(z). \end{aligned}$$

From the above identity and Proposition 2.1, we conclude that P_0: $L^p(D) \to A^p(D)$ is bounded. □

We next prove a preliminary result on the operators R and E.

PROPOSITION 2.3. *With the notation above, the following statements hold*:

(a) *For* $0 < p \le \infty$, *we have that* $R : \mathcal{H}^p(D_e) \to A_0^p(D)$ *is a surjective contraction.*

(b) *The operator* E: $L_0^p(D) \to L^p(\partial D_e)$ $(E : A_0^p(D) \to \mathcal{H}^p(D_e))$ *is an isometry.*

(c) *If* $f \in L^2(\partial D_e) \ominus \mathcal{H}^2(\partial D_e)$, *then* $R(f) \in L_0^2(D) \ominus A_0^2(D)$, *where* $\mathcal{H}^2(\partial D_e)$ *is the space of boundary value functions in* $\mathcal{H}^2(D_e)$.

Proof. By definition we have that, for each $f \in \mathcal{H}^p(D_e)$,

$$R(f)(z) = f(z, 0), \quad z \in D.$$

It is easy to see, using Proposition 2.1, that

$$\int_D |f(z, 0)|^p\, \omega(z)\, dv(z) \le \int_{\partial D_e} |f|^p\, d\sigma.$$

The proof of (a) follows.

With the same reasoning (see Proposition 2.1), we have (b).

Now we prove (c). Let $f \in L^2(\partial D_e) \ominus \mathcal{H}^2(D)$. For any $g \in A_0^2(D)$, we have

$$\begin{aligned}
&\int_D R(f)(z)\overline{g}(z)\, \omega(z)\, dv(z) \\
&= \int_D \frac{1}{2\pi} \int_0^{2\pi} f\left(z, \sqrt{r(z)}e^{i\theta}\right) d\theta\, \overline{g}(z)\, \omega(z)\, dv(z) \\
&= \int_{\partial D_e} f(z, z_{n+1})\overline{E(g)}(z, z_{n+1})\, d\sigma(z, z_{n+1}) \\
&= 0
\end{aligned}$$

since $E(g) \in \mathcal{H}^2(D_e)$. Hence (c) follows. Therefore the proof of Proposition 2.2 is complete. □

Let $\mathcal{L}^2(\partial D_e) = \{\overline{f}\colon f \in L^2(\partial D_e) \ominus \mathcal{H}^2(D_e)\}$, and let $\mathcal{L}^p(\partial D_e)$ be the closure of $\mathcal{L}^2(\partial D_e)$ in $L^p(\partial D_e)$ with norm $\|\cdot\|_{L^p(\partial D_e)}$ for $1 \le p \le 2$.

Let $\mathcal{L}^2(D) = \{\overline{f}\colon f \in L_0^2(D) \ominus A_0^2(D)\}$; for each $1 \le p < \infty$, we let $\mathcal{L}^p(D)$ denote the closure of $L_0^p(D) \cap \mathcal{L}^2(D)$ in $L_0^p(D)$. It is easy to see that $\mathcal{L}^1(D)$ can be identified with the space of complex conjugates of functions in closure of $A^2(D)^\perp$ under $L^1(D)$ norm.

COROLLARY 2.4. *For* $0 < p < \infty$, *we have*

(i) $L_0^p(D) = L^p(D) = R(L^p(\partial D_e))$;

(ii) $A_0^p(D) = A^p(D) = R(\mathcal{H}^p(D_e)$;
(iii) $\mathcal{L}^p(D) = R(\mathcal{L}^p(\partial D_e))$ *when* $1 \le p \le 2$.

Proof. It is easy to see that (i) and (ii) are direct consequence of Proposition 2.1. Part (iii) follows from Propositions 2.1, 2.2 and 2.3; we leave the details to the reader. □

Next we give some applications of the properties of the operators E and R.

THEOREM 2.5. *Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let $f \in A_0^1(D)$. Then there are a sequence of positive numbers $\{\lambda_j\}$ and two sequence of functions $\{f_n\}$, $\{g_n\} \subset A_0^2(D)$ such that*

(i) $\|f_j\|_{A^2}\|g_j\|_{A^2} \approx 1, \quad j = 1, 2, \ldots;$
(ii) $f = \sum_{j=1} \lambda_j f_j g_j$;
(iii) $\sum_{j=1}^{\infty} \lambda_j \approx \|f\|_{A_0^1(D)}$.

Proof. The proof of this theorem is a direct consequence of the analogous results for Hardy space given in Section 5 of [KL1] and of Proposition 2.1. □

As usual, we let $h_f(u) = S(f\overline{u})$ denote the small Hankel operator on the Hardy space, and we let $h_f^0 = P_0(f\overline{u})$ denote the small Hankel operator on the Bergman space, respectively. First we prove:

THEOREM 2.6. *Let D be either a smoothly bounded strictly pseudoconvex domain or a convex domain of finite type in $\mathbb{C}^n$. Then, for $1 < p < \infty$, we have that if $f \in \mathcal{H}^2(\partial D_e)$ and $h_f\colon \mathcal{H}^p(\partial D_e) \to L^p(\partial D_e)$ is bounded (compact), then $h^0_{R(f)}\colon A_0^p(D) \to L_0^p(D)$ is bounded (compact).*

Remark. The reader will note that some results in this paper are not asserted (or proved) for finite type domains in $\mathbb{C}^2$. This may seem surprising because the "regular" domains that we discuss below certainly include the finite type domains in $\mathbb{C}^2$. The problem is that the extension/restriction results treated in the present paper would require us to know something about finite type domains in $\mathbb{C}^3$, and that is largely unexplored territory.

Proof. Let p' be the conjugate index of p with $1 < p < \infty$. Under the hypotheses we have the expected duality between $\mathcal{H}^p(\partial D_e)$ and $\mathcal{H}^{p'}(\partial D_e)$ (see [NRSW], [MS]). Let $g \in A^2(D)$. We know that $E(g) \in \mathcal{H}^2(\partial D_e)$. Thus

$$\int_D |h^0_{R(f)}(g)(z)|^p dv_0(z)$$

$$
\begin{aligned}
&= \int_D |P_0(R(f)\bar{g})(z)|^p dv_0(z) \\
&= \int_D |S(E(R(f))\,\overline{E(g)})(z,0)|^p dv_0(z) \\
&\le \int_{\partial D_e} |S(E(R(f))\,\overline{E(g)})|^p d\sigma \\
&\le C_p \sup\left\{\left|\int_{\partial D_e} S(E(R(f))\,\overline{E(g)})\bar{v}\,d\sigma\right| : v \in \mathcal{H}^{p'}(\partial D_e) : \|v\|_{p'} = 1\right\} \\
&= C_p \sup\left\{\left|\int_{\partial D_e} E(R(f)\,\overline{E(g)}\bar{v}\,d\sigma\right| : v \in \mathcal{H}^{p'}(\partial D_e) : \|v\|_{p'} = 1\right\} \\
&= C_p \sup\left\{\left|\int_{\partial D_e} E(R(f)\,\overline{E(g)E(R(v))}d\sigma\right| : v \in \mathcal{H}^{p'}(\partial D_e) : \|v\|_{p'} = 1\right\} \\
&= C_p \sup\left\{\left|\int_{\partial D_e} f\,\overline{E(g)E(R(v))}d\sigma\right| : v \in \mathcal{H}^{p'}(\partial D_e) : \|v\|_{p'} = 1\right\} \\
&\le C_p \int_{\partial D_e} |S(f\,\overline{E(g)})|^p d\sigma \\
&= C_p \int_{\partial D_e} |h_f(E(g))|^p d\sigma.
\end{aligned}
$$

Therefore $\mathrm{h}_{R(f)}$ is bounded (compact) on $A_0^p(D)$ if h_f is bounded (compact) on $\mathcal{H}^p(D_e)$. The proof is complete. □

THOEREM 2.7. *Let D be a bounded domain in $\mathbb{C}^n$ and let $0 < p < \infty$. If $f \in A^2(D)$ and $h_{E(f)} \in S_p(\mathcal{H}^2(\partial D_e), L^2(\partial D_e))$, then $h_f^0 \in S_p(A^2(D), L^2(D))$.*

Proof. This follows directly from the fact that for each n the nth singular value of h_f^0 is no bigger than the nth singular value of $h_{E(f)}$. This fact is a direct consequence of the interpretation of the singular values as approximation numbers and the identity

$$h_{E(f)}(E(u))(z, z_{n+1}) = h_{E(f)}(E(u))(z, 0) = h_f^0(u)(z)$$

for all $u \in A^2(D)$. □

3. Real variable analysis

Let D be either a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ or a smoothly bounded convex domain of finite type in $\mathbb{C}^n$. It is easy to check that if D is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ (convex domain of finite type) then D_e is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^{n+1}$ (convex of finite type in $\mathbb{C}^{n+1}$.). We shall describe a homogeneous structure with respect to a quasimetric d which is related to the complex structure, and the Lebesgue surface measure over ∂D_e defined in [KL1,2]. We let $BMO(\partial D_e)$, $VMO(\partial D_e)$, and the real variable Hardy space $H^p(\partial D_e)$ be defined with respect to that structure of homogeneous type (see [KL2] for definitions.)

Next we show that a quasimetric on ∂D_e can be defined in terms of the quasimetric on ∂D.

Let d_0 be a quasimetric on ∂D. We extend d_0 from ∂D to D in a natural way by letting

$$d_0(z, w) = d_0(\pi(z), \pi(w)) + |r(z) - r(w)|$$

when $r(z, w) = |r(z)| + |r(w)| + |z - w| \leq \delta_0$; if $r(z, w) > 4\delta_0$, then we let $d_0(z, w) = |z - w|$, where δ_0 is a fixed positive number depending only on D. We shall define a function $d : \partial D_e \times \partial D_e \to \mathbb{R}_+$ based on d_0 as follows:

$$d((z, z_{n+1}), (w, w_{n+1})) = d_0(z, w) + |z_{n+1} - w_{n+1}|^2 + |\overline{w}_{n+1}(z_{n+1} - w_{n+1})| \quad (3.1)$$

We may arrange the definition of $d_0(z, w)$ when $r(z, w) \in (\delta_0, 4\delta_0)$ so that d is well defined on $\partial D_e \times D_e$ and is a quasimetric on $\partial D_e \times \partial D_e$. We shall require the following proposition.

PROPOSITION 3.1. *Let D be either a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ or a convex domain of finite type in $\mathbb{C}^n$. Then the quasimetric d on ∂D_e defined as above is comparable to the one formulated in* [KL1,2].

Proof. The proof is a straightforward calculation. We omit the details. □

Following the notation in [BL1], we let $z_0 \in \partial D$; the Carleson region $C_r(z_0)$ is defined as follows:

$$C_r(z_0) = C(z_0, r) = \{z \in D : r(z) \leq r,\ \pi(z) \in B(z_0, r) \subset \partial D\}.$$

The notation $BMO(D)$ was used to denote the function space of all $L^1(D)$ with

$$\|f\|_{BMO(D)} = \sup_{C_r(z_0)} \frac{1}{|C_r(z_0)|} \int_{C_r(z_0)} \left| f(w) - \frac{1}{C_r(z_0)} \int_{C_r(z_0)} f(z)dv(z) \right| dv(z) < \infty.$$

Further, $VMO(D)$ denotes the function space consisting of functions with vanishing mean oscillation with respect the above family of Carleson regions. It was also shown in [BL1] that, $BMO(D)$ ($VMO(D)$) is equivalent to the function space BMO (VMO) with respect to the tents, or with respect to hyperbolic balls. In [LUL], $L_b^1(D)$ spaces are introduced as subspaces of $L_0^1(D)$ consisting of all functions

$$u = \sum_{j=1}^{\infty} \lambda_j a_j, \quad (3.2)$$

where $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1$ ($\lambda_j \geq 0$), and a_j are function with support in some hyperbolic ball $B_\epsilon(z_j) = B(z_j, \epsilon r(z_j))$ with $\int_D a_j(z) dv_0(z) = 0$ and $|a_j| \leq |B_j|^{-1}$. The norm of u is defined as follows:

$$\|u\|_{L_b^1} = \inf\left\{\sum_{j=1}^{\infty} \lambda_j : u = \sum_{j=1}^{\infty} \lambda_j a_j\right\}$$

It is obvious that

$$(L_b^1(D))^* = BMO(D), \quad VMO(D)^* = L_b^1(D). \tag{3.3}$$

Moreover,

$$E(L_b^1(D)) \subset H^1(\partial D_e) \tag{3.4}$$

and

$$R(BMO(\partial D_e) \subset BMO(D), \quad E(BMO(D) \cap C(D)) \subset BMO(\partial D_e). \tag{3.5}$$

Let $H^2(\partial D_e)$ be the atomic Hardy space on ∂D_e. In [KL3], Krantz and Li gave a factorization theorem for functions in $H^1(\partial D_e) \cap \mathcal{L}^1(\partial D_e)$ which played a useful role when applied to the study of the Corona problem in several complex variables (refer to line (2.1) for the definition of D_e). The purpose of this section is to give the analogous Bergman space version of the theorem in Section 4 of [KL3]. Indeed, we shall decompose a function $f \in \mathcal{L}^1(D) \cap L_b^1(D)$ as an infinite sum of the products of (i) functions in $A^p(D)$ with (ii) functions in $\mathcal{L}^{p'}(D)$ where $1/p + 1/p' = 1$. In other words, we shall prove the following theorem:

THEOREM 3.2. *Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let $f \in \mathcal{L}^1(D) \cap L_b^1(D)$. Then there are a sequence of positive numbers $\{\lambda_j\}$, a sequence of functions $\{f_j\} \subset A^p(D)$, and a sequence of functions $\{g_j\} \subset \mathcal{L}^{p'}(D)$, such that*

(i) *(i)* $\|f_j\|_{A^p} \|g_j\|_{L^{p'}} \approx 1, \quad j = 1, 2, \ldots;$
(ii) $f = \sum_{j=1} \lambda_j f_j g_j;$
(iii) $\sum_{j=1}^{\infty} \lambda_j \approx \|f\|_{L_b^1(D)}.$

Proof. We note that a variant of Theorem 3.2 was proved in [LUL]; that references considers a different Bergman projection. Here we present a new proof by using the idea of restriction.

Let $f \in L_b^1(D) \cap \mathcal{L}^1(D)$. It is sufficient to decompose an atom into a sum of the above products with at most a fixed number of terms. Let a be an atom with support on $B_\epsilon(z_0)$. Then, by (3.4), we have that $E(a) \in H^1(\partial D_e)$ is an atom. By the proof of theorems in Section 4 in [KL3], we have

$$(E(a) - \overline{P}(E(a)) = \sum_{j=1}^{M} g_j h_j, \quad \|g_j\|_p \|h_j\|_{p'} \approx 1,$$

where M is a fixed number depending only on D_e, $g_j \in \mathcal{H}^p(\partial D_e)$ and $g_j \neq 0$. (In fact, the g_j are each some power of the Levi polynomial), and $h_j = g_j^{-1} H_j$ where $H_j \in \mathcal{L}^1(\partial D_e)$.

Now we let

$$\tilde{g}_j(z) = g_j(z, 0) = R(g_j)(z) \neq 0, \quad \tilde{h}_j(z) = \tilde{g}_j^{-1} R(H_j).$$

The same argument shows that

$$\|\tilde{g}\|_{L^p(D)} \|\tilde{h}_j\|_{L^{p'}(D)} \leq C.$$

Then

$$E(a) - \overline{P}(E(a)) = \sum_{j=1}^{M} H_j = \sum_{j=1}^{M} \tilde{g}_j \tilde{h}_j$$

and

$$a - \overline{P_0}(a) = \sum_{j=1}^{M} \tilde{g}_j \tilde{h}_j$$

and, since $f \in \mathcal{L}^1(D)$, we have

$$\begin{aligned} f &= \sum_{j=1}^{\infty} \lambda_j a_j \\ &= \sum_{j=1}^{\infty} \lambda_j (a_j - P_0(a)) \\ &= \sum_{j=1}^{\infty} \lambda_j \sum_{\ell=1}^{M} \tilde{g}_{\ell j} \tilde{h}_{\ell j} \end{aligned}$$

To complete the proof of Theorem 3.2, we need the following theorem.

THEOREM 3.3. *Let $1 < p < \infty$ and let p' be its conjugate exponent. Let D be a regular domain. Let $f \in A^p(D)_0$ and $g \in \mathcal{L}^{p'}(D)$. Then $fg \in L^1_b(D) \cap \mathcal{L}^1(D)$.*

We let $H_f = (I - S)S$ and $H_g^0 = (I - P_0) M_{\overline{g}} P_0$ denote the big Hankel operators on the Hardy and Bergman spaces, respectively.

With the same proof as in [KL3], we have that Theorem 3.3 is a corollary of the following theorem.

THEOREM 3.4. *Let D be either a bounded strictly pseudoconvex or convex finite type domain in $\mathbb{C}^n$ with smooth boundary. If $f \in BMO(D) \cap L^p(D)$, then the big Hankel operator $H_f^0 = (I - P_0) M_{\overline{f}} P_0$ is bounded on $L^p(D)$.*

The above theorem was proved in [BL1] (for strictly pseudoconvex domains in $\mathbb{C}^n$ and finite type domains in $\mathbb{C}^2$) and in [LIH] (in the strictly pseudoconvex case) independently. We shall now see how to use the idea of restriction and the results on Hardy spaces in [KL2] to give a new proof of the above theorem. To achieve this goal, we need the following lemma.

LEMMA 3.5. *Let D be as in Theorem 3.4. Then for any $f \in BMO(D) \cap L^p(D)$ and any $c > 0$ there is an $f_0 \in C(D) \cap BMO(D)$ such that*

$$\|f_0\|_{L^p} + \|f_0\|_{BMO(D)} \leq C, \quad \|E(f_0)\|_{BMO(\partial D_e)} \leq C\|f\|_{BMO(D)},$$

the multiplication operator M_{f-f_0} is bounded on $A^p(D)$ and

$$\|M_{f-f_0}P_0\|_{\mathrm{op}} \leq C_p.$$

Proof. Let

$$f_0(z) = \frac{1}{|C_{r(z)}(z)|} f(w)dv_0(w).$$

Then by Lemmas (2.1) and (2.16) in [BL1], we have

$$\|f_0\|_{L^p} + \|f_0\|_{BMO(D)} \leq C, \text{ and } \|M_{f-f_0}P_0\|_{\mathrm{op}} \leq C_p.$$

By (3.5), we have

$$\|E(f_0)\|_{BMO(\partial D_e)} \leq C\|f\|_{BMO(D)}$$

and the proof of the lemma is complete. □

LEMMA 3.6. *Let D be either a smoothly bounded strictly pseudoconvex domain or convex domain of finite type in $\mathbb{C}^n$. Then for $1 < p < \infty$, we have $f \in L^2(D)$ and if $H_{E(f)}$: $\mathcal{H}^p(D_e) \to L^p(\partial D_e)$ is bounded (compact), then H_f^0: $A_0^p(D) \to L_0^p(D)$ is bounded (compact).*

Proof. Let $g \in A^2(D)$. We know that $E(g) \in \mathcal{H}^2(\partial D_e)$. Thus

$$\begin{aligned}
&\int_D |H_f^0(g)(z)|^p dv_0(z) \\
&\quad = \int_D |(\overline{f}g - P_0(\overline{f}\,g)(z)|^p dv_0(z) \\
&\quad = \int_D |(\overline{f}S(E(g))(z,0) - S(\overline{E(f)}\,E(g))(z,0)|^p dv_0(z) \\
&\quad = \int_{\partial D_e} |(\overline{E(f)}S(E(g))(z,z_{n+1}) - S(\overline{E(f)}\,E(g))(z,z_{n+1})|^p d\sigma(z,z_{n+1}).
\end{aligned}$$

Therefore H_f^0 is bounded (compact) on $A_0^p(D)$ if $H_{E(f)}$ is bounded (compact) on $\mathcal{H}^p(D_e)$. So the proof of (b) is complete. Thus the theorem is proved. □

Now we are ready to prove Theorem 3.4.

Proof of Theorem 3.4. By Lemma 3.6, we have that $H^0_{f_0}$ is bounded on $A^p_0(D)$, with

$$\|H^0_{f_0}\| \le C_p \|f\|_{BMO(D)}.$$

Thus

$$\|H^0_f\| \le C(\|M_{f-f_0}P_0\| + \|H^0_{f_0}\|) \le C_p\|f\|_{BMO(D)}$$

and the proof of Theorem 3.4 is complete. □

COROLLARY 3.7. *Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and let $f \in L^p(D)$. Then H^0_f is bounded from $A^p(D)$ to $L^p(D)$ if and only if $(I - P_0)f \in BMO(D) \cap L^p(D)$ for all $1 < p < \infty$.*

Note. This is the main theorem (Theorem 3.3) in [LUL].

Proof. Since $H^0_f = H^0_{f-P_0(f)}$, we have that if $f - P_0(f) \in BMO(D) \cap L^p(D)$, then $H^0_f\colon A^p(D) \to L^p(D)$ is bounded for all $1 < p < \infty$.

Conversely, suppose that H^0_f is bounded on $A^p_0(D)$. Then, for any $u \in A^p_0(D)$ and $g_0 \in L^{p'}(D)$, we have

$$\begin{aligned}|\langle H^0_f(u), g\rangle| &= |\langle \overline{f}u - P_0(\overline{f}u), g\rangle| \\ &= |\langle\langle u(\overline{g} - \overline{P_0(g)}), f\rangle| \\ &= |\langle\langle u(\overline{g} - \overline{P_0(g)}), (I - P_0)f\rangle|\end{aligned}$$

By the Factorization Theorem 3.2, and $(I - P_0)f \in \mathcal{L}^1(D) \cap L^p(D)$ we have $(I - P_0)f \in L^1_b(D))^* = BMO(D) \cap \mathcal{L}^1(D)$ since $L^1_b(D)^* = BMO(D)$. Therefore $(I - P_0)f \in \mathcal{L}^p(D) \cap BMO(D)$, and the proof is complete. □

THEOREM 3.8. *Let D be either a smoothly bounded strictly pseudoconvex or convex finite type domain in $\mathbb{C}^n$ and let $0 < p < \infty$. Let $f \in \mathcal{H}^2(\partial D_e)$. If $H_f \in S_p(\mathcal{H}^2(D_e), L^2(\partial D_e))$, then $H^0_{R(f)} \in S_p(A^2_0(D), L^2_0(D))$.*

Proof. These assertions follow similarly as the proof of Theorem 2.7. □

4. Final remarks

It seems natural to consider function spaces $R(H^1(\partial D_e))$ and $R(BMO(\partial D_e))$. From the preceding sections, we see that

$$L^1_b(D) \subset R(H^1(\partial D_e), \quad R(BMO(\partial D_e)) \subset BMO(D).$$

One may use these containments to prove that

$$R(H^1(\partial D_e))^* = R(BMO(\partial D_e))$$

Further, one may obtain the following result that is similar to the theorem in Section 2 of [KL3].

THOEREM 4.1. *Let D be either a smoothly bounded strictly pseudoconvex or convex domain of finite type in $\mathbb{C}^n$. Then we have the following consequences*:

(a) *If $f \in L^1(D)$ and $f \geq 0$, then $f \in R(H^1(\partial D_e)$ if and only if $f \log^+ f \in L^1(D)$*;

(b) *For any $f \in L^1(D)$, there are $g \in R(BMO(\partial D_e))$ and $h \in R(H^1(\partial D_e)$ such that $f = g\,h$ and $\|f\|_{L^1(D)} \approx \|g\|_{R(BMO(\partial D_e))} \|h\|_{L \log L}$.*

This theorem is similar to one that appeared in Section 2 of [KL3]; now it may be proved using the restriction method. We leave the details to the interested reader.

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Steven G. Krantz, Department of Mathematics, Washington University, St. Louis, MO 63130
sk@math.wustl.edu

Song-Ying Li, Department of Mathematics, University of California, Irvine, CA 92697
sli@math.uci.edu

Richard Rochberg, Department of Mathematics, Washington University, St. Louis, MO 63130
rr@math.wustl.edu