ILLINOIS JOURNAL OF MATHEMATICS
Volume 43, Number 2, Summer 1999

GENUS n BANACH SPACES

P. G. CASAZZA AND M. C. LAMMERS

ABSTRACT. We show that the classification problem for genus n Banach spaces can be reduced to the unconditionally primary case and that the critical case there is $n = 2$. It is further shown that a genus n Banach space is unconditionally primary if and only if it contains a complemented subspace of genus $(n-1)$. We begin the process of classifying the genus 2 spaces by showing they have a strong decomposition property.

1. Introduction

It is well known that a Banach space with a basis has uncountably many non-equivalent normalized bases [13]. However, there are spaces with normalized unconditional bases that are unique up to equivalence. G. Köthe and O. Toeplitz [9] showed that ℓ_2 has a unique unconditional basis and two papers by Lindenstrauss and Pelczynski [10] and Lindenstrauss and Zippin [11] showed that the complete list of spaces with a unique unconditional bases is ℓ_1, ℓ_2 and c_0.

One quickly notices that a unique normalized unconditional basis must be symmetric. This leads us to explore uniqueness up to a permutation. That is, two normalized unconditional bases are said to be equivalent up to a permutation if there exists a permutation of one which is equivalent to the other. Since the list of normalized unconditional bases that are actually unique is now complete we will use the phrase unique unconditional basis for unique up to a permutation. Edelstein and Wojtaszczyk [6] showed that direct sums of ℓ_1, ℓ_2 and c_0 have unique unconditional bases and in 1985 Bourgain, Casazza, Lindenstrauss, and Tzafriri [2] showed that 2-convexified Tsirelson T^2 and $(\sum_{n=1}^{\infty} \oplus \ell_1)_{c_0}$, $(\sum_{n=1}^{\infty} \oplus \ell_2)_{c_0}$, $(\sum_{n=1}^{\infty} \oplus \ell_2)_{\ell_1}$, $(\sum_{n=1}^{\infty} \oplus c_0)_{\ell_1}$, (along with their complemented subspaces with unconditional bases) all have unique unconditional bases, while somewhat surprisingly $(\sum_{n=1}^{\infty} \oplus \ell_1)_{\ell_2}$ and $(\sum_{n=1}^{\infty} \oplus c_0)_{\ell_2}$ do not. More recently Casazza and Kalton [4], [5] showed that other Tsirelson type spaces, certain Nakano spaces, and some c_0 sums of ℓ_{p_n} with $p_n \to 1$ have unique unconditional bases.

In [2] they define a new class of Banach spaces. A Banach space X is said to be of *genus n* if it and all its complemented subspaces with unconditional bases have a

Received January 16, 1998.
1991 Mathematics Subject Classification. Primary 46B15, 46B07.
The first-named author was supported by a grant from the National Science Foundation.

© 1999 by the Board of Trustees of the University of Illinois
Manufactured in the United States of America

unique normalized unconditional basis and there are exactly n different complemented subspaces with unconditional bases, up to isomorphism. For example, the space $\ell_1 \oplus \ell_2$ is a Banach space of genus 3, the three different complemented subspaces being $\ell_1 \oplus \ell_2$, ℓ_1 and ℓ_2. It was shown in the memoir [2] that 2-convexified Tsirelson space, T^2, is a Banach space of infinite genus. In fact T^2 has uncountably many non-isomorphic complemented subspaces with unconditional bases and every unconditional basis of a complemented subspace is unique. It is not known if there are any Banach spaces of countable genus, i.e., of genus ω. Our results apply to this case as well so we include it in this paper.

At this point even the genus 2 spaces are unclassified, although there is a conjecture that they are precisely the ones we already know(see Appendix 5). In Section 2 we show that the problem of classifying genus n spaces reduces to classifying the unconditionally primary genus n spaces. We then characterize the unconditionally primary genus n spaces as those which contain a complemented subspace of genus $n-1$. This basically shows that the backbone of this classification problem is really the genus 2 case.

In Section 3 we show that all Banach spaces of finite genus have the property that any subsequence of the original basis must contain a further subsequence equivalent to the unit vector basis of c_0, ℓ_1 or ℓ_2. This is particularly important to the genus 2 case for it classifies such spaces into these three cases.

In Section 4 we first show that if the only spaces of genus 2 containing c_0 are those conjectured by the memoir [2] $((\sum_{n=1}^{\infty} \oplus \ell_1^n)_{c_0}, (\sum_{n=1}^{\infty} \oplus \ell_2^n)_{c_0})$, then the only space of genus 2 containing ℓ_1 are the duals of these spaces. In other words it is enough to consider only the "c_0 case". The remaining part of this section deals with decomposing genus 2 spaces containing c_0. This decomposition relies heavily on a result of Wojtaszczyk so we give some details of this for clarity.

Finally we end with an appendix of a conjectured list of all genus n spaces for $1 \leq n \leq 6$. We divide the genus n spaces into those which are unconditionally primary and those which are not.

2. Reducing genus n spaces

We start with a simple observation. If (x_n) and (y_n) are sequences in Banach spaces X and Y respectively, we write $(x_n) \sim (y_n)$ to mean that $Tx_n = y_n$ defines an isomorphism from span$[x_n]$ to span$[y_n]$. Also we write $(x_n) \sim_\pi (y_n)$ if there is a permutation π of the natural numbers so that $(x_n) \sim (y_{\pi(n)})$.

PROPOSITION 2.1. *If X has an unconditional basis and X^* has a unique normalized unconditional basis, then X has a unique normalized unconditional basis.*

Proof. Since X^* is separable, every unconditional basis for X is shrinking. So if (x_n, x_n^*) and (y_n, y_n^*) are normalized unconditional bases for X then (x_n^*) and (y_n^*) are bounded unconditional bases for X^*. Hence $(x_n^*) \sim_\pi (y_n^*)$ and so $(x_n) \sim_\pi (y_n)$. □

We had to assume that X has an unconditional basis above since ℓ_1 has preduals without unconditional bases. We immediately get:

COROLLARY 2.2. *If X has a unconditional basis and X^* is genus n, then X is of genus $\leq n$, for all $n \leq \omega$.*

Recall that a Banach space X is said to be *primary* if whenever $X \cong Y \oplus Z$, then either $X \cong Y$ or $X \cong Z$. We say that X is *unconditionally primary* if whenever $X \cong Y \oplus Z$ and Y, Z have unconditional bases, then either $X \cong Y$ or $X \cong Z$. Now we wish to give a characterization of primary genus n Banach spaces. For this we need a recent result of Kalton [8].

THEOREM 2.3. *If X is a Banach space with an unconditional basis and X has only countably many non-isomorphic complemented subspaces with unconditional bases then $X \cong X^2$.*

The impact of the theorem is clear, for it follows that genus n spaces are isomorphic to their squares. We are now ready for our characterization of unconditionally primary spaces of finite genus.

PROPOSITION 2.4. *Let X be a Banach space of genus n. Then X is unconditionally primary if and only if X contains a complemented subspace of genus $n - 1$.*

Proof. $\Rightarrow$ Let Z be a complemented subspace of X with a unique unconditional basis of maximal genus m and $m < n$. If $m < n - 1$ there exists a complemented subspace Y of X with an unique unconditional basis not equivalent to a complemented subspace of Z and not isomorphic to X. Using the theorem of Kalton, we have $Y \cong Y \oplus Y$ and Y is complemented in X. It follows that $X \cong X \oplus Y$. Similarly $X \cong X \oplus Y \cong X \oplus Y \oplus Z$. So $Y \oplus Z$ is complemented in X and has genus $k \leq m$ = genus(Y). But if $k = m$ then $Y \oplus Z \cong Y$ contradicting our assumption that Z does not imbed complementably into Y. Hence, $k > m$. But Z was the maximal complemented space with genus m and $m < n$ so $Y \oplus Z$ must be of genus n and hence isomorphic to X. So X is not primary contradicting our assumption. Therefore we must have $m = n - 1$.

$\Leftarrow$ By way of contradiction, suppose W is a complemented subspace of X and W is genus $n - 1$. Now let $X \cong Y \oplus Z$, where neither Y nor Z is isomorphic to X. Then Y and Z are genus m_1 and genus m_2 respectively with m_1, $m_2 < n$.

Since there are n-distinct unique unconditional bases for complemented subspaces of X, $n - 1$ of them must be in W and the remaining one is the basis for X. Hence if (y_n) is a basis for Y then because Y has genus m_1 with $m_1 < n$ it can not be a basis for X hence it must be equivalent to a subsequence of the basis for W. Similarly for Z.

This implies that Y and Z are both complemented subspaces of W with unique unconditional bases, hence $Y \oplus Z$ is a complemented subspace of $W \oplus W \cong W$ by the preceding theorem. It follows that X is a complemented subspace of W. This is clearly a contradiction since X is genus n and W is genus $n-1$. □

Now we do the reduction of classifying genus n Banach spaces to the unconditionally primary case.

THEOREM 2.5. *Every Banach space X of genus n can be decomposed into $X \cong X_1 \oplus X_2 \oplus \cdots \oplus X_m$, where each X_i is unconditionally primary.*

Proof. If X is unconditionally primary, we are done. Otherwise by definition $X \cong Y \oplus Z$ where neither Y nor Z is isomorphic to X and both Y and Z have unique unconditional bases. Also Y and Z are genus $< n$. Now iterate this process until it stops. □

Theorem 2.5 tells us that to classify all Banach spaces of genus n we only need to classify the unconditionally primary Banach spaces of genus n. From the results of [2], we know that the following spaces are unconditionally primary (See Section 5):

$$\left(\sum_n \oplus E_n\right)_{c_0}, \left(\sum_n \oplus F_n\right)_{\ell_1},$$

where, for $1 \le n < \infty$,

$$E_n = \ell_1^n \text{ or } \ell_1, \text{ or } \ell_2^n \text{ or } \ell_2,$$

and

$$F_n = \ell_\infty^n \text{ or } c_0, \text{ or } \ell_2^n \text{ or } \ell_2.$$

It is natural then to conjecture that such iterations are the only way to produce unconditionally primary spaces. So we end this section with the following conjecture.

CONJECTURE 2.6. *X is unconditionally primary and genus n if and only if there is an unconditionally primary space Y of genus $< n$ with unconditional basis (y_i) and one of the following holds:*

(1)

$$X \cong \left(\sum \oplus Y\right)_{c_0} \text{ or } X \cong \left(\sum \oplus Y\right)_{\ell_1}$$

(2)

$$X \cong \left(\sum \oplus Y_n\right)_{c_0} \text{ or } X \cong \left(\sum \oplus Y_n\right)_{\ell_1};$$

where $Y_n = \text{span}[y_1, y_2, \dots y_n]$.

This gives an indication of the role that Banach spaces of genus 2 may play in the bigger picture of classifying all Banach spaces of finite genus. Conjecture 2.6 would look much more tractible if a conjecture of [2] were true. That is, in [2], it is asked if X having a unique normalized unconditional basis implies that $c_0(X)$ also has a unique normalized unconditional basis? Recently, Casazza and Kalton [5] showed that this is false by showing that c_0 sums of the original Tsirelson space fails to have a unique normalized unconditional basis while in an earlier paper [4] they showed that Tsirelson's space and its dual do have unique normalized unconditional bases.

3. Genus n spaces contain c_0, ℓ_1 or ℓ_2

If we consider only spaces of finite genus we get the following result.

THEOREM 3.1. *If X is finite genus n then every normalized unconditional basis for a complemented subspace of X has a subsequence which is permutatively equivalent to the unit vector basis of c_0, ℓ_1 or ℓ_2.*

To prove the theorem we need three propositions from [2]. The first gives a condition on an unconditional basis which implies the unconditional basis has a permutation which is subsymmetric. Recall that an unconditional basis (x_n) is *subsymmetric* if it is equivalent to all its subsequences.

PROPOSITION 3.2 [2, Proposition 6.2]. *Let X be a Banach space with an unconditional basis (x_n). Suppose that every subsequence of (x_n) contains a further subsequence which is permutatively equivalent to (x_n). Then there exists a permutation π of the integers such that $(x_{\pi(n)})$ is a subsymmetric basis.*

The next two propositions generalize results on Banach spaces with symmetric bases to Banach spaces which have subsymmetric bases. Actually Proposition 3.4 generalizes a result on homogeneous bases (bases which are equivalent to all of their normalized block bases) but it is a well-known result of Zippin that bases with this property must be equivalent to the unit vector basis of c_0 or ℓ_p for some $1 \leq p < \infty$ and therefore are symmetric.

PROPOSITION 3.3 [2, Proposition 6.3]. *Let X be a Banach space with a subsymmetric basis (x_n). Let $(\sigma_j)_{j=1}^{\infty}$ be mutually disjoint subsets of the integers so that $\max(\sigma_j) < \min(\sigma_{j+1})$, for all j. If we let $U = \operatorname{span}[u_j = \sum_{i\in\sigma_j} x_i]$, then $X \oplus U$ is isomorphic to X.*

PROPOSITION 3.4 [2, Proposition 6.4]. *Let X be a Banach space with a normalized unconditional basis (x_n). Suppose that for every normalized block basis with constant coefficients (u_j), there exists a permutation π of the integers so that $(u_{\pi(j)})$ is equivalent to $(x_n)_{n=1}^{\infty}$. Then $(x_n)_{n=1}^{\infty}$ is equivalent to the unit vectors in c_0 or ℓ_p for $1 \leq p < \infty$.*

Combining Proposition 3.3 and Proposition 3.4 one can obtain the following immediate corollary [2].

COROLLARY 3.5. *If X has a subsymmetric basis and a unique unconditional basis up to permutation, then X is isomorphic to c_0, ℓ_1 or ℓ_2.*

Proof. If (u_j) is a constant coefficient block basis of the subsymmetric basis $(x_n)_{n=1}^{\infty}$ then by Proposition 3.3 span$[u_j] \oplus X$ is isomorphic to X. By the uniqueness of the unconditional basis for X the basis $((u_j), (x_n))$ is permutatively equivalent to $(x_n)_{n=1}^{\infty}$. Hence by Proposition 3.4, $(x_n)_{n=1}^{\infty}$ is equivalent to the unit vector basis of c_0 or ℓ_p. However for $p \neq 1,2$, ℓ_p does not have a unique unconditional basis up to a permutation. This implies that $(x_n)_{n=1}^{\infty}$ must be equivalent to the unit vector basis of c_0 ,ℓ_1 or ℓ_2. □

In particular, if X has a subsymmetric basis and is genus n, then X is isomorphic to c_0, ℓ_1 or ℓ_2. Now we are ready for the proof of the main theorem in this section.

Proof of Theorem 3.1. By Corollary 3.5 it is enough to show that every normalized unconditional basis $(x_n)_{n=1}^{\infty}$ for a complemented subspace of X has a subsymmetric subsequence. Since X is genus n there are only n different normalized unconditional bases for complemented subspaces of X. Let $(x_m^i)_{m=1}^{\infty}$, $1 \leq i \leq n$, be a representative of each of these n different bases.

CLAIM 3.6. *(x_n) has a subsequence $(x_n(1))$ with the property that every subsequence of $(x_n(1))$ has a further subsequence permutatively equivalent to $(x_n(1))$.*

Proof. Either (x_n) has the required property, and we are done, or (x_n) has a subsequence (x_n^1) which has no further subsequence equivalent to (x_n). Now, either (x_n^1) satisfies the claim or it has a subsequence (x_n^2) which contains no further subsequence equivalent to (x_n^1). Continuing, we find either a sequence satisfying the claim or we can find sequences $((x_k^i)_{k=1}^{\infty})_{i=0}^{n-1}$, where $x_k^0 = x_k$, satisfying:

(1) $(x_k^i)_{k=1}^{\infty}$ is a subsequence of $(x_k^{i-1})_{k=1}^{\infty}$ for all $1 \leq i \leq n-1$.

(2) $(x_k^i)_{k=1}^{\infty}$ has no subsequence equivalent to $(x_k^{i-1})_{k=1}^{\infty}$ for all $1 \leq i \leq n-1$.

Since X is genus n, it follows that $((x_k^i)_{k=1}^{\infty})_{i=0}^{n-1}$ must have exhausted the list, up to permutative equivalence, of all unconditional bases for a complemented subspace of X. But then by (2), every subsequence of $(x_k^{n-1})_{k=1}^{\infty}$ is permutatively equivalent to $(x_k^{n-1})_{k=1}^{\infty}$. □

By Claim 3.6 and Proposition 3.2, (x_n) has a subsequence with a permutation, call it (y_n), so that y_n is subsymmetric. By Corollary 3.5, span$[y_n]$ is isomorphic to c_0, ℓ_1 or ℓ_2.

The above argument works for Banach spaces of genus ω also. To do this we need another result of Kalton [8].

PROPOSITION 3.7. *If X has an unconditional basis and at most countably many subsequences of this basis span non-isomorphic Banach spaces, then X is isomorphic to its hyperplanes.*

COROLLARY 3.8. *If a Banach space X is of genus ω, then every normalized unconditional basis for X has a subsequence permutatively equivalent to the unit vector basis of c_0, ℓ_1 or ℓ_2.*

Proof. We will just note the changes required in the argument of Corollary 3.5. Actually it is only Claim 3.6 that needs to be altered since the rest of the proof works perfectly well in this case. Let (x_n) be the unique normalized unconditional basis for X and $(y_i^k)_{k,i=1}^\infty$ be a complete list of unconditional bases for complemented subspaces of X. Now either (x_n) has the required property or (x_n) has a subsequence (x_n^1) which has no further subsequence equivalent to (y_n^1). Continue as in the proof of Claim 3.6, only now the process does not stop so we construct infinitely many subsequences with the properties:

(1) $(x_k^i)_{k=1}^\infty$ is a subsequence of $(x_k^{i-1})_{k=1}^\infty$.
(2) $(x_k^i)_{k=1}^\infty$ has no subsequence equivalent to $(x_k^{i-1})_{k=1}^\infty$ or $(y_k^i)_{k=1}^\infty$.

Now choose the diagonal elements from these subsequences, $(x_k^k) = (z_k)$ to get a subsequence of (x_n). There must be an i so that $(y_k^i)_{k=1}^\infty \sim (z_k)_{k=1}^\infty$. But $(z_k)_{k=i+1}^\infty$ is a subsequence of $(x_k^i)_{k=1}^\infty$ which has no subsequence equivalent to $(y_k^i)_{k=1}^\infty$. Finally, applying Proposition 3.7 we get the contradiction that $(z_k)_{k=1}^\infty$ is equivalent to $(z_k)_{k=i+1}^\infty$ (since they span isomorphic spaces which by definition have unique unconditional bases) while one of these is permutatively equivalent to $(y_k^i)_{k=1}^\infty$ and the other is not. □

One should note that this does not say that every subsequence of the unconditional basis of a genus n space must contain the same ℓ_p unit vector basis. Clearly $\ell_1 \oplus c_0$ has subsquences of the unconditional basis equivalent to both that of c_0 and ℓ_1. However for a genus 2 space this is precisely the case and hence we can classify the genus 2 spaces into the categories of containing c_0, ℓ_1, or ℓ_2. We consider two of these cases in the next section.

4. Genus 2 spaces containing c_0

First we notice by duality that if we can classify all spaces of genus 2 containing c_0, then we also get the desired result for those spaces containing ℓ_1.

PROPOSITION 4.1. *The only genus 2 spaces containing complemented c_0 are*

$$\left(\sum_{n=1}^{\infty} \oplus \ell_2^n\right)_{c_0} \quad \textit{and} \quad \left(\sum_{n=1}^{\infty} \oplus \ell_1^n\right)_{c_0}$$

if and only if the only spaces of genus 2 *containing complemented* ℓ_1 *are*

$$\left(\sum_{n=1}^{\infty} \oplus \ell_2^n\right)_{\ell_1} \quad \textit{and} \quad \left(\sum_{n=1}^{\infty} \oplus \ell_\infty^n\right)_{\ell_1}$$

Proof. $\Leftarrow$ Follows from Corollary 2.2.

$\Rightarrow$ We use the results of James [7] that an unconditional basis for a space X is boundedly complete iff the space does not contain c_0, and the dual result relating shrinking bases and ℓ_1. Let $(x_n)_{n=1}^{\infty}$ be a normalized unconditional basis for a genus 2 space X containing ℓ_1 complemented and let (x_n^*) be the associated biorthogonal functions of $(x_n)_{n=1}^{\infty}$. Then c_0 cannot embed into X or X would be at least genus 3. Hence $(x_n)_{n=1}^{\infty}$ is boundedly complete, and so if Y =span$[x_n^*]$, then $Y^* \cong X$.

CLAIM 4.2. *Y is genus* 2.

Proof. Let (y_n) be a normalized unconditional basis for a complemented subspace of Y and let (y_n^*) be the associated biorthogonal functions. Because the space spanned by (y_n) cannot contain complemented ℓ_1, again because X did not contain complemented c_0, (y_n) is a shrinking basis for the span$[y_n]$. Since $Y^* \cong X$ and the basis is shrinking it follows that (y_n^*) is a normalized unconditional basis for a complemented subspace of X. Therefore, either $(y_n^*) \sim_\pi (x_n)$ for some permutation π or $(y_n^*) \sim (e_n)_{\ell_1}$. So $(y_n) \sim_\pi (x_n^*)$ or $(y_n) \sim (e_n)_{c_0}$. Hence Y is genus 2. □

Now since $(x_n)_{n=1}^{\infty}$ has subsequences equivalent to $(e_n)_{\ell_1}$, c_0 embeds into Y. Hence Y is a genus 2 space containing c_0. So Y is isomorphic to

$$\left(\sum_{n=1}^{\infty} \oplus \ell_2^n\right)_{c_0} \quad \text{or} \quad \left(\sum_{n=1}^{\infty} \oplus \ell_1^n\right)_{c_0}$$

and X is isomorphic to one of the duals of these two. That is,

$$\left(\sum_{n=1}^{\infty} \oplus \ell_2^n\right)_{\ell_1} \quad \text{or} \quad \left(\sum_{n=1}^{\infty} \oplus \ell_\infty^n\right)_{\ell_1}. \qquad \square$$

One of the major difficulties in working with the genus n spaces is that although every subsequence of the basis is equivalent to one of n specified unconditional bases, there is no uniform constant of equivalence. For example, in

$$\left(\sum \oplus \ell_1^n\right)_{c_0}$$

the natural basis of $\ell_1^n \oplus c_0$ becomes "badly" equivalent to the unit vector basis of c_0 as n increases. Our next goal is to produce a uniform constant in genus 2 spaces for subsequences of the unconditional basis which are equivalent to the whole basis. That is, there is a constant K so that any subsequence of the original basis of a genus 2 space X that spans a space isomorphic to the original space, is K-uniformly equivalent to the original basis. Then we will use this constant to show that X has a UFDD (unconditional finite dimensional decomposition) of a very strong form.

In order to do this we first need a theorem of Wojtowicz [15] which also appears in [14] and implicitly in [12]. We should mention that Wojtowicz's theorem can be applied to any Banach space with an unconditional basis not just those of genus 2. In fact the theorem was used in [14] for a result on quasi-Banach spaces. Although we will not reproduce the proof of this theorem we do need to present some of the terminology and results from bipartite graph theory in order to state the stronger version of Wojtowicz's theorem which we need and which he actually proved. The following can be found in Wojtaszczyk's paper [14] and in more detail in [3].

A bipartite graph G consists of two disjoint sets N and M, and any set $E(G)$ of ordered pairs from $N \cup M$ with the property that one element in the ordered pair is from N and one is from M. We denote $N \cup M$ by $V(G)$. We call the elements of $V(G)$ the vertices of the graph while $E(G)$ is called the edge set of G. A subset $A \subset V(G)$ is called one sided if $A \subset N$ or $A \subset M$. Let A be a one sided subset of $V(G)$ we say A is matchable if there exists a 1-1 map ψ: $A \to$ V(G) such that $(a, \psi(a)) \in$ E(G) for all $a \in A$ and we call ψ a matching of A.

We now give a version of the classical Schroeder-Bernstein theorem of set theory, which has been observed by Banach [1], in the language of bipartite graph theory.

THEOREM 4.3. *Let M, N and $E(G)$ form a bipartite graph G. If both M and N are matchable then there exists a matching of N, ψ such that $\psi(N) = M$.*

Now we are ready to state Wojtowicz's theorem and sketch how it is proved. This will elucidate the quantitative estimates needed. We change the statement slightly, for the original theorem was stated for quasi-Banach spaces and here we are only concerned with Banach spaces.

THEOREM 4.4. *If $(x_n)_{n\in N}$ and $(y_m)_{m\in M}$ are normalized 1-unconditional bases for Banach spaces X and Y, and each is equivalent to a permutation of a subsequence of the other, (that is, $(x_n)_{n\in N} \sim (y_{\sigma(n)})_{n\in N}$ for a 1-1 map $\sigma : N \to M$ and $(y_m)_{m\in M} \sim (x_{\gamma(m)})_{m\in M}$ for a 1-1 map $\gamma : M \to N$) then $(x_n)_{n\in N}$ and $(y_m)_{m\in M}$ are permutatively equivalent to each other.*

One should note that although σ and γ are 1-1 maps they need not be onto, while the conclusion of the theorem implies that there exists a 1-1 and onto map for the equivalence of (x_n) and (y_n).

Wojtaszczyk [14] uses bipartite graph theory and the classical Schroeder-Bernstein theorem to obtain his result. In particular he creates a bipartite graph G with $V(G) = N \cup M$ and $E(G) = \{(n, \sigma(n))\}_{n\in N} \cup \{(m, \gamma(m))\}_{m\in M}$. Since both M and N are matchable there exists a 1-1 map $\Psi : N \underset{\text{onto}}{\to} M$, and a partition of N, $N = N_1 \cup N_2$, and hence a partition of M, $M = \Psi(N_1) \cup \Psi(N_2)$ so that $(x_n)_{n\in N_1}$ is equivalent to $(y_m)_{m\in\Psi(N_1)}$ and $(x_n)_{n\in N_2}$ is equivalent to $(y_m)_{m\in\Psi(N_2)}$. In particular

$$\Psi(n) = \begin{cases} \sigma(n) & \text{if } n \in N_1 \\ \gamma^{-1}(n) & \text{if } n \in N_2 \end{cases}$$

and therefore $(x_n)_{n\in N}$ and $(y_m)_{m\in M}$ are permutatively equivalent to each other.

If we consider the constants of equivalence K_1 and K_2 such that $(x_n)_{n\in N} \sim_{K_1} (y_{\sigma(n)})_{n\in N}$ and $(y_m)_{m\in M} \sim_{K_2} (x_{\gamma(m)})_{m\in M}$ by the 1-unconditionality of the basis one can obtain

$$\begin{aligned}
\left\|\sum_{n\in N} a_n x_n\right\| &\le \left\|\sum_{n\in N_1} a_n x_n\right\| + \left\|\sum_{n\in N_2} a_n x_n\right\| \\
&\le K_1 \left\|\sum_{n\in N_1} a_n y_{\Psi(n)}\right\| + K_2 \left\|\sum_{n\in N_2} a_n y_{\Psi(n)}\right\| \\
&\le K_1 \left\|\sum_{n\in N} a_n y_{\Psi(n)}\right\| + K_2 \left\|\sum_{n\in N} a_n y_{\Psi(n)}\right\| \\
&\le (K_1 + K_2) \left\|\sum_{n\in N} a_n y_{\Psi(n)}\right\| .
\end{aligned}$$

The inequality in the other direction can be produced in a similar way. The arguments above yield the following.

THEOREM 4.5. *Let $(x_n)_{n\in N}$ and $(y_m)_{m\in M}$ be normalized 1-unconditional bases for Banach spaces X and Y. If K_1 and K_2 are constants such that*
(1) $(x_n)_{n\in N} \sim_{K_1} (y_{\sigma(n)})_{n\in N}$ and
(2) $(y_m)_{m\in M} \sim_{K_2} (x_{\gamma(m)})_{m\in M}$
then $(x_n)_{n\in N}$ and $(y_m)_{m\in M}$ are $(K_1 + K_2)$-permutatively equivalent to each other.

An immediate corollary to this theorem is the form we will need for the proof of the theorem below.

COROLLARY 4.6. *If $(x_n)_{n=1}^{\infty}$ is a normalized 1-unconditional basis and (x_{n_i}) is a subsequence of $(x_n)_{n=1}^{\infty}$ which has a further subsequence which is K-equivalent to a permutation of $(x_n)_{n=1}^{\infty}$, then (x_{n_i}) is $(K + 1)$-equivalent to a permutation of $(x_n)_{n=1}^{\infty}$.*

We are now ready to present a decomposition theorem for Banach spaces of genus 2. To do this we first produce a uniform constant for subbases of the original basis that span a space isomorphic to the original space. This relies heavily on the theorem above and the fact that any subsequence of an unconditional basis for a genus 2 Banach space X containing c_0 which is not equivalent to the unit vectors in c_0, must be a basis for a space isomorphic to X. This is clear since any subsequence of the basis spans a complemented subspace with an unconditional basis. We start by producing the necessary uniform equivalence constant.

THEOREM 4.7. *Let X be a Banach space of genus 2 containing c_0 and let $(x_n)_{n=1}^{\infty}$ be a normalized 1-unconditional basis for X. Then there exists a natural number K such that for any subsequence (x_{n_i}) of $(x_n)_{n=1}^{\infty}$ which spans a space isomorphic to X, (x_{n_i}) is K-permutatively equivalent to (x_n).*

Proof. Assume no such K exists and then we will proceed by induction on K to get a contradiction. By Corollary 4.6 there exists a subsequence (x_n^1) of $(x_n)_{n=1}^{\infty}$ such that

(i) span$[(x_n^1)]$ is isomorphic to X and
(ii) no further subsequence of (x_n^1) is 2-permutatively equivalent (i.e., $K = 2$) to $(x_n)_{n=1}^{\infty}$.

If such a sequence did not exist and all subsequences that spanned the space had a further subsequence that was 2-permutatively equivalent to $(x_n)_{n=1}^{\infty}$, then by the corollary all subsequences of this type would be 3-permutatively equivalent to $(x_n)_{n=1}^{\infty}$. Let k_0=2 and choose $k_1 > k_0$ such that

$$\left\| \sum_{n=1}^{k_1} x_n^1 \right\| \geq 2.$$

Now since (x_n^1) is permutatively equivalent to $(x_n)_{n=1}^{\infty}$ and we have assumed no K exists satisfying the theorem, there exists (x_n^2) a subsequence of (x_n^1) such that

(i) span$[(x_n^2)]$ is isomorphic to X and
(ii) no further subsequence of (x_n^2) is $(k_0+k_1)^2$-permutatively equivalent to $(x_n)_{n=1}^{\infty}$.

Without loss of generality we may assume that the support of $(x_n^2) > k_0+k_1$. Now choose k_2 so that $k_2 > k_1$ and

$$\left\| \sum_{n=1}^{k_2} x_n^2 \right\| \geq 2^2.$$

Proceeding in this manner we can generate subsequences (x_n^i) of $(x_n)_{n=1}^{\infty}$ such that for all $i \in \mathbb{N}$,

(1) (x_n^{i+1}) is a subsequence of (x_n^i) with support of $(x_n^{i+1}) > k_0+k_1+k_2+\cdots+k_i = K_i$,

(2) $\|\sum_{n=1}^{k_i} x_n^i\| \geq 2^i$,
(3) no subsequence of (x_n^i) is $(K_{i-1})^2$- permutatively equivalent to $(x_n)_{n=1}^{\infty}$ and
(4) span$[(x_n^i)]$ is isomorphic to X.

First consider the sequence $(z_n) = ((x_n^j)_{n=1}^{k_j})_{j=1}^{\infty}$. This is a normalized basis for a complemented subspace of X and because of (2) above it can not span a space isomorphic to c_0. Since X is genus 2, (z_n) must be a basis for the whole space. Also by genus 2, we know that $((x_n^j)_{n=1}^{k_j})_{j=1}^{\infty}$ is D-permutatively equivalent to $(x_n)_{n=1}^{\infty}$ for some D and some permutation π. Now consider the sequence $(y_n^i) = (x_n^i)_{n=1}^{K_{i-1}} \cup ((x_n^j)_{n=1}^{k_j})_{j=i+1}^{\infty}$ for each $i \in \mathbb{N}$. Again this is a basis for the whole space and is a subsequence of (x_n^i) by (1).

If we let (a_n) be any set of scalars we have

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} a_n x_{\pi(n)}\right\| &\leq \left\|\sum_{n=1}^{K_{i-1}} a_n x_{\pi(n)}\right\| + \left\|\sum_{n=K_{i-1}+1}^{\infty} a_n x_{\pi(n)}\right\| \\ &\leq K_{i-1}\left\|\sum_{n=1}^{K_{i-1}} a_n y_n^i\right\| + D\left\|\sum_{n=K_{i-1}}^{\infty} a_n z_n\right\| \\ &\leq K_{i-1}\left\|\sum_{n=1}^{\infty} a_n y_n^i\right\| + D\left\|\sum_{n=1}^{\infty} a_n y_n^i\right\| \\ &\leq (K_{i-1}+D)\left\|\sum_{n=1}^{\infty} a_n y_n^i\right\| \qquad (1)\end{aligned}$$

This follows from the fact that (z_n) and (y_n^i) are the same sequence for $n > K_{i-1}$. Similarly we get

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} a_n y_n^i\right\| &= \left\|\sum_{n=1}^{K_{i-1}} a_n x_n^i + \sum_{n=K_{i-1}+1}^{\infty} a_n z_n\right\| \\ &\leq \left\|\sum_{n=1}^{K_{i-1}} a_n x_n^i\right\| + \left\|\sum_{n=K_{i-1}+1}^{\infty} a_n z_n\right\| \\ &\leq K_{i-1}\left\|\sum_{n=1}^{\infty} a_n x_{\pi(n)}\right\| + D\left\|\sum_{n=1}^{\infty} a_n x_{\pi(n)}\right\| \\ &\leq (K_{i-1}+D)\left\|\sum_{n=1}^{\infty} a_n x_{\pi(n)}\right\| \qquad (2)\end{aligned}$$

By the two inequalities above we have that (y_n^i) is $(K_{i-1}+D)$-equivalent to a permutation of $(x_n)_{n=1}^{\infty}$. But the fact that (y_n^i) is a subsequence of (x_n^i) and from (3) above we have

$$(K_{i-1} + D) > (K_{i-1})^2.$$

This is a contradiction since K_i goes to ∞ and D is fixed. □

Below we will refer to this constant as K_0 for a given Banach space of genus 2 with unconditional basis $(x_n)_{n=1}^{\infty}$. Now we show that spaces of genus 2 which contain c_0 have a strong decomposition property.

A sequence $(E_n)_{n=1}^{\infty}$ of finite dimensional subspaces of a Banach space X is called an *unconditional finite dimensional decomposition* for X, denoted UFDD, if for each $x \in X$ there is a unique choice of $x_n \in E_n$ so that

$$x = \sum_{n=1}^{\infty} x_n$$

and the series is unconditionally convergent in X. In this case we write

$$X \cong \sum_{n=1}^{\infty} \oplus E_n.$$

If the E_n's are not necessarily finite dimensional we call this an *unconditional Schauder decomposition*. If there is a $K \geq 1$ so that E_n is K-isomorphic to X, for all $n = 1, 2, 3, \ldots$, we say that X has an *unconditional decomposition into copies of itself.*

THEOREM 4.8. *If X is a Banach space of genus 2 containing c_0, then there exists $X \cong \sum \oplus E_n$, an unconditional finite dimensional decomposition satisfying*:

(1) *Each E_n is dimension n and has a unconditional basis $(x_i^n)_{i=1}^n$.*
(2) *$((x_i^n)_{i=1}^n)_{n=1}^{\infty}$ is an unconditional basis for X.*
(3) *There is a constant $K \geq 1$ so that for all $n < m$,*

$$(x_i^n)_{i=1}^n \sim_K (x_i^m)_{i=1}^n.$$

(4) *For all $n_1 < n_2 \ldots$,*

$$X \cong_K \sum \oplus E_{n_i}.$$

Proof. Let $(x_n)_{n=1}^{\infty}$ be a normalized unconditional basis for X. By Theorem 4.7, $(x_n)_{n=1}^{\infty}$ is K_0-permutatively equivalent to $(x_n)_{n=m}^{\infty}$, for $m = 1, 2, \ldots$. Then letting $m = 2$ above, we can choose $n_1, n_2 > 1$ so that $(x_i)_{i=1}^2$ is K_0-equivalent to a permutation of $\{x_{n_1}, x_{n_2}\}$. Continuing we can find $n_3, n_4, n_5 > \max\{n_1, n_2\}$ so that $(x_i)_{i=1}^3$ is K_0-equivalent to a permutation of $\{x_{n_3}, x_{n_4}, x_{n_5}\}$. After taking permutations we have found $((x_i^n)_{i=1}^n)_{n=1}^{\infty}$, a permutation of a subsequence of $(x_n)_{n=1}^{\infty}$ so that (1) holds with $E_n = \text{span}[(x_i^n)_{i=1}^n]$. Now (2) of the theorem is immediate and (3) follows from the choice of $(x_i^n)_{i=1}^n$. Finally, since X is not isomorphic to c_0,

$$\sup_n \left\| \sum_{i=1}^{n} x_i \right\| = +\infty.$$

Therefore, for every $n_1 < n_2 \ldots$,

$$\sup_{n_i} \left\| \sum_{j=1}^{n_i} x_j \right\| = +\infty$$

and so $Y = \sum \oplus E_{n_i}$ is not isomorphic to c_0. Hence $Y \cong X$. But Y is the span of a subsequence of the basis for X. So again by Theorem 4.7, $Y \cong_{K_0} X$. □

We immediately have:

COROLLARY 4.9. *If X is a Banach space of genus 2 containing c_0 then X has an unconditional Schauder decompositon into copies of itself, i.e., $X \cong \sum \oplus X$.*

The following result of Kalton [8] is the corresponding result for genus ω Banach spaces.

PROPOSITION 4.10. *If X has an unconditional basis which has only countably many non-isomorphic subsequences, then X has an unconditional Schauder decomposition into copies of itself.*

The next step to classifying genus 2 spaces would be to use the strong decomposition result in Theorem 4.8 to show that genus 2 spaces containing c_0 must be of the form $X = (\sum \oplus E_n)_{c_0}$. After this there are enough tools available to complete the classification.

5. Appendix

The following is a conjectured list of all genus n spaces for $1 \le n \le 6$.
Genus 1: It is known [2] that the only genus 1 spaces are

$$c_0,\ \ell_1,\ \ell_2.$$

Genus 2: The spaces

$$\left(\sum_{n=1}^{\infty} \oplus \ell_2^n\right)_{c_0},\ \left(\sum_{n=1}^{\infty} \oplus \ell_1^n\right)_{c_0},\ \left(\sum_{n=1}^{\infty} \oplus \ell_2^n\right)_{\ell_1},\ \left(\sum_{n=1}^{\infty} \oplus \ell_\infty^n\right)_{\ell_1}$$

are known [2] to be genus 2 and have been conjectured [2] to be the only spaces of genus 2. They are all unconditionally primary.

Genus 3: The spaces

$$c_0 \oplus \ell_1,\ c_0 \oplus \ell_2,\ \ell_1 \oplus \ell_2.$$

are known [6] to be genus 3.

We conjecture that the only unconditionally primary spaces of genus 3 are

$$\left(\sum_{k=1}^{\infty} \oplus \left(\sum_{n=1}^{\infty} \oplus \ell_\infty^n\right)_{\ell_1}^k\right)_{c_0},\ \left(\sum_{k=1}^{\infty} \oplus \left(\sum_{n=1}^{\infty} \oplus \ell_1^n\right)_{\ell_\infty}^k\right)_{\ell_1}.$$

Genus 4: We conjecture that the only genus 4 spaces that are not unconditionally primary are

$$\left(\sum_{n=1}^{\infty} \ell_2^n\right)_{\ell_1} \oplus \left(\sum_{n=1}^{\infty} \ell_\infty^n\right)_{\ell_1},\ \left(\sum_{n=1}^{\infty} \ell_2^n\right)_{\ell_1} \oplus \left(\sum_{n=1}^{\infty} \ell_\infty^n\right)_{\ell_1},$$

and the only genus 4 spaces that are unconditionally primary are

$$\left(\sum_{k=1}^{\infty} \oplus \left(\sum_{j=1}^{\infty} \oplus \left(\sum_{n=1}^{\infty} \oplus \ell_1^n\right)_{\ell_\infty}^j\right)_{\ell_1}^k\right)_{c_0},$$

$$\left(\sum_{k=1}^{\infty} \oplus \left(\sum_{j=1}^{\infty} \oplus \left(\sum_{n=1}^{\infty} \oplus \ell_\infty^n\right)_{\ell_1}^j\right)_{\ell_\infty}^k\right)_{\ell_1}.$$

Genus 5: We conjecture that the only spaces of genus 5 that are not unconditionally primary are

$$\ell_2 \oplus \left(\sum_{n=1}^{\infty} \oplus \ell_1^n\right)_{c_0},\ c_0 \oplus \left(\sum_{n=1}^{\infty} \oplus \ell_\infty^n\right)_{\ell_1}$$

$$\ell_1 \oplus \left(\sum_{n=1}^{\infty} \oplus \ell_1^n\right)_{c_0},\ \ell_2 \oplus \left(\sum_{n=1}^{\infty} \oplus \ell_\infty^n\right)_{\ell_1}$$

$$\ell_2 \oplus \left(\sum_{n=1}^{\infty} \oplus \ell_2^n\right)_{c_0},\ c_0 \oplus \left(\sum_{n=1}^{\infty} \oplus \ell_2^n\right)_{\ell_1}$$

$$\ell_1 \oplus \left(\sum_{n=1}^{\infty} \oplus \ell_2^n\right)_{c_0},\ \ell_2 \oplus \left(\sum_{n=1}^{\infty} \oplus \ell_2^n\right)_{\ell_1},$$

and the only spaces of genus 5 that are unconditionally primary are

$$\Bigg(\sum_{l=1}^{\infty}\oplus\sum_{k=1}^{\infty}\oplus\bigg(\sum_{j=1}^{\infty}\oplus\bigg(\sum_{n=1}^{\infty}\oplus\ell_1^n\bigg)_{\ell_\infty}^{j}\bigg)_{\ell_1}^{k}\Bigg)_{c_0}^{l}\Bigg)_{\ell_1}$$

$$\Bigg(\sum_{l=1}^{\infty}\oplus\sum_{k=1}^{\infty}\oplus\bigg(\sum_{j=1}^{\infty}\oplus\bigg(\sum_{n=1}^{\infty}\oplus\ell_\infty^n\bigg)_{\ell_1}^{j}\bigg)_{\ell_\infty}^{k}\Bigg)_{\ell_1}^{l}\Bigg)_{c_0}$$

$$\bigg(\sum_{k=1}^{\infty}\oplus\bigg(\sum_{n=1}^{\infty}\oplus\ell_2^n\bigg)_{\ell_1}^{k}\bigg)_{c_0},\quad\bigg(\sum_{k=1}^{\infty}\oplus\bigg(\sum_{n=1}^{\infty}\oplus\ell_2^n\bigg)_{\ell_\infty}^{k}\bigg)_{\ell_1}.$$

Genus 6: We conjecture the only spaces of genus 6 that are not unconditionally primary are

$$\bigg(\sum_{n=1}^{\infty}\oplus\ell_2^n\bigg)_{c_0}\oplus\Bigg(\sum_{k=1}^{\infty}\oplus\bigg(\sum_{j=1}^{\infty}\oplus\bigg(\sum_{n=1}^{\infty}\oplus\ell_1^n\bigg)_{\ell_\infty}^{j}\bigg)_{\ell_1}^{k}\Bigg)_{c_0}$$

$$\bigg(\sum_{n=1}^{\infty}\oplus\ell_2^n\bigg)_{\ell_1}\oplus\Bigg(\sum_{k=1}^{\infty}\oplus\bigg(\sum_{j=1}^{\infty}\oplus\bigg(\sum_{n=1}^{\infty}\oplus\ell_\infty^n\bigg)_{\ell_1}^{j}\bigg)_{\ell_\infty}^{k}\Bigg)_{\ell_1},$$

and that the only spaces of genus 6 that are unconditionally primary are

$$\Big(\sum\oplus\ell_2\Big)_{\ell_1},\ \Big(\sum\oplus c_0\Big)_{\ell_1},\ \Big(\sum\oplus\ell_2\Big)_{c_0},\ \Big(\sum\oplus\ell_1\Big)_{c_0}$$

$$\Bigg(\sum_{m=1}^{\infty}\oplus\Bigg(\sum_{l=1}^{\infty}\oplus\sum_{k=1}^{\infty}\oplus\bigg(\sum_{j=1}^{\infty}\oplus\bigg(\sum_{n=1}^{\infty}\oplus\ell_1^n\bigg)_{\ell_\infty}^{j}\bigg)_{\ell_1}^{k}\Bigg)_{c_0}^{l}\Bigg)_{\ell_1}^{m})_{c_0}$$

$$\Bigg(\sum_{m=1}^{\infty}\oplus\Bigg(\sum_{l=1}^{\infty}\oplus\sum_{k=1}^{\infty}\oplus\bigg(\sum_{j=1}^{\infty}\oplus\bigg(\sum_{n=1}^{\infty}\oplus\ell_\infty^n\bigg)_{\ell_1}^{j}\bigg)_{\ell_\infty}^{k}\Bigg)_{\ell_1}^{l}\Bigg)_{c_0}^{m}\Bigg)_{\ell_1}.$$

References

1. S. Banach, *Un théorém sur les transformations biunivoques*, Fundementa Math. **6** (1924), 236–239.
2. J. Bourgain, P. G. Casazza, J. Lindenstrauss and L. Tzafriri, *Banach spaces with a unique unconditional basis up to a permutation*, Mem. Amer. Math. Soc., No. 322, 1985.

3. R. A. Brauldi, "Transversal theory in graphs" in *Studies in Graph Theory Part I*, D. R. Fulkerstone, ed., MAA Studies in Math., vol 12, (1975), pp. 23–88.
4. P. G. Casazza and N. J. Kalton, *Uniqueness of unconditional bases in Banach spaces*, Israel J. Math, to appear.
5. ———, *Uniqueness of unconditional bases in c_0 products*, preprint.
6. I. S. Edelstein and P. Wojtaszczyk, *On projections and unconditional bases in direct sums of Banach spaces*, Studia Math. **56** (1976), 263–276.
7. R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. **52** (1950), 518–527.
8. N. J. Kalton, *A Remark on Banach spaces isomorphic to their squares*, preprint.
9. G. Köthe and O. Toeplitz, *Lineare Raume mit unendlich vielen Koordinaten und Ringen unendlicher Matrizen*, J. Reine Angew. Math. **171** (1934), 193–226.
10. J. Lindenstrauss and A. Pelczynski, *Absolutely summing operators in L_p space and their applications*, Studia Math **29** (1968), 315–349.
11. J. Lindenstrauss and M. Zippin, *Banach spaces with a unique unconditional basis*, J. Funct. Anal. **3** (1969), 115–125.
12. B. S. Mityagin, *Equivalence of bases in Hilbert scales* (in Russian), Studia Math. **37** (1970), 111–137.
13. A. Pelczynski and I. Singer, *On non-equivalent bases and conditional bases in Banach spaces.*, Studia Math. **25** (1964), 5–25.
14. P. Wojtaszczyk, *Uniqueness of unconditional bases in quasi-Banach spaces with applications to Hardy spaces,II*, Israel J. Math. **97** (1997), 253–280.
15. M. Wojtowicz, *On Cantor-Bernstein type theorems in Riesz spaces*, Indag. Math. **91** (1988), 93–100.

P. G. Casazza, Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65211
pete@casazza.math.missouri.edu

M. C. Lammers, Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65211
lammers@sol.cms.uncwil.edu