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SIMPLICIAL CURRENTS[1]

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0. Introduction

For a smooth manifold X, the deRham theorem provides a quasi-isomorphism from the complex $\Omega^*(X)$ of differential forms to the complex of (smooth) singular cochains on X. Furthermore (under this isomorphism) the wedge-product in $\Omega^*(X)$ induces the cup-product in cohomology; but $\Omega^*(X)$ has the advantage of being an associative, graded commutative algebra already on the chain level.

In the dual case the deRham theorem gives a quasi-isomorphism from the complex of (smooth) singular chains on X to the complex $\Omega_*(X)$ of compactly supported currents on X. (We use this non-standard notation rather than $\mathcal{D}'(X)$ or $\mathcal{D}'_*(X)$.) The dual of the wedge-product is a map

$$\wedge' \colon \Omega_*(X) \to \Omega_*(X) \hat{\otimes} \Omega_*(X)$$

(where $\hat{\otimes}$ denotes the completed, projective tensor-product), and this is in the appropriate sense an associative and graded commutative coproduct. Furthermore there is a commutative diagram

$$\begin{array}{ccc} H(\Omega_*(X)) & \xrightarrow{\wedge'} & H(\Omega_*(X)\hat{\otimes}\Omega_*(X)) \\ \| & & \uparrow\cong \\ H(\Omega_*(X)) & \xrightarrow{\text{coproduct}} & H(\Omega_*(X)) \otimes H(\Omega_*(X)) \end{array} \tag{0.1}$$

proving that $\wedge'$ identifies with the usual coproduct in homology.

The deRham theorem has a natural and frequently used extension to the category of simplicial manifolds, i.e., simplicial objects in the differentiable category. Here the complex $\Omega^* \|X\|$ of simplicial differential forms, as defined in [5], plays the role of the differential forms on a manifold. That is, $\Omega^*\|X\|$ is an associative, graded commutative algebra, and the cohomology identifies with the cohomology algebra of the (fat) realization $\|X\|$ (see Section 3 for the definitions).

The aim of the following is to introduce a complex $\Omega_* \|X\|$ of *simplicial currents* on a simplicial manifold X, with properties similar to the complex of currents on

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a manifold. To define the simplicial currents we equip the space $\Omega^*\|X\|$ with a natural Frechét topology, and as a first definition we let $\Omega_*\|X\|$ be the dual space. This definition has the advantage that the "simplicial deRham theorem" for $\Omega^*\|X\|$ immediately gives the following corollary.

COROLLARY 0.2 (SIMPLICIAL DERHAM THEOREM FOR CURRENTS). *There is a natural isomorphism*

$$H(\Omega_*\|X\|) \cong H_*(\|X\|).$$

There is however another, more concrete definition of $\Omega_*\|X\|$ given as follows:

THEOREM 0.3. *There is an isomorphism of chain complexes*

$$\Omega_*\|X\| \cong \bigoplus_k \Omega_*(\Delta^k \times X_k)/\overline{\mathrm{span}}_{\mathbf{C}}\{(\varepsilon^i \times \mathrm{id})_*T - (\mathrm{id} \times \varepsilon_i)_*T \mid T \in \Omega_n(\Delta_{k-1} \times X_k)\}.$$

We proceed to prove that $\Omega_*\|X\|$ posesses a suitable coalgebra structure.

THEOREM 0.4. *The dual of the wedge-product*

$$\wedge'\colon\ \Omega_*\|X\| \to \Omega_*\|X\|\hat{\otimes}\Omega_*\|X\|$$

identifies in homology with the coproduct.

Notice that this coproduct is again associative and graded commutative at the chain level. This is of interest even for X a discrete simplicial set. In the proof of this theorem, we shall generalize the constructions to bisimplicial manifolds, and in passing we obtain Künneth formulas for the (co)homology of simplicial forms and currents. These are used to establish the analogue of the diagram (0.1).

Another construction of a complex of simplicial currents is suggested in [7], in the case of a discrete simplicial set. We conclude the treatment of simplicial currents with a proof that this complex embeds in $\Omega_*\|X\|$ as a dense subcomplex with the same homology. More precisely we extend the complex in [7] to general simplicial manifolds, which gives a complex

$$\mathcal{A}_n(X) = \bigoplus_{k=0}^{\infty}\bigoplus_{l=0}^{\infty} \Omega^{k-l}(\Delta^k) \otimes \Omega_{n-l}(X_k),$$

with differential

$$\partial(\omega \otimes S) = (-1)^l d\omega \otimes S + \sum_{i=0}^{k}(-1)^i(\varepsilon^i)^*\omega \otimes (\varepsilon_i)_*S + (-1)^l\omega \otimes \partial S,$$

and we prove:

THEOREM 0.5. *There is a quasi-isomorphism $\mathcal{A}_*(X) \to \Omega_*\|X\|$, which embeds $\mathcal{A}_*(X)$ as a dense subspace.*

Hence in particular our complex $\Omega_*\|X\|$ can be regarded as a completion of $\mathcal{A}_*(X)$ with a suitable topology.

In a subsequent paper we shall apply these constructions for the study of classifying spaces.

Another kind of deRham theorem for currents on simplicial sets has been proved by H. Scheerer, K. Schuch and E. Vogt in their preprint *Tame homotopy theory via de Rham currents*, Freie Universität Berlin, Preprint no. A91-20, Berlin 1991.

1. Some notation

This section is intended to fix the notation for the usual spaces of differential forms and currents.

Forms and currents. Let M be a smooth manifold of dimension m. When nothing else is stated, all manifolds will be second countable. Recall that the *differential p-forms* on M are the smooth sections $\Omega^p(M) = \Omega^0(\bigwedge^p T^*M)$, while the *p-forms of compact support* are the sections with compact support $\Omega_c^p(M) = \Omega_c^0(\bigwedge^p T^*M)$. In local coordinates $x_1, \ldots, x_m$ over $U \subseteq M$, every p-form can be written in the form $\phi = \sum_I f_I dx_I$, where $I = (i_1, \ldots, i_p)$ is a sequence with $i_1 < \cdots < i_p$, $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ and $f_I \in C^\infty(U)$.

Now (cf. [3, chap. 17]) $\Omega^p(M)$ is made into a separable Frechét space in a standard fashion, and so are the spaces ($K \subseteq M$ compact)

$$\Omega_K^p(M) = \{\phi \in \Omega^p(M) \mid \operatorname{supp}(\phi) \subseteq K\}.$$

Furthermore $\Omega_c^p(M)$ is topologized as the strict inductive limit of the spaces $\Omega_K^p(M)$. The *p-currents with compact support* is the (topological) dual $\Omega_p(M) = \Omega^p(M)'$ and the *p-currents* (with general support) is the dual space $\overline{\Omega}_p(M) = \Omega_c^p(M)'$. The canonical inclusion $\Omega_p(M) \subseteq \overline{\Omega}_p(M)$ identifies $\Omega_p(M)$ with the compactly supported currents. The exterior differentials d in $\Omega^*(M)$ and $\Omega_c^*(M)$, define dual differentials $\partial = d'$ in $\Omega_*(M)$ and $\overline{\Omega}_*(M)$. Note that in the notation of deRham/Schwartz $\partial = b$ and

$$\Omega^*(M) = \mathcal{E}(M) \text{ and } \Omega_c^*(M) = \mathcal{D}(M)$$

$$\Omega_*(M) = \mathcal{E}'(M) \text{ and } \overline{\Omega}_*(M) = \mathcal{D}'(M)$$

Unless otherwise specified, the spaces of currents carry the *strong dual topology* (see [10, Chapter 19]). In this topology the spaces are locally convex, complete. By $V \hat{\otimes} W$ we denote the *completed, projective* tensor product of V and W. (We will always use the projective topology on tensor products; see [10, Chapter 43]). There are wedge- and cap-products

$$\wedge\colon\ \Omega^p(M)\hat{\otimes}\Omega^q(M) \to \Omega^{p+q}(M); \qquad \wedge\colon\ \Omega_p(M)\hat{\otimes}\Omega^q(M) \to \Omega_{p-q}(M). \quad (1.1)$$

Also there is a monomorphism (provided M is orientable)

$$T\colon\ \Omega^{m-p}_{\mathrm{loc}}(M) \rightarrowtail \overline{\Omega}_p(M); \quad T_\omega(\phi) = \int_M \phi \wedge \omega. \quad (1.2)$$

Here $\Omega^*_{\mathrm{loc}}(M)$ denotes the locally integrable forms on M. This map preserves the differential up to a sign, $\partial T_\omega = (-1)^p T_{d\omega}$. If ω has compact support, so has T_ω.

Forms and currents on a product space. Let M and N denote smooth manifolds of dimension m and n respectively. A form $\phi \in \Omega^k(M \times N)$ is called a *(p, q)-form* (with $p+q = k$) if in local coordinates $\{x_i\}$ and $\{y_i\}$ it is of the form $\phi = \sum_{I,J} f_{I,J}dx_I \wedge dy_J$. The space of (p, q)-forms is denoted $\Omega^{p,q}(M \times N)$, and is a closed subspace of $\Omega^k(M \times M)$, which in fact is the direct sum

$$\Omega^k(M \times N) = \bigoplus_{p+q=k} \Omega^{p,q}(M \times N). \quad (1.3)$$

The *cross-product*

$$\times\colon\ \Omega^p(M)\hat{\otimes}\Omega^q(N) \xrightarrow{\cong} \Omega^{p,q}(M \times N); \qquad \phi \times \psi = \pi^*_M(\phi) \wedge \pi^*_N(\psi) \quad (1.4)$$

is a topological isomorphism. We obtain from (1.3) a bicomplex structure on $\Omega^*(M \times N)$ with (continuous) differentials defined by $d_M(\phi\times\psi) = (d\phi)\times\psi$ and $d_N(\phi\times\psi) = (-1)^p\phi \times d\psi$. Also from (1.3) we get a corresponding decomposition of currents

$$\Omega_k(M \times N) = \bigoplus_{p+q=k} \Omega_{p,q}(M \times N), \quad (1.5)$$

where

$$\begin{aligned} \Omega_{p,q}(M \times N) &= \Omega^{p,q}(M \times N)' \\ &= \big\{T \in \Omega_k(M \times N) \bigm| T|\Omega^{r,s}(M \times N) = 0 \text{ unless } (r,s) = (p,q)\big\}. \end{aligned}$$

We let $\partial_M = d'_M$ and $\partial_N = d'_N$, imposing a bicomplex structure on $\Omega_*(M \times N)$ as well. Using the tensor-product of distributions, one can define the *cross-product* of currents, which is the topological isomorphism characterized by

$$\begin{gathered} \times\colon\ \Omega_p(M)\hat{\otimes}\Omega_q(N) \xrightarrow{\cong} \Omega_{p,q}(M \times N) \\ \langle S \times T, \phi \times \psi\rangle = \langle S, \phi\rangle \cdot \langle T, \psi\rangle. \end{gathered} \quad (1.6)$$

The monomorphism (1.2) preserves the cross-product up to a sign, $T_\omega \times T_\tau = (-1)^{pq} \cdot T_{\omega\times\tau}$. Finally we have the *slant product* between forms and currents defined by

$$\begin{gathered} /\colon\ \Omega^{p,q}(M \times N)\hat{\otimes}\Omega_p(M) \to \Omega^q(N) \\ (\phi \times \psi)/T = \langle T, \phi\rangle \cdot \psi. \end{gathered} \quad (1.7)$$

2. Forms and currents on a geometric simplex

Before giving the definition of simplicial currents in the next section, we need to study the topological vector spaces of forms and currents on the product of a manifold and a geometric simplex.

The symbol Δ^k will denote (a version of) the *standard simplex*

$$\Delta^k = \{x \in \mathbf{R}^k \mid \sum_i x_i \leq 1; \quad x_i \geq 0 \,\forall i\},$$

that is, the convex hull of the set $\{e_0, e_1, \dots, e_k\} \subseteq \mathbf{R}^k$, where $e_0 = 0$ and $\{e_1, \dots, e_k\}$ is the standard basis.

In the following, M is a smooth manifold.

Differential forms on $\Delta^k \times M$. The *smooth p-forms* on $\Delta^k \times M$ is the quotient space

$$\Omega^p(\Delta^k \times M) = \Omega^p(\mathbf{R}^k \times M)/\{\phi \in \Omega^p(\mathbf{R}^k \times M) \mid \phi|\Delta^k \times M = 0\} \tag{2.1}$$

with the quotient topology derived from the usual Frechét topology on $\Omega^p(\mathbf{R}^k \times M)$. We shall give a different definition of this topology. First notice:

LEMMA 2.2. *$\Omega^p(\Delta^k \times M)$ is a separable Frechét space. Furthermore it is nuclear, Montel and in particular reflexive.*

Proof. We divide by a closed subspace (it is an intersection of kernels of Dirac p-currents), hence the first statement follows. Since $\Omega^p(\mathbf{R}^k \times M)$ is nuclear, this also implies that $\Omega^p(\Delta^k \times M)$ is nuclear [10, Prop. 50.1]. Finally [10, Prop. 50.2, Cor. 3] and [10, Prop. 33.2, Cor. 1] implies that $\Omega^p(\Delta^k \times M)$ is a Montel space. □

Next there is a canonical identification of $\Omega^p(\Delta^k \times M)$ as a vector space with

$$\tilde{\Omega}^p(\Delta^k \times M) = \{\phi\colon \Delta^k \times M \to \wedge^p(\mathbf{R}^k \times TM)^* \mid \exists \bar{\phi} \in \Omega^p(\mathbf{R}^k \times M)\colon \phi = \bar{\phi}|\Delta^k \times M\}, \tag{2.3}$$

that is the space of extendable p-forms on $\overset{\circ}{\Delta}{}^k \times M$. We shall give $\tilde{\Omega}^p(\Delta^k \times M)$ the topology defined by the separating family of semi-norms

$$p^N_{h,\eta,K}(\phi) = \sum_{i=0}^{\binom{n}{p}} \sup_{|\alpha| \leq N} \sup_{\Delta^k \times h(K)} |D^\alpha(\pi_i \circ \eta \circ \phi \circ h^{-1})|. \tag{2.4}$$

Here $h\colon U \subseteq M \to \mathbf{R}^n$ is a chart for M, and $K \subseteq U$ is compact. Also $\eta\colon \bigwedge^p T^*M|U \to U \times \mathbf{C}^{\binom{n}{p}}$ is a trivialization of $\bigwedge^p T^*M$ over U, and $\pi_i\colon U \times \mathbf{C}^{\binom{n}{p}} \to \mathbf{C}$ denotes the i-th projection.

PROPOSITION 2.5. *$\tilde{\Omega}^p(\Delta^k \times M)$ is a Frechét space and the canonical linear isomorphism $\Omega^p(\Delta^k \times M) \xrightarrow{\cong} \tilde{\Omega}^p(\Delta^k \times M)$ is a homeomorphism.*

Proof. To prove that $\tilde{\Omega}^p(\Delta^k \times M)$ is metrizable, we observe that the topology can be defined by countably many semi-norms if we proceed as in [3, Chapter 17.1–17.2].

In order to demonstrate completeness, let us assume for simplicity that $p = 0$ and M is a point. Note that the topology on $\tilde{\Omega}^0(\Delta^k)$ is defined by the norms

$$\|f\|_N = \sum_{|\alpha| \leq N} \|D^\alpha f\|_\infty$$

where $\|\cdot\|_\infty$ denotes the ∞-norm on $C(\Delta^k)$, and that the topology is characterized by the property

$$f_n \to f \Leftrightarrow D^\alpha f_n \to D^\alpha f \text{ uniformly on } \Delta^k \text{ for any multi-index } \alpha.$$

If (f_n) is a Cauchy sequence in $\tilde{\Omega}^0(\Delta^k)$, it is easy to construct a limit function. In fact each sequence $(D^\alpha f_n)$ is Cauchy in $\Omega^0(\Delta^k)$, and in particular is convergent. Put

$$g_\alpha = \lim_{n\to\infty} D^\alpha f_n \in C(\Delta^k).$$

Now $g_\alpha = D^\alpha g_0$ on $\mathring{\Delta}^k$, since $\frac{\partial}{\partial x_i} D^\alpha f_n$ converge uniformly, hence

$$\frac{\partial}{\partial x_i} g_\alpha = \frac{\partial}{\partial x_i} \lim_{n\to\infty} D^\alpha f_n = \lim_{n\to\infty} \frac{\partial}{\partial x_i} D^\alpha f_n \quad \text{on} \quad \mathring{\Delta}^k,$$

whence the assertion follows by induction. We conclude, that $D^\alpha f_n \to g_\alpha = D^\alpha g_0$ on $\mathring{\Delta}^k$, and it remains to show, that $g_0 \in \tilde{\Omega}^0(\Delta^k)$, that is g_0 extends to a smooth function on $\mathbf{R}^k$. Now Whitney's extension theorem [3, 16.4, Prop. 6] states that, if we write

$$g_\alpha(x) = \sum_{|\alpha+\beta|\leq N} g_{\alpha+\beta}(x_0) \cdot \frac{(x-x_0)^\beta}{\beta!} + R_{\alpha,N}(x, x_0),$$

then g_0 extends if we can show

$$R_{\alpha,N}(x, x_0)/|x - x_0|^N \to 0 \text{ for } x \to x_0 \text{ in } \Delta^k.$$

We already know, that this is true for x_0 in the interior by Taylor's theorem. In general we can adapt the proof of Taylor's theorem to this situation. With $x_0 \in \Delta^k$ and $x \in \mathring{\Delta}^k$, define $\gamma\colon [0, 1] \to \Delta^k$, $\quad \gamma(t) = (1-t) \cdot x_0 + t \cdot x$. Put $G_\alpha = g_{\alpha\gamma}$, then

$$G_\alpha^{(k)}(t) = \sum_{|\alpha+\beta|\leq k} \frac{k!}{\beta!} \cdot \gamma(t)^\beta \cdot g_\alpha(\gamma(t)). \tag{$*$}$$

In fact this is true for $t \in]0,1]$ by the chain rule, and in general by continuity. By the Taylor Theorem in one variable,

$$G_\alpha(1) = \sum_{k=0}^{N} \frac{1}{k!} \cdot G_\alpha^{(k)}(\varepsilon) + \frac{1}{(N+1)!} \int_\varepsilon^1 (1-t)^{N-1}(G_\alpha^{(N)}(t) - G_\alpha^{(N)}(\varepsilon))\, dt.$$

By continuity this is also true if $\varepsilon = 0$. Using this formula together with (*) we find

$$R_{\alpha,N}(x, x_0)/|x - x_0|^N \leq \sum_{|\alpha+\beta|=N} \frac{N!}{\beta!} \max_{t\in[0,1]} |g_{\alpha+\beta}(\gamma(t)) - g_{\alpha+\beta}(x_0)|,$$

which finally (using continuity) yields the desired property.

For the second claim of the proposition, notice that i is trivially continuous: Indeed it follows readily from the definitions of the seminorms that the restriction map $\Omega^p(\mathbf{R}^k \times M) \to \tilde{\Omega}^p(\Delta^k \times M)$ is continuous, and i is the induced map $\Omega^p(\Delta^k \times M) \to \tilde{\Omega}^p(\Delta^k \times M)$. Finally the open mapping theorem implies, that i is a homeomorphism. □

In the following we will identify $\Omega^p(\Delta^k \times M)$ and $\tilde{\Omega}^p(\Delta^k \times M)$.

Currents on $\Delta^k \times M$. The *p-currents* (with compact support) on $\Delta^k \times M$ is defined as the dual space

$$\Omega_p(\Delta^k \times M) = \Omega^p(\Delta^k \times M)', \tag{2.6}$$

with the strong dual topology.

LEMMA 2.7. *$\Omega_p(\Delta^k \times M)$ is a complete, nuclear, Montel and in particular a reflexive vector space.*

Proof. [10, Theorem 32.2, Cor. 2], [10, Prop. 50.6] and [10, Prop. 36.10]. □

The currents on $\Delta^k \times M$ are more conveniently described by the following proposition, which gives an alternative definition of $\Omega_p(\Delta^k \times M)$.

PROPOSITION 2.8. *There is a natural isomorphism of vector spaces*

$$\Omega_p(\Delta^k \times M) \cong \{T \in \Omega_p(\mathbf{R}^k \times M) \mid \operatorname{supp}(T) \subseteq \Delta^k \times M\}.$$

Proof. The projection $\Omega^p(\mathbf{R}^k \times M) \to \Omega^p(\Delta^k \times M)$ induces by transposition a (continuous), injective map Φ: $\Omega_p(\Delta^k \times M) \to \Omega_p(\mathbf{R}^k \times M)$, which satisfies $\operatorname{supp}\Phi(T) \subseteq \Delta^k \times M$. On the other hand consider $T \in \Omega_p(\mathbf{R}^k \times M)$ with $\operatorname{supp}(T) \subseteq \Delta^k \times M$ and $\phi \in \Omega^p(\mathbf{R}^k \times M)$ with $\phi|\Delta^k \times M = 0$. We claim that $\langle T, \phi\rangle = 0$. Using a suitable partition of unity, we can assume that $M = \mathbf{R}^n$. Writing $\phi = \sum_I f_I dx_I$ we get $D^\alpha f_I|\Delta^k \times M = 0$ for all multi-indices α, and this implies [8, Theorem 2.3.3] that $\langle T, f_I dx_I\rangle = \langle T \wedge dx_I, f_I\rangle = 0$. So $\langle T, \phi\rangle = 0$ as we claimed, and consequently T defines an element $\tilde{T} \in \Omega_p(\Delta^k \times M)$, with $\Phi(\tilde{T}) = T$. □

The properties of forms and currents on $\Delta^k \times M$. We observe that $\Omega^*(\Delta^k \times M)$ and $\Omega_*(\Delta^k \times M)$ has natural structures of bicomplexes. The differentials will be denoted d_Δ and d_X, respectively ∂_Δ and ∂_X. The total differentials are $d = d_\Delta + d_X$ and $\partial = \partial_\Delta + \partial_X$. The isomorphisms in Proposition 2.5 and Proposition 2.8 provides two alternative definitions of these bicomplex structures.

A map $f\colon \Delta^k \times M \to \Delta^l \times N$ which extends to a smooth map $\bar{f}\colon \mathbf{R}^k \times M \to \mathbf{R}^l \times N$ induces the maps of forms and currents

$$f^*\colon \Omega^p(\Delta^l \times N) \to \Omega^p(\Delta^k \times M) \quad \text{and} \quad f_*\colon \Omega_p(\Delta^k \times M) \to \Omega_p(\Delta^l \times N), \tag{2.9}$$

which are independent of the extension $\bar{f}$. Also there are well-defined wedge- and cap-products

$$\begin{aligned} \wedge\colon \Omega^p(\Delta^k \times M) \otimes \Omega^q(\Delta^k \times M) &\to \Omega^{p+q}(\Delta^k \times M) \\ \wedge\colon \Omega_p(\Delta^k \times M) \otimes \Omega^q(\Delta^k \times M) &\to \Omega_{p-q}(\Delta^k \times M), \end{aligned} \tag{2.10}$$

as well as cross-products

$$\begin{aligned} \times\colon \Omega^p(\Delta^k) \otimes \Omega^q(M) &\to \Omega^{p+q}(\Delta^k \times M) \\ \times\colon \Omega_p(\Delta^k) \otimes \Omega_q(M) &\to \Omega_{p+q}(\Delta^k \times M). \end{aligned} \tag{2.11}$$

(For the second one use Proposition 2.8 together with the property that $\operatorname{supp}(S \times T) \subseteq \operatorname{supp}(S) \times \operatorname{supp}(T)$.)

LEMMA 2.12. *The cross-products* (2.11) *induce topological isomorphisms*

$$\times\colon \Omega^p(\Delta^k)\hat{\otimes}\Omega^q(M) \xrightarrow{\cong} \Omega^{p,q}(\Delta^k \times M),$$
$$\times\colon \Omega_p(\Delta^k)\hat{\otimes}\Omega_q(M) \xrightarrow{\cong} \Omega_{p,q}(\Delta^k \times M).$$

Proof. We already know that $\Omega^p(\Delta^k)$ and $\Omega^q(M)$ are nuclear. The map

$$\times\colon \Omega^p(\Delta^k) \times \Omega^q(M) \to \Omega^{p,q}(\Delta^k \times M)$$

is continuous, because $\times\colon \Omega^p(\mathbf{R}^k) \times \Omega^q(M) \to \Omega^{p,q}(\mathbf{R}^k \times M)$ is continuous. Proceeding as in the proof of [10, theorem 51.6] it remains to show that

$$\times\colon \Omega^p(\Delta^k) \otimes \Omega^q(M) \to \Omega^{p,q}(\Delta^k \times M)$$

is injective with a dense image. The image is dense because $\times\colon \Omega^p(\mathbf{R}^k) \otimes \Omega^q(M) \to \Omega^{p,q}(\mathbf{R}^k \times M)$ has a dense image. To prove injectivity we refer to Proposition 2.5, which implies that $\times$ can be identified with a restriction of the cross-product

$$\times\colon \Omega^p(\mathring{\Delta}^k) \otimes \Omega^q(M) \to \Omega^{p,q}(\mathring{\Delta}^k \times M),$$

which is injective.

By duality we get a topological isomorphism

$$\times'\colon \Omega_{p,q}(\Delta^k \times M) \xrightarrow{\cong} \Omega_p(\Delta^k)\hat{\otimes}\Omega_q(M).$$

It follows (using the proof of proposition 2.8) that $\times' \circ \times = \mathrm{id}$, and thus also $\times$ is a topological isomorphism. □

Finally let us recall the monomorphism

$$T\colon \Omega_{\mathrm{loc}}^{k-p}(\mathbf{R}^k) \rightarrowtail \overline{\Omega}_p(\mathbf{R}^k), \quad T_\omega(\phi) = \int_{\mathbf{R}^k} \phi \wedge \omega,$$

and consider the canonical inclusion

$$\Omega^{k-p}(\Delta^k) \hookrightarrow \Omega_{\mathrm{loc}}^{k-p}(\mathbf{R}^k), \quad [\omega] \mapsto 1_{\Delta^k} \cdot \omega.$$

Proposition 2.8 implies, that the composed mapping yields a monomorphism

$$T\colon \Omega^{k-p}(\Delta^k) \rightarrowtail \Omega_p(\Delta^k). \tag{2.13}$$

In particular the constant function 1_{Δ^k} is regarded as a k-current, and is just integration over Δ^k. This map does not preserve the differential, an explicit formula (which will enter in Section 5) can be derived from Stokes' theorem:

LEMMA 2.14.

$$\partial T_\omega = (-1)^p T_{d\omega} + \sum_{i=0}^{k} (-1)^i (\varepsilon^i)_* T_{(\varepsilon^i)^*\omega}.$$

Proof. With $\omega \in \Omega^{k-p}(\Delta^k)$ and $\phi \in \Omega^{p-1}(\Delta^k)$ we have

$$\begin{aligned}
\langle \partial T_\omega, \phi\rangle &= \int_{\Delta^k} d\phi \wedge \omega = \int_{\Delta^k} d(\phi\wedge\omega) + (-1)^p \int_{\Delta^k} \phi \wedge d\omega \\
&= (-1)^p \langle T_{d\omega}, \phi\rangle + \sum_{i=0}^{k} (-1)^i \int_{\Delta^{k-1}} (\varepsilon^i)^*\phi \wedge (\varepsilon^i)^*\omega \\
&= (-1)^p \langle T_{d\omega}, \phi\rangle + \sum_{i=0}^{k} (-1)^i \langle (\varepsilon^i)_* T_{(\varepsilon^i)^*\omega}, \phi\rangle.
\end{aligned}$$

□

Remarks 2.15.

(1) Notice that all the preceding immediately generalize to the case, where Δ^k is replaced by the convex hull of $k+1$ points $\{x_0, \dots, x_k\}$ in general position. Once a bijection $\{e_0, \dots, e_k\} \to \{x_0, \dots, x_k\}$ is chosen, this yields spaces which are canonically isomorphic to the above .

(2) Observe that the "Poincaré lemma" operators

$$h_j\colon \Omega^n(\Delta^k \times M) \to \Omega^{n-1}(\Delta^k \times M), \qquad i = 0, \dots, k, \tag{2.16}$$

as defined in [5] for example, are continuous; in particular $\Omega_*(\Delta^k \times M)$ is chain homotopic to $\Omega_*(M)$, and thus has the correct homology groups.

3. Simplicial deRham theory

With the definitions of Section 2, we are ready to introduce the extended simplicial deRham theory. Let $\boldsymbol{\Delta}$ denote the *simplicial category* of ordered sequences $[n] = \{0, 1, 2, \dots, n\}$ with *weakly increasing* functions $f\colon [n] \to [m]$ as morphisms. Also let $\boldsymbol{\Delta}_0$ denote the category with morphisms restricted to the *strictly* increasing functions. A *simplicial manifold* is a functor $X\colon \boldsymbol{\Delta}^{\mathrm{op}} \to \mathcal{M}$, where $\mathcal{M}$ is the category of smooth manifolds, while a Δ-*manifold* is a functor $X\colon \boldsymbol{\Delta}_0^{\mathrm{op}} \to \mathcal{M}$. In both cases X is determined by a sequence $X_0, X_1, X_2, \dots$ of manifolds together with face maps $\varepsilon_i\colon X_k \to X_{k-1}$, and, in the case of a simplicial manifold, also degeneracy maps $\eta_i\colon X_k \to X_{k+1}$ satisfying the usual relations (see e.g. [6, Def. 2.5]).

In the following X will denote a Δ-manifold (or the underlying Δ-manifold of a simplicial manifold), but the constructions, except when otherwise stated, has "normal" (in the phraseology of [6]) counterparts for simplicial manifolds.

Simplicial forms and currents. As in [5] (with a minor change of notation) define the *simplicial n-forms* as the space

$$\Omega^n \|X\| = \Big\{\phi \in \prod_k \Omega^n(\Delta^k \times X_k) \;\Big|\; (\varepsilon^i \times \mathrm{id})^*\phi^{(k)} = (\mathrm{id} \times \varepsilon_i)^*\phi^{(k-1)}\Big\}. \tag{3.1}$$

This should be thought of as the space of forms on the (fat) realization

$$\|X\| = \coprod_k \Delta^k \times X_k/(\varepsilon^i(t), x) \sim (t, \varepsilon_i(x)).$$

If X is a simplicial manifold, the corresponding "normal" space is to be denoted $\Omega^n|X|$, and can be regarded as the space of forms on the "geometric" realization $|X|$ of X. We endow $\Omega^n\|X\|$ with the topology induced from the product topology on the space $\prod_k \Omega^n(\Delta^k \times X_k)$.

LEMMA 3.2. *$\Omega^n\|X\|$ is a separable Frechét space, which is also nuclear, Montel and hence reflexive.*

Proof. That $\Omega^n\|X\|$ is a Frechét space follows from Lemma 2.2, since a countable product of Frechét spaces is a Frechét space, and $\Omega^n\|X\|$ is a closed subspace of $\prod_k \Omega^n(\Delta^k \times X_k)$ (it is the intersection of the kernels of the operators $(\varepsilon^i \times \mathrm{id})^* - (\mathrm{id} \times \varepsilon_i)^*$). The other properties follows as in the proof of Lemma 2.2. □

Definition 3.3. The space of *simplicial n-currents* is the dual space $\Omega_n\|X\| = \Omega^n\|X\|'$ with the strong dual topology.

LEMMA 3.4. *$\Omega_n\|X\|$ is complete, nuclear Montel space; in particular it is reflexive.*

Proof. As for Lemma 2.7. □

Clearly $\Omega^n\|\cdot\|$ and $\Omega_n\|\cdot\|$ are functorial with respect to smooth simplicial maps. Also observe that the Hahn-Banach theorem implies that the evaluation pairing

$$\langle , \rangle\colon \Omega_p\|X\| \otimes \Omega^p\|X\| \to \mathbf{C} \tag{3.5}$$

is non-degenerate.

We now give an alternative, more suggestive definition of $\Omega_n\|X\|$ as the following quotient space.

DEFINITION 3.6.

$$\tilde{\Omega}_n\|X\| = \bigoplus_k \Omega_n(\Delta^k \times X_k)/\overline{\operatorname{span}}_{\mathbf{C}}\{(\varepsilon^i \times \mathrm{id})_*T - (\mathrm{id} \times \varepsilon_i)_*T \mid T \in \Omega_n(\Delta_{k-1} \times X_k)\}$$

with the quotient topology.

Here we can take the weak or strong closure indifferently, since $\bigoplus_k \Omega_n(\Delta^k \times X_k)$ is reflexive; see [10, Prop. 36.2]. We shall now see, that $\Omega_n\|X\|$ and $\tilde{\Omega}_n\|X\|$ can be identified.

THEOREM 3.7. *There is a natural, continuous isomorphism $\tilde{\Omega}_n\|X\| \xrightarrow{\cong} \Omega_n\|X\|$.*

Proof. Consider the inclusion $i\colon \Omega^n\|X\| \hookrightarrow \prod_k \Omega^n(\Delta^k \times X_k)$, and the transposed map $i'\colon \bigoplus_k \Omega_n(\Delta^k \times X_k) \to \Omega_n\|X\|$. This map is onto (by the Hahn-Banach theorem), while $\ker i' = \Omega^n\|X\|^\circ$, the polar of $\Omega^n\|X\|$ (cf. [10, Chap. 19]). Put

$$\mathcal{N}_n\|X\| = \overline{\operatorname{span}}_{\mathbf{C}}\{(\varepsilon^i \times \mathrm{id})_*T - (\mathrm{id} \times \varepsilon_i)_*T \mid T \in \Omega_n(\Delta_{k-1} \times X_k)\}.$$

We want to prove, that $\mathcal{N}_n\|X\| = \ker i'$. Since $\mathcal{N}_n\|X\| \subseteq \ker i'$, there is an induced map

$$i'\colon \bigoplus_k \Omega_n(\Delta^k \times X_k)/\mathcal{N}_n\|X\| \to \Omega_n\|X\|.$$

We have to show that i' is one-to-one. For this proof let us put the weak topology on the dual spaces. It suffices to prove, that the weak transposed i'' is onto [10, Prop. 35.4]. There is a commutative diagram

$$\begin{tikzcd}
\Omega_n \|X\|' \arrow[r, "i''"] & \left(\bigoplus_k \Omega_n(\Delta^k \times X_k)/\mathcal{N}_n\|X\|\right)' \arrow[d, "\pi'"', "\cong"] \\
\Omega^n \|X\| \arrow[u, "j", "\cong"'] \arrow[r, "\cong", "j_1"'] & \mathcal{N}_n\|X\|^\circ
\end{tikzcd}$$

Here j is the canonical isomorphism defined by $j(\phi)(T) = \langle T, \phi\rangle$, and j_1 is induced by the similar isomorphism $j_1\colon \prod_k \Omega^n(\Delta^k \times X_k) \xrightarrow{\cong} (\bigoplus_k \Omega_n(\Delta^k \times X_k))'$; in fact the induced map is clearly well defined and injective. It is also surjective, indeed let $\xi \in \mathcal{N}_n\|X\|^\circ$, then there exists a form $\phi \in \prod_k \Omega^n(\Delta^k \times X_k)$ with $j_1(\phi) = \xi$. This is actually a *simplicial* form since

$$\langle T, (\varepsilon^i \times \mathrm{id})^*\phi - (\mathrm{id} \times \varepsilon_i)^*\phi\rangle = \langle j_1(\phi), (\varepsilon^i \times \mathrm{id})_*T - (\mathrm{id} \times \varepsilon_i)_*T\rangle = 0$$

and $\Omega_n(\Delta^{k-1} \times X_k)$ separates points in $\Omega^n(\Delta^{k-1} \times X_k)$. Finally π' is induced by the projection

$$\pi\colon \bigoplus_k \Omega_n(\Delta^k \times X_k) \to \bigoplus_k \Omega_n(\Delta^k \times X_k)/\mathcal{N}_n\|X\|.$$

π' is an isomorphism by [10, prop. 35.5]. That $\pi' \circ i'' \circ j = j_1$ is seen as follows:

$$\langle \pi' \circ i'' \circ j(\phi), T\rangle = \langle j(\phi), i' \circ \pi(T)\rangle = \langle i' \circ \pi(T), \phi\rangle = \langle T, \phi\rangle = \langle j(\phi), T\rangle.$$

We conclude that i'', and hence also i' is an isomorphism. □

Next observe, that the bigradings pass to the simplicial forms and currents, precisely let

$$\Omega^{p,q}\|X\| = \left(\prod_k \Omega^{p,q}(\Delta^k \times X_k)\right) \cap \Omega^n\|X\|,$$

$$\Omega_{p,q}\|X\| = \Omega^{p,q}\|X\|' = \left\{T \in \Omega_n\|X\| \;\middle|\; T(\Omega^{r,s}\|X\|) = 0 \text{ unless } (r,s) = (p,q)\right\}.$$

These are closed subspaces. It is easy to see, that we get bicomplexes $(\Omega^{**}\|X\|, d_\Delta, d_X)$ and $(\Omega_{**}\|X\|, \partial_\Delta, \partial_X)$ with total complexes $(\Omega^*\|X\|, d)$ and $(\Omega_*\|X\|, \partial)$. We can also define a bicomplex structure on $\tilde{\Omega}_*\|X\|$, compatible with the isomorphism in Theorem 3.7. Finally there are well-defined wedge- and cap products

$$\wedge\colon \Omega^p\|X\| \otimes \Omega^q\|X\| \to \Omega^{p+q}\|X\|; \qquad \wedge\colon \Omega_p\|X\| \otimes \Omega^q\|X\| \to \Omega_{p-q}\|X\| \quad (3.8)$$

obeying the usual relations.

Homology. We proceed to observe that our complex $\Omega_*\|X\|$ has the desired homology groups. This is in fact an easy corollary to the simplicial deRham theorem in [5].

There is another kind of deRham double complex $(A^{**}(X), \delta, d_X)$ associated with X. Namely $A^{p,q}(X) = \Omega^q(X_p)$ with $d_X = (-1)^p d$ and $\delta = \sum_{i=0}^{q+1}(-1)^i \varepsilon_i^*$. The total complex is denoted by $(A^*(X), d)$. Let $(A_{**}(X), \delta, \partial_X)$ denote the dual double complex with total complex $(A_*(X), \partial)$.

PROPOSITION 3.9. *There are natural isomorphisms $H(A^*(X), d) \cong H^*(\|X\|)$ and dually $H(A_*(X), \partial) \cong H_*(\|X\|)$. Under these identifications, the evaluation pairing $A_*(X) \otimes A^*(X) \to \mathbf{C}$ induces the Kronecker product $H_*(\|X\|) \otimes H^*(\|X\|) \to \mathbf{C}$.*

Proof. This is a standard fact with forms and currents on X_p replaced by the singular cochains and chains. Hence this version follows from the deRham theorem(s). □

We quote from [5]:

THEOREM 3.10 (SIMPLICIAL DERHAM THEOREM). *For each q the chain complexes $(\Omega^{*,q}\|X\|, d_\Delta)$ and $(A^{*,q}(X), \delta)$ are naturally chain homotopy equivalent. In fact there are natural maps of bicomplexes*

$$\mathcal{J}\colon \Omega^{**}\|X\| \to A^{**}(X), \qquad \mathcal{E}\colon A^{**}(X) \to \Omega^{**}\|X\|$$

such that $\mathcal{J} \circ \mathcal{E} = \mathrm{id}$, *and chain homotopies* $s\colon \Omega^{p,q}\|X\| \to \Omega^{p-1,q}\|X\|$ *such that*

$$\mathcal{E} \circ \mathcal{J} - \mathrm{id} = s d_\Delta + d_\Delta s, \qquad s d_X = d_X s.$$

These maps are defined by

$$\begin{aligned} \mathcal{J}(\phi) &= \phi^{(p)}/1_{\Delta^p}; && \phi \in \Omega^{p,q}\|X\| \\ \mathcal{E}(\omega)^{(k)} &= p! \cdot \sum_{|I|=p} \alpha_I \times \mu_I^*(\omega); && \omega \in A^{p,q}(X) \qquad (k \geq p) \\ s(\phi)^{(k)} &= \sum_{0 \leq |I| < p} |I|! \cdot \alpha_I \wedge h_I(\phi); && \phi \in \Omega^{p,q}\|X\|. \end{aligned}$$

Here $I = (i_0, i_1, \ldots, i_{|I|})$ denotes a sequence of integers such that $0 \leq i_0 < i_1 < \cdots i_{|I|} \leq k$. Also

$$h_I = h_{i_{|I|}} \circ \cdots \circ h_{i_0} \text{ and } \mu_I = \varepsilon_{j_{k-|I|}} \circ \cdots \circ \varepsilon_{j_1},$$

where $0 \leq j_{k-|I|} < \cdots < j_1 \leq k$ is the complementary sequence to I. Finally α_I is the form

$$\alpha_I = \sum_{j=0}^{|I|} (-1)^j t_{i_j} dt_{i_0} \wedge \cdots \wedge \widehat{dt_{i_j}} \wedge \cdots \wedge dt_{i_{|I|}}.$$

The notation is from [5]. We now observe, that all the operators $\mathcal{J}$, $\mathcal{E}$ and s are continuous, so by the first definition (3.3) of simplicial currents we immediately get

COROLLARY 3.11 (SIMPLICIAL DERHAM THEOREM FOR CURRENTS). *For each q the chain complexes $(\Omega_{*,q}\|X\|, \partial_\Delta)$ and $(A_{*,q}(X), \delta)$ are naturally chain homotopy equivalent. In fact there are natural maps of bicomplexes*

$$\mathcal{J}'\colon A_{**}(X) \to \Omega_{**}\|X\|, \qquad \mathcal{E}'\colon \Omega_{**}\|X\| \to A_{**}(X)$$

such that $\mathcal{E}' \circ \mathcal{J}' = \mathrm{id}$, and chain homotopies $s'\colon \Omega_{p,q}\|X\| \to \Omega_{p+1,q}\|X\|$ such that

$$\mathcal{J}' \circ \mathcal{E}' - \mathrm{id} = s'\partial_\Delta + \partial_\Delta s', \qquad s'\partial_X = \partial_X s'.$$

Proof. The pairing (3.5) is non-degenerate. □

In terms of the second definition of simplicial currents (3.6) these maps are defined by

$$\mathcal{J}'(T) = 1_{\Delta^p} \times T; \qquad T \in A_{p,q}(X)$$
$$\mathcal{E}'(T) = p! \cdot \sum_{|I|=p} (\mu_I)_*(T \wedge \alpha_I); \qquad T \in \Omega_{p,q}(\Delta^k \times X_k) \qquad (k \geq p)$$
$$s'(T) = \sum_{0 \leq |I| < p} |I|! \cdot h'_I(T \wedge \alpha_I); \qquad T \in \Omega_{p,q}(\Delta^k \times X_k).$$

ADDENDUM 3.12. *Under the resulting isomorphisms $H^*(\|X\|) \cong H^*(\Omega^*\|X\|, d)$ and $H_*(\|X\|) \cong H_*(\Omega_*\|X\|, \partial)$ the cup- and cap-products are induced by the wedge-product as in* (3.8).

Proof. That the cohomology isomorphism is multiplicative is shown in [5]. This implies, that also the cap-product is preserved, as follows from Proposition 3.9. □

Remarks 3.13.

(1) In the next section we shall prove a similar result for the coproduct.
(2) Inspired by the above one might consider using the complex $\tilde{\Omega}_*\|X\|_0$ defined by

$$\bigoplus_k \Omega_*(\Delta^k \times X_k)/\operatorname{span}_{\mathbb{C}}\big\{(\varepsilon^i \times \mathrm{id})_* T - (\mathrm{id} \times \varepsilon_i)_* T \;\big|\; T \in \Omega_n(\Delta_{k-1} \times X_k)\big\}.$$

as a definition of simplicial currents. There is a projection $\tilde{\Omega}_*\|X\|_0 \to \tilde{\Omega}_*\|X\|$. We do not know if this is an isomorphism, but at least it is a quasi-isomorphism. Indeed there are well-defined maps $\mathcal{J}'\colon A_*(X) \to \tilde{\Omega}_*\|X\|_0$, $\mathcal{E}'\colon \tilde{\Omega}_*\|X\|_0 \to A_*(X)$ and $s'\colon \tilde{\Omega}_*\|X\|_0 \to \tilde{\Omega}_*\|X\|_0$, defined as the maps above; and one proves directly that $\mathcal{E}' \circ \mathcal{J}' = \mathrm{id}$, $\mathcal{J}' \circ \mathcal{E}' - \mathrm{id} = s'\partial_\Delta + \partial_\Delta s'$ and $s'\partial_X = \partial_X s'$.

(3) Consider in particular a *discrete* simplicial set S. If S is uncountable, e.g., as the singular complex $S_p = \mathrm{Map}(\Delta^p; X)$ of a space X, then the properties of Lemma 3.2 fails. However we still have local convexity, and this means that Corollary 3.11 is still valid, because we only need that the pairing (3.5) is non-degenerate.

4. Künneth Theorems and the coproduct

We now turn to the material needed for Theorem 0.4.

Bisimplicial manifolds. The theory in Section 3 easily extends to bisimplicial manifolds. We leave the details to the reader, but let us briefly describe the constructions. If $(X_{**}, \varepsilon_i', \varepsilon_j'')$ is a bisimplicial manifold, the proper definitions are

$$\Omega^n\|X\| = \left\{ \phi \in \prod_{k,l} \Omega^n(\Delta^k \times \Delta^l \times X_{kl}) \;\middle|\; \begin{array}{l} (\varepsilon^i \times \mathrm{id} \times \mathrm{id})^*\phi^{(k,l)} = (\mathrm{id} \times \mathrm{id} \times \varepsilon_i')^*\phi^{(k-1,l)} \text{ and} \\ (\mathrm{id} \times \varepsilon^j \times \mathrm{id})^*\phi^{(k,l)} = (\mathrm{id} \times \mathrm{id} \times \varepsilon_j'')^*\phi^{(k,l-1)} \end{array} \right\}, \tag{4.1}$$

$$\begin{aligned} \Omega_n\|X\| = \Omega^n\|X\|' \cong \bigoplus_{k,l} \Omega_n(\Delta^k \times \Delta^l \times X_{kl})/ \\ \overline{\mathrm{span}}_{\mathbf{C}}\{(\varepsilon^i \times \mathrm{id} \times \mathrm{id})_*T - (\mathrm{id} \times \mathrm{id} \times \varepsilon_i')_*T \mid T \in \Omega_n(\Delta^{k-1} \times \Delta^l \times X_{kl})\} \\ \cup\{(\mathrm{id} \times \varepsilon^j \times \mathrm{id})_*T - (\mathrm{id} \times \mathrm{id} \times \varepsilon_j'')_*T \mid T \in \Omega_n(\Delta^k \times \Delta^{l-1} \times X_{kl})\}. \end{aligned} \tag{4.2}$$

These spaces are graded over $\mathbf{Z}^3$ with differentials d_Δ', d_Δ'' and d_X in $\Omega^*\|X\|$ respectively ∂_Δ', ∂_Δ'' and ∂_X in $\Omega_*\|X\|$, with the obvious notation. The simplicial deRham theorems generalizes to show, that the (co)homology of these groups is the singular (co)homology of the realization $\|X\|$. As in Section 3, this goes through a suitable complex $A^*(X)$.

This extension is useful for studying products. Thus if X and Y are simplicial manifolds, $X \times Y$ denotes the bisimplicial product. For a bisimplicial manifold Z, the corresponding (diagonal) simplicial manifold is denoted δZ. There is a natural continuous map

$$\delta\colon \Omega^*\|Z\| \to \Omega^*\|\delta Z\|; \qquad \delta(\phi)^{(k)} = (\delta^k)^*(\phi^{(k,k)}), \tag{4.3}$$

where $\delta^k\colon \Delta^k \times Z_k \to \Delta^k \times \Delta^k \times Z_k$ is the obvious map. The latter also induces a map $\delta\colon \|\delta Z\| \to \|Z\|$, and it is standard, that this induce an isomorphism in homology. Furthermore these maps are compatible with the deRham isomorphisms.

(To see this, use the description of the deRham isomorphism on the chain level used in [5, proof of Theorem 2.14]; see also [1].) For technical reasons we also need the following construction: From the simplicial manifold X we can construct bisimplicial manifolds $LX = X \times *$ and $RX = * \times X$, where $*$ denotes the simplicial manifold with $*_k =$ point for all k. There are natural continuous maps $\pi_L\colon \|LX\| \to \|X\|$ and $\pi_R\colon \|RX\| \to \|X\|$, and these induce isomorphisms in homology. Clearly we also have natural continuous maps

$$\pi_L^*\colon \Omega^*\|X\| \to \Omega^*\|LX\| \text{ and } \pi_R^*\colon \Omega^*\|X\| \to \Omega^*\|RX\|, \tag{4.4}$$

and again the maps are compatible under the deRham isomorphisms. All this applies to Δ-manifolds as well, except that for these it may not be true, that δ, π_L and π_R induce isomorphisms in homology.

Cross-products and Künneth formulas. We shall now prove suitable Künneth formulas for simplicial manifolds. To this end we introduce the cross-products

$$\times\colon \Omega^m\|X\| \otimes \Omega^n\|Y\| \to \Omega^{m+n}\|X \times Y\|, \tag{4.5}$$

$$\times_\delta\colon \Omega^m\|X\| \otimes \Omega^n\|Y\| \to \Omega^{m+n}\|\delta(X \times Y)\| \tag{4.6}$$

defined by

$$(\phi \times \psi)^{(k,l)} = \phi^{(k)} \times \psi^{(l)}; \qquad \phi \times_\delta \psi = \pi_X^*(\phi) \wedge \pi_Y^*(\psi).$$

Here π_X and π_Y denotes the obvious projections (note that these are simplicial maps). With $\Delta\colon X \to \delta(X \times X)$ denoting the diagonal map of X, we obtain the usual relation

$$\phi \wedge \psi = \Delta^*(\phi \times_\delta \psi). \tag{4.7}$$

The two cross-products are related by

$$\delta(\phi \times \psi) = \phi \times_\delta \psi. \tag{4.8}$$

There is a cross-product of currents, defined similar to (4.5):

$$\times\colon \Omega_m\|X\| \otimes \Omega_n\|Y\| \to \Omega_{m+n}\|X \times Y\|. \tag{4.9}$$

Indeed it follows from (4.2) that the cross-product

$$\times\colon \Big(\bigoplus_k \Omega_m(\Delta^k \times X_k)\Big) \otimes \Big(\bigoplus_l \Omega_n(\Delta^l \times Y_l)\Big) \to \bigoplus_{k,l} \Omega_{m+n}(\Delta^k \times \Delta^l \times X_k \times Y_l)$$

induces a map as in (4.9). The analogue of (4.7) for currents is the commutative diagram

$$\begin{array}{ccc} \Omega_n\|X\| & \xrightarrow{\wedge'} & \bigoplus_{k+l=n} \Omega_k\|X\|\hat{\otimes}\Omega_l\|X\| \\ \big\downarrow{\scriptstyle \Delta_*} & & \big\downarrow{\scriptstyle \times} \\ \Omega_n\|\delta(X \times X)\| & \xrightarrow{\delta'} & \Omega_n\|X \times X\|. \end{array} \tag{4.10}$$

Here δ' and, as we shall see, also $\times$ are quasi-isomorphisms, and this will lead to Theorem 0.4. First we need the following Künneth formulas:

PROPOSITION 4.11. *For Δ-manifolds X, Y there are commutative diagrams*

$$\begin{tikzcd} H(\Omega^*\|X\|) \otimes H(\Omega^*\|Y\|) \arrow[r, "\times"] \arrow[d, "\cong"] & H(\Omega^*\|X \times Y\|) \arrow[d, "\cong"] \\ H^*(\|X\|) \otimes H^*(\|Y\|) \arrow[r, "\times"] & H^*(\|X \times Y\|) \end{tikzcd} \tag{*}$$

and

$$\begin{tikzcd} H(\Omega_*\|X\|) \otimes H(\Omega_*\|Y\|) \arrow[r, "\times"] & H(\Omega_*\|X \times Y\|) \\ H_*(\|X\|) \otimes H_*(\|Y\|) \arrow[r, "\times"] \arrow[u, "\cong"] & H_*(\|X \times Y\|). \arrow[u, "\cong"] \end{tikzcd} \tag{**}$$

In particular the cross-products (4.5) *and* (4.9) *are quasi-isomorphisms, where in the case of* (4.5) *it is assumed that either $H_*(\|X\|)$ or $H_*(\|Y\|)$ is of finite type.*

Proof. We observe, that the diagram (*) can be composed of two diagrams as follows:

$$\begin{tikzcd} H(\Omega^*\|X\|)\otimes H(\Omega^*\|Y\|) \arrow[r, "\pi_L^*\otimes\pi_R^*"] \arrow[d] & H(\Omega^*\|LX\|)\otimes H(\Omega^*\|RY\|) \arrow[r, "\times"] \arrow[d] & H(\Omega^*\|X \times Y\|) \arrow[d] \\ H^*(\|X\|)\otimes H^*(\|Y\|) \arrow[r, "\pi_L^*\otimes\pi_R^*"] & H^*(\|LX\|)\otimes H^*(\|RY\|) \arrow[r, "\times"] & H^*(\|X \times Y\|) \end{tikzcd}$$

where the cross-products are defined by

$$\phi \times \psi = \pi_X^*(\phi) \wedge \pi_Y^*(\psi),$$
$$a \times b = \|\pi_X\|^*(a) \cup \|\pi_Y\|^*(b).$$

and where $\pi_X\colon X \times Y \to LX$ and $\pi_Y\colon X \times Y \to RY$ are the projections (note that these are in fact bisimplicial maps). We already know, that the left square commutes. The right square commutes by naturality and multiplicativity of the simplicial deRham isomorphisms.

If we assume that $H_*(\|X\|)$ is of finite type, so that the horizontal maps in (*) are isomorphisms, we can easily show the commutativity of (**). Indeed we have the relation

$$\langle S \times T, \phi \times \psi\rangle = \langle S, \phi\rangle \cdot \langle T, \psi\rangle,$$

and since the Kronecker pairing is preserved by the simplicial deRham isomorphisms, the commutativity of (*) implies, that also (**) is commutative.

In the general case, we apply Lemma 4.12 below, and use the fact that homology commutes with direct limits. □

LEMMA 4.12. *We have isomorphisms of chain complexes*:

(1) $\Omega_*\|X\| = \varinjlim_U \Omega_*\|U\|$ *and* $\Omega_*\|X \times Y\| = \varinjlim_U \Omega_*\|U \times Y\|$.

(2) $C_*(\|X\|) = \varinjlim_U C_*(\|U\|)$ *and* $C_*(\|X \times Y\|) = \varinjlim_U C_*(\|U \times Y\|)$.

Here the direct limit is taken over the open, Δ-submanifolds of finite type, ordered by inclusion. (C_* denotes the singular chain complex functor.)

Proof. Since the other statements are completely analogous, we shall restrict ourselves to prove that $\Omega_*\|X\| = \varinjlim_U \Omega_*\|U\|$. This will follow, if we can show, that every $T \in \Omega_n\|X\|$ is in the image of one of the injective maps $i_*\colon \Omega_n\|U\| \to \Omega_n\|X\|$ where $i\colon U \hookrightarrow X$ is the inclusion and U is a Δ-submanifold of the above type.

Consider $T \in \Omega_n\|X\|$, represented by $T_k \in \Omega_n(\Delta^k \times X_k)$; $k \geq 0$. We use the identification in Proposition 2.8. Define the compact sets $C_k = \pi_k(\mathrm{supp}(T_k))$, where $\pi_k\colon \Delta^k \times X_k \to X_k$ is the projection. We shall apply the following property of a manifold M:

> *For any compact $K \subseteq M$ there is an open neighborhood $V \subseteq M$ of K such that $\bar{V}$ is a compact manifold with boundary in X.*

This follows from a small variation of [2, Lemma 7.11]; indeed one can easily adapt the proof to construct a smooth, proper map $f\colon M \to \mathbf{R}_+$ such that $f|K \equiv 0$. We can then choose $V = f^{-1}([0, a[)$, where a is a regular value. We can now construct a suitable Δ-submanifold $U \subseteq X$ as follows: Since $C_k = \emptyset$ if k is greater than, say, N, we choose $U_k = \emptyset$ if $k > N$. Now choose a neighborhood $U_N \supseteq C_N$ of the above type. Then U_k is defined inductively as similar neighborhoods:

$$U_k \supseteq \varepsilon_0(\bar{U}_{k+1}) \cup \cdots \cup \varepsilon_{k+1}(\bar{U}_{k+1}) \cup C_k; \qquad k = N-1, \ldots, 0.$$

Clearly $U = \{U_k\}$ is a Δ-submanifold.

Since $\mathrm{supp}(T_k) \subseteq \Delta^k \times C_k \subseteq \Delta^k \times U_k$, there are well-defined restrictions

$$T_k|\Delta^k \times U_k \in \Omega_n(\Delta^k \times U_k),$$

with $\mathrm{supp}(T_k|\Delta^k \times U_k) = \mathrm{supp}\, T_k$. It follows readily, that $i_*(T_k|\Delta^k \times U_k) = T_k$. To see this choose a smooth function $\psi\colon X_k \to [0, 1]$ with $\mathrm{supp}(\psi) \subseteq U_k$, but $\psi|C_k \equiv 1$, and use $\langle T_k|\Delta^k \times U_k, \phi\rangle = \langle T_k, (\psi \circ \pi_k) \cdot \phi\rangle$ for all $\phi \in \Omega^n(\Delta^k \times U_k)$.

It remains to show that $H_*(\|U\|)$ is of finite type. By the construction, $H_*(U_k)$ is of finite type. Consider the double complex $A_{**}(U)$; for one of the spectral sequences we have $E^1_{p,q} = H_q(U_p)$, and since this is finite dimensional, $E^r_{p,q}$ is finite dimensional for all $r \geq 1$. Since the spectral sequence converges to $H_*(\|U\|)$, we are done. □

Remark 4.13. There is no obvious way to construct U in this proof in the "normal" case, i.e., such that U is stable with respect to the degeneracy maps, and it may not be true that $\Omega_*|X| = \varinjlim_U \Omega_*|U|$. Nevertheless the analogue of Proposition 4.11 is valid, because the cross-products are compatible with the projections $\Omega_*\|\cdot\| \to \Omega_*|\cdot|$, respectively $\|\cdot\| \to |\cdot|$.

Coproduct. We shall see that the dual to the wedge-product

$$\wedge' \colon \Omega_*\|X\| \to \Omega_*\|X\|\hat{\otimes}\Omega_*\|X\|$$

induces the coproduct in homology in the following sense:

THEOREM 4.14. *There is a commutative diagram*

$$\begin{tikzcd}
H(\Omega_*\|X\|) \arrow[r, "\wedge'"] \arrow[d, equal] & H(\Omega_*\|X\|\hat{\otimes}\Omega_*\|X\|) \\
H(\Omega_*\|X\|) \arrow[r, "coproduct"] & H(\Omega_*\|X\|) \otimes H(\Omega_*\|X\|). \arrow[u, "\cong"']
\end{tikzcd}$$

Proof. Consider the diagram

$$\begin{tikzcd}
H(\Omega_*\|X\|) \arrow[r, "\wedge'"] \arrow[d, equal] & H(\Omega_*\|X\|\hat{\otimes}\Omega_*\|X\|) \arrow[r, "\times", "\cong"'] & H(\Omega_*\|X \times X\|) \arrow[d, equal] \\
H(\Omega_*\|X\|) & H(\Omega_*\|X\|) \otimes H(\Omega_*\|X\|) \arrow[u, "\cong"'] \arrow[r, "\times", "\cong"'] & H(\Omega_*\|X \times X\|) \\
H_*(\|X\|) \arrow[u, "\cong"'] \arrow[r, "\text{coproduct}"] & H_*(\|X\|) \otimes H_*(\|X\|) \arrow[u, "\cong"'] \arrow[r, "\times", "\cong"'] & H_*(\|X \times X\|). \arrow[u, "\cong"']
\end{tikzcd}$$

The upper right square commutes by definition. The lower right square commutes by prop. 4.11. The "exterior" diagram commutes by (4.10) and the definition of the coproduct. Lemma 4.15 below implies that $\times \colon H(\Omega_*\|X\|\hat{\otimes}\Omega_*\|X\|) \to H(\Omega_*\|X \times X\|)$ is an isomorphism.

We conclude that the rectangle on the left commutes, and this ends the proof. □

LEMMA 4.15. *The maps*

$$\times \colon \Omega^*\|X\|\hat{\otimes}\Omega^*\|Y\| \to \Omega^*\|X \times Y\|,$$
$$\times \colon \Omega_*\|X\|\hat{\otimes}\Omega_*\|Y\| \to \Omega_*\|X \times Y\|$$

are quasi-isomorphisms.

Proof. This follows from a comparison with the complexes $A^*(\cdot)$. Indeed there is a commutative diagram

$$\begin{array}{ccc} \Omega^*\|X\|\hat{\otimes}\Omega^*\|Y\| & \xrightarrow{\times} & \Omega^*\|X\times Y\| \\ \Big\downarrow{\scriptstyle \mathcal{J}\hat{\otimes}\mathcal{J}} & & \Big\downarrow{\scriptstyle \mathcal{J}} \\ A^*(X)\hat{\otimes}A^*(Y) & \xrightarrow[\cong]{\times} & A^*(X\times Y). \end{array}$$

(It is clear how to define the cross-product between the spaces $A^*(\cdot)$, and also that this is a topological isomorphism.) Since $\mathcal{J}\hat{\otimes}\mathcal{J}$ and $\mathcal{J}$ are quasi-isomorphisms we are done.

The proof for currents is similar. □

Remark 4.16. It is very likely, that the maps in Lemma 4.15 are in fact (topological) isomorphisms, but we can only prove, that they are injective (which is trivial).

5. Another complex of simplicial currents

In [7], another complex of simplicial currents is defined. A straightforward generalization to arbitrary Δ-manifolds is the complex

$$\mathcal{A}_n(X) = \bigoplus_{k=0}^{\infty}\bigoplus_{l=0}^{\infty}\Omega^{k-l}(\Delta^k)\otimes\Omega_{n-l}(X_k), \tag{5.1}$$

with differential

$$\partial(\omega\otimes S) = (-1)^l d\omega\otimes S + \sum_{i=0}^{k}(-1)^i(\varepsilon^i)^*\omega\otimes(\varepsilon_i)_*S + (-1)^l\omega\otimes\partial S. \tag{5.2}$$

We are going to compare $\mathcal{A}_*(X)$ to $\Omega_*\|X\|$. There are maps

$$I\colon A_n(X)\to\mathcal{A}_n(X); \qquad I(S) = 1_{\Delta^k}\otimes S, \quad S\in\Omega_{n-k}(X_k)$$
$$\Phi\colon \mathcal{A}_n(X)\to\Omega_n\|X\|; \qquad \Phi(\omega\otimes S) = T_\omega\times S.$$

Clearly $\Phi\circ I = \mathcal{I}$, and furthermore I and Φ are chain maps, as follows from straightforward calculations (use 2.14 for Φ).

THEOREM 5.3. (1) *Φ is a quasi-isomorphism.*
(2) *Φ is injective.*
(3) ImΦ *is a dense subset of* $\Omega_n\|X\|$.
(4) *The pairing*

$$\mathcal{A}_n(X)\otimes\Omega^n\|X\|\to\mathbf{C}; \qquad \langle T,\phi\rangle = \langle\Phi(T),\phi\rangle$$

is non-degenerate.

Proof. (1) We prove the equivalent statement that I is a quasi-isomorphism. Introduce the following bicomplexes:

$$\begin{aligned}
\mathcal{A}_{p,q} &= \bigoplus_{k=0}^{\infty} \Omega^{-p}(\Delta^k) \otimes \Omega_q(X_k); \qquad p \le 0, \quad q \ge 0 \\
\partial'(\omega \otimes S) &= (-1)^{p+k} d\omega \otimes S \\
\partial''(\omega \otimes S) &= \sum_{i=0}^{k} (-1)^i (\varepsilon^i)^* \omega \otimes (\varepsilon_i)_* S + (-1)^{p+k} \omega \otimes \partial S
\end{aligned}$$

and

$$\begin{aligned}
A_{pq} &= \begin{cases} A_q(X) & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases} \\
\partial' &= 0, \qquad \partial'' = d.
\end{aligned}$$

Then I induces a map $I\colon A_{**} \to \mathcal{A}_{**}$ of bicomplexes. Let us recall that

$$H(\Omega^*(\Delta^k), d) \cong \mathbf{C}$$

in degree 0, generated by 1_{Δ^k}. Using this, it is straightforward to see that the induced map

$$I\colon\ H(A_{*q}, \partial') \to H(\mathcal{A}_{*q}, \partial')$$

identifies with the identity on $A_q(X)$, which by a spectral sequence argument proves our claim.

(2) Assume that $\Phi(\sum_{k=0}^{N} c_k) = 0$, where $c_k \in \bigoplus_{l=0}^{\infty} \Omega^{k-l}(\Delta^k) \otimes \Omega_{n-l}(X_k)$. Let $\tilde{c}_k \in \Omega_n(\Delta^k \times X_k)$ denote the image of c_k under the map $\omega \otimes S \mapsto T_\omega \times S$.

Assume inductively that $\tilde{c}_k = 0$ if $k > p$. We claim that also $\tilde{c}_p = 0$. It is sufficient to show that $\tilde{c}_p | \mathring{\Delta}^p \times X_p = 0$, since $\tilde{c}_p$ is smooth in Δ^p (for this use the isomorphism $\Omega_*(\Delta^p \times X_p) \cong \Omega_*(\Delta^p; \Omega_*(X_p))$, which follows from Lemma 2.12 together with [10, prop. 50.5]). Thus consider $\phi \in \Omega^n(\Delta^p \times X_p)$ with $\operatorname{supp}\phi \subseteq \mathring{\Delta}^p \times X_p$. We can extend ϕ to a simplicial form $\bar{\phi}$ by Lemma 5.5 below. Since $\sum_{k=0}^{N} \tilde{c}_k$ annihilates $\Omega^n \|X\|$ (cf. the proof of Prop. 3.7), we get

$$0 = \left\langle \sum_{k=0}^{N} \tilde{c}_k, \bar{\phi} \right\rangle = \sum_{k=0}^{N} \langle \tilde{c}_k, \bar{\phi}^{(k)} \rangle = \langle \tilde{c}_p, \phi \rangle.$$

We conclude that $\tilde{c}_p = 0$ and hence $c_k = 0$ for all k.

(3) In fact $\mathcal{A}_n(X)$ is dense in $\bigoplus_k \Omega_n(\Delta^k \times X_k)$. This follows easily, combining the isomorphism Lemma 2.12 with the fact that the image of $\Omega^{k-l}(\Delta^k)$ is dense in $\Omega_l(\Delta^k)$. This implies that $\mathcal{A}_n(X)$ is dense in $\tilde{\Omega}_n \|X\|$. But the surjective map $\tilde{\Omega}_n \|X\| \to \Omega_n \|X\|$ is continuous.

(4) Follows immediately from (2) and (3), since (3.5) is non-degenerate. $\square$

Remark 5.4. Observe that we have proved that $\mathcal{A}_n(X)$ is dense in $\tilde{\Omega}_n\|X\|$ as well as in $\Omega_n\|X\|$. In particular we can regard $\Omega_n\|X\|$ and $\tilde{\Omega}_n\|X\|$ as completions of $\mathcal{A}_n\|X\|$ in suitable topologies.

LEMMA 5.5. *Consider $\phi \in \Omega^n(\Delta^k \times X_k)$. If $\operatorname{supp}\phi \subseteq \mathring{\Delta}^k \times X_k$, then ϕ extends to a simplicial n-form $\bar{\phi} \in \Omega^n\|X\|$ such that $\bar{\phi}^{(k)} = \phi$ and $\bar{\phi}^{(p)} = 0$ if $p < k$.*

Proof. First some notation, following [6, Chapter 2]. For a sequence $I = (i_0, \dots, i_k)$ of integers satisfying $0 \le i_0 < \dots < i_k \le p$ we introduce

$$\mu_I = \varepsilon_{j_l} \dots \varepsilon_{j_1} \colon X_p \to X_k,$$

where $p \ge j_1 > \dots > j_l \ge 0$ is the complementary sequence to I. Also define (in barycentric coordinates)

$$\begin{aligned} \Delta_I^p &= \{(t_0, \dots, t_p) \in \Delta^p \mid \exists s\colon t_{i_s} > 0\} \\ &= \Delta^p - \{(t_0, \dots, t_p) \in \Delta^p \mid t_{i_0} = \dots = t_{i_k} = 0\}. \end{aligned}$$

That is, we subtract the $l - 1 = p - k - 1$-dimensional face opposite to the k-dimensional face represented by I. We then define a projection

$$\pi_I\colon \Delta_I^p \to \Delta^k; \qquad \pi_I(t_0, \dots, t_p) = \frac{1}{\sum_s t_{i_s}}(t_{i_0}, \dots, t_{i_k}).$$

Note that if $\phi \in \Omega^*(\Delta^k)$, the pullback $\pi_I^*(\phi)$ will usually not extend to a smooth form on Δ^p. To remedy this, we shall multiply $\pi_I^*(\phi)$ by a (suitably chosen) function $u_I \in C^\infty(\Delta^p)$, which is 0 near the face we subtract. To define u_I consider a smooth function $\psi\colon [0, 1] \to [0, 1]$ such that $\psi \equiv 0$ near 0, but $\psi(1) = 1$. We define

$$u_I(t_0, \dots, t_p) = \psi(\sum_s t_{i_s}).$$

Put $\tilde{u}_I = u_I \times 1_{X_p}$; we may then define

$$\bar{\phi}^{(p)} = \begin{cases} \sum_I (\pi_I \times \mu_I)^* \phi \cdot \tilde{u}_I & \text{if } p \ge k \\ 0 & \text{if } p < k. \end{cases}$$

It is easy to see that $\bar{\phi}$ extends ϕ; since $u_{01\dots k}(t_0, \dots, t_k) = \psi(\sum_s t_s) = \psi(1) = 1$ we get

$$\bar{\phi}^{(k)} = (\pi_{01\dots k} \times \mu_{01\dots k})^* \phi \cdot \tilde{u}_{01\dots k} = \mathrm{id}^*(\phi) \cdot 1 = \phi.$$

We shall verify that

$$(\varepsilon^i \times \mathrm{id})^* \bar{\phi}^{(p)} = (\mathrm{id} \times \varepsilon_i)^* \bar{\phi}^{(p-1)}.$$

This is trivial if $p \leq k$, so assume $p > k$. We then have to prove that

$$\sum_I (\varepsilon^i \times \mathrm{id})^*((\pi_I \times \mu_I)^*\phi \cdot \tilde{u}_I) = \sum_J (\mathrm{id} \times \varepsilon_i)^*((\pi_J \times \mu_J)^*\phi \cdot \tilde{u}_J),$$

which can be rewritten as

$$\sum_I ((\pi_I \circ \varepsilon^i) \times \mu_I)^*\phi \cdot (\varepsilon^i \times \mathrm{id})^*\tilde{u}_I = \sum_J (\pi_J \times (\mu_J \circ \varepsilon_i))^*\phi \cdot (\mathrm{id} \times \varepsilon_i)^*\tilde{u}_J.$$

To see this, we need some more notation: For each $i = 0 \dots p$, let $\varepsilon^i(I)$ denote the sequence

$$\varepsilon^i(I) = (i_0, \dots, i_s, i_{s+1} + 1, \dots, i_k + 1); \qquad \text{where } i_s < i \leq i_{s+1}.$$

It is straightforward to verify the following relations:

(1) $\mu_{\varepsilon^i(I)} = \mu_I \circ \varepsilon_i$,
(2) $\pi_{\varepsilon^i(I)} \circ \varepsilon^i = \pi_I$,
(3) $\pi_I \circ \varepsilon^i(\Delta^{p-1}) \subseteq \partial\Delta^k$ if $i \in I$,
(4) $(\varepsilon^i)^* u_{\varepsilon^i(I)} = u_I$.

Using these, we can finish the proof. The point is that $J \mapsto \varepsilon^i(J)$ gives a 1-1 correspondence between the sequences J and the sequences I, which does *not* contain i. Applying (1)–(4) then immediately gives the result (note in particular that (3) implies $((\pi_I \circ \varepsilon^i) \times \mu_I)^*\phi = 0$ if $i \in I$).

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