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THREE VIEWPOINTS ON THE INTEGRAL GEOMETRY OF FOLIATIONS

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ABSTRACT. We deal with three different problems of the multidimensional integral geometry of foliations. First, we establish asymptotic formulas for integrals of powers of curvature of foliations obtained by intersecting a foliation by affine planes. Then we prove an integral formula for surfaces of contact of an affine hyperplane with a foliation. Finally, we obtain a conformally invariant integral-geometric formula for a foliation in three-dimensional space.

1. Introduction

As the extrinsic geometry of hypersurfaces is easier to understand than the geometry of submanifolds of codimension greater than one, in the integral geometry of foliations in a Euclidean space a great deal of attention was paid to the case of hyperfoliations [3], [12], [14] (but compare with [15], [16]), where generalizations of many classical results were developed.

The present paper deals mainly with the integral-geometric invariants of foliations and submanifolds of codimension greater then 1. It consists of three principal parts.

The classical integral-geometric formulas concerning foliations deal with curvature integral of "weight" bounded by a dimension of a section. In the first part we obtain asymptotic formulas for divergent integrals of powers of curvature of foliations constructed by intersecting a given foliation with affine subspaces. The main result of that part is that the coefficient of the principal term of the asymtotics *does not depend* on an exponent, but only on the extrinsic geometry of foliation in a nice way (for the precise formulations see Theorems 1 and 2).

In the second part (see Theorem 3) we consider surfaces of contact of a foliation with affine hyperplanes. It turns out that the total absolute curvature of a foliation can be expressed in terms of the integrals over surfaces of contact of certain functions of angles between such a surface and a leaf.

The third part of the paper is devoted to the *conformal* integral geometry of a two-dimensional foliation in a three-dimensional conformal space of constant curvature. We obtain a formula which connects an integral of a local conformal invariant of a foliation and the numbers of proper points of its contact with spheres in a spirit

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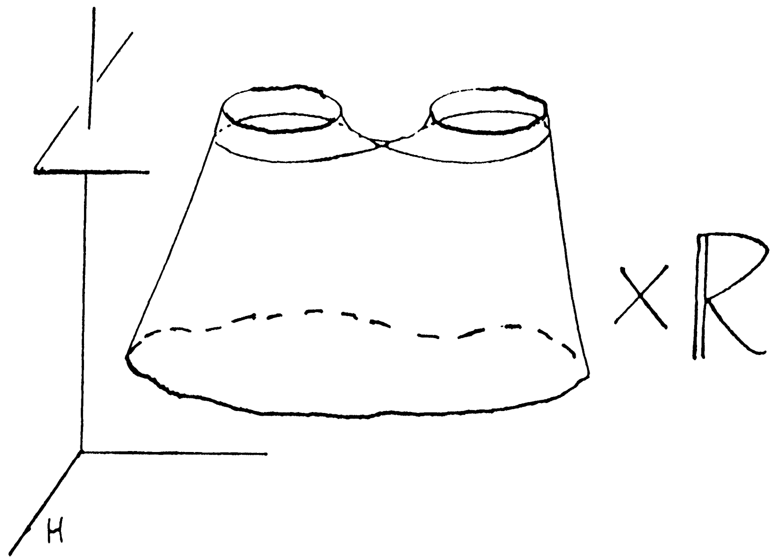

Fig. 1a. One leaf of $\mathcal{F}^2$. The foliation is invariant by vertical translations in $\mathbb{R}^3$.

of the exchange theorem [3], [11] for the metric case (Theorem 4). The integrand $|k_2 - k_1|^3\, dV$ can be viewed as a foliation analogue of the well-known Willmore integrand.

All the objects (foliations, manifolds, maps) are considered to be smooth enough (C^∞, for instance), all the integrals are computed with respect to the standard measure of a corresponding space, if it is not indicated explicitly. We often use Sard's lemma to omit "bad sets" of measure zero without references. Normally we do not compute constants C_i depending only on dimensions.

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Asymptotic formulas for integrals of powers of curvature. Let $\mathcal{F}^n \subset U \subset \mathbb{R}^{n+p}$ be a smooth foliation in a domain U in a Euclidean space $\mathbb{R}^{n+p}$ ($n \geq 2$, $p \geq 1$). For any $(p+1)$-dimensional affine subspace π in $\mathbb{R}^{n+p}$ consider the intersection $\pi \cap \mathcal{F}^n$. We obtain a foliation by curves in π (in general, with singularities).

Figure 1 (a, b, c, d) shows two generic types of singularities arising when one intersects a foliation $\mathcal{F}^2 \subset \mathbb{R}^4$ with 3-dimensional affine subspaces. The foliation is obtained by a vertical parallel translation of a leaf in $\mathbb{R}^3$ (Fig. 1a, 1c) and then by multiplying that by a line. The intersection with $\pi = H \times \mathbb{R}$ in a neighbourhood of a singular set is shown in Fig. 1b, 1d. It can be seen that these two cases are the

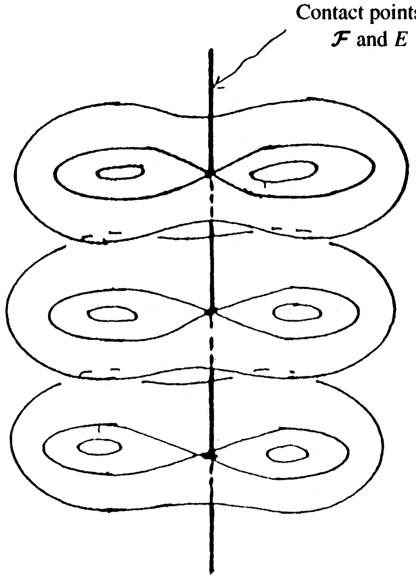

Fig. 1b. Intersection of $\mathcal{F}^2$ with $\pi = H \times \mathbb{R}$.

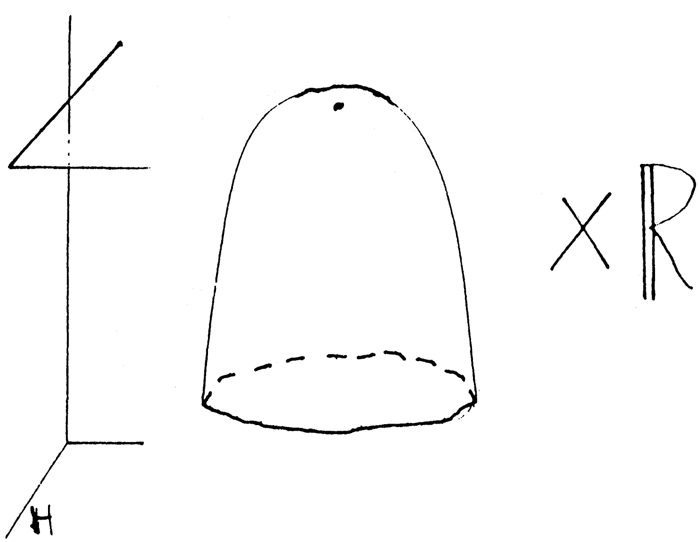

Fig. 1c. One leaf of $\mathcal{F}^2$. The foliation is invariant by vertical translations in $\mathbb{R}^3$.

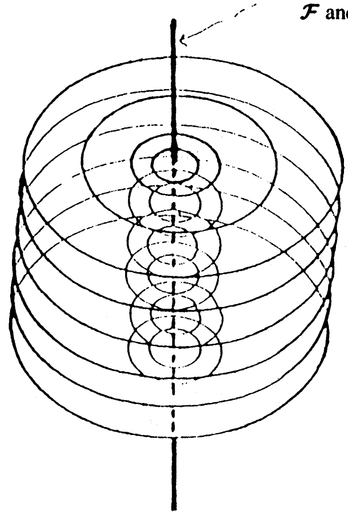

Fig. 1d. Intersection of $\mathcal{F}^2$ with $\pi = H \times \mathbb{R}$.

only generic cases for $\mathcal{F}^2 \subset \mathbb{R}^4$. Indeed, the set of tangent subspaces to leaves is a 4-dimensional submanifold in an affine Grassmannian $\mathcal{A}(2,4)$ of dimension 6, while the set of 2-dimensional affine subspaces in π is a 3-dimensional affine Grassmannian $\mathcal{A}(2,3)$. In general, these two sets intersect in a curve and the structure of the intersection looks like Fig. 1b or 1d depending on an index of a point of contact of $\mathcal{F}^2$ with π.

Let k be the curvature of curves of this induced foliation in smooth points. We are interested in studying the behavior of the integral

$$F_m = \int_{\mathcal{A}(p+1,n+p)} \left(\int_{\pi \cap U} k^m (\pi \cap \mathcal{F}) d\mathcal{L} \right) d\pi,$$

where $\mathcal{A}(p+1, n+p)$ is the manifold of all affine $(p+1)$-dimensional subspaces in $\mathbb{R}^{n+p}$ with its standard measure $d\pi$ [17], the inner integral is computed with respect to Lebesgue $(p+1)$-dimensional measure $d\mathcal{L}$ in $\pi \cap U$ and $m \geq 0$.

Similarly we will study the integral

$$S_m = \int_{\mathcal{A}(p+1,n+p)} \left(\int_{\pi \cap \mathcal{S}} k^m ds \right) d\pi$$

for a smooth n-submanifold $\mathcal{S}^n$ in a Euclidean space where the inner integral is computed with respect to the arc length ds of the curve of intersection.

Notice that in the case $p > 1$ the curvature is always nonnegative, but in the case $p = 1$ we shall consider both the integrals of $|k|^m$ and of k^m when m is an odd integer. In the last case we suppose the foliation to be transversely orientable and fix a normal vector field to $\mathcal{F}^n$ (to $\mathcal{S}^n$ in the case of one submanifold respectively). Then the sign of curvature can be defined in a standard way.

The geometric meaning of the integrals F_0 and S_0 both for a foliation $\mathcal{F}^n \subset U$ and for a submanifold $\mathcal{S}^n \subset \mathbb{R}^{n+p}$ (that is, the integral over $\mathcal{A}(p+1, n+p)$ of the volume or of the length of the curve of intersection respectively) is well known: it is just the volume of the domain U or of the submanifold $\mathcal{S}^n$ respectively up to dimensional constants [17].

The integral F_1 for the case of codimension 1 was studied in [3] and [12]. It was proved that

$$F_1 = \text{const} \int_U h_1 dV,$$

where h_1 is the integral of the absolute value of a normal curvature over the unit sphere in the tangent space and dV is the volume element. For submanifolds the similar equality

$$S_1 = \text{const} \int_{\mathcal{S}} h_1 dV$$

can be deduced from [12] (see also [18], [19]).

For a surface $\mathcal{S}^2 \subset \mathbb{R}^3$ and $m = 2$, it was proved in [6] using a kinematic approach that the integral S_2 is equal to $\int_{\mathcal{S}}(3H^2 - K)dS$ up to a constant, where H and K are the mean and the Gauss curvatures of $\mathcal{S}$ respectively. The similar formula $S_2 = \int_{\mathcal{S}}(3\|H\|^2 - \frac{2}{n^2}\text{Scal})dS$, where Scal is the scalar curvature of $\mathcal{S}$, holds for multidimensional case as can be seen below (formula (6) and Lemma 5). The kinematic formulas for powers of the mean curvature of hypersurfaces have been given in [24]. If m is not an integer and the integrals S_m and F_m converge, they are equal to const $\int \|h(X, X)\|^m$ where the integrals are computed over the unit tangent bundle of the foliation and of the submanifold respectively.

It is clear that in general for large m, the inner integral in F_m and the integral S_m may diverge. For these cases the integrals

$$F_m(M) = \int_{\mathcal{A}(p+1,n+p)} \left(\int_{\pi \cap \{k \le M\}} k^m(\pi \cap \mathcal{F}) d\mathcal{L} \right) d\pi$$

$$S_m(M) = \int_{\mathcal{A}(p+1,n+p)} \left(\int_{\pi \cap \{k \le M\} \cap \mathcal{S}} k^m ds \right) d\pi$$

are considered and the *asymptotic formulas* for $M \to \infty$ are obtained (the similar idea of truncating the integrand was used in [1]).

Introduce the following local invariants of a foliation $\mathcal{F}$ and of a submanifold $\mathcal{S}$:

$$\Phi_1(x) = \int_{\mathbb{P}T_x\mathcal{F}} K_n(X)\,|K_n(X)|dX \quad \text{for the case} \quad p = 1$$
$$\Phi_2(x) = \int_{\mathbb{P}T_x\mathcal{S}} \|h(X,X)\|^3 dX$$

where $\mathbb{P}T_x\mathcal{F}$ and $\mathbb{P}T_x\mathcal{S}$ are the projective spaces over the tangent spaces to the foliation and to the submanifold respectively at the point x, X takes values in these projective spaces, $K_n(X)$ is the normal curvature of the leaf passing through x with respect to the fixed field of normal vectors, and $\|h(X,X)\|$ is the length of the normal curvature vector in the direction X.

The main results of this part are the following theorems:

THEOREM 1 (FOLIATION CASE). *Let $\mathcal{F}^n \subset U \subset \mathbb{R}^{n+p}$ be a smooth foliation with a bounded second fundamental form in a bounded domain U in a Euclidean space $\mathbb{R}^{n+p}$. Then:*

(1) *For $m < 2$ the integral $F_m(M)$ converges to F_m (where $F_0 =$ const Vol(U) and $F_1 = \text{const} \int_U h_1 dV$) when $M \to \infty$.*

(2) *For $m \geq 2$ the integral $F_m(M)$ diverges and when $M \to \infty$ we have the asymptotic formulas*

$$F_2(M) = \log M \left(C_1 \int_U \left(3\|H\|^2 - \frac{2}{n^2} Scal\right) dV + \bar{o}(1)\right)$$
$$F_m(M) = M^{m-2} \left(C_2 \int_U \left(3\|H\|^2 - \frac{2}{n^2} Scal\right) dV + \bar{o}(1)\right) \quad \textit{for} \quad m > 2,$$

where $\|H\|$ is the length of the mean curvature vector, Scal is the scalar curvature of the foliation and dV is the volume element on U. In the case that $p = 1$ and m is an odd integer the last formula holds for the absolute value of curvature. Taking into account the sign one has

$$\int_{\mathcal{A}(p+1,n+p)} \left(\left(\int_{\pi\cap\{|k|\leq M\}} k^m(\pi \cap \mathcal{F})\right) d\mathcal{L}\right) d\pi = M^{m-2}\,(C_3 \int_U \Phi_1 dV + \bar{o}(1)).$$

THEOREM 2 (SUBMANIFOLD CASE). *Let $\mathcal{S}^n \subset \mathbb{R}^{n+p}$ be a regular bounded submanifold with a bounded second fundamental form in a Euclidean space $\mathbb{R}^{n+p}$ and dV its volume element. Then:*

(1) *For $m < 3$ the integral $S_m(M)$ converges to S_m (where $S_0 =$ const Vol$(\mathcal{S})$ and $\mathcal{S}_1 = \text{const} \int_{\mathcal{S}} h_1 dV$) when $M \to \infty$.*

(2) *For $m \geq 3$ the integral $S_m(M)$ diverges and when $M \to \infty$ we have the asymptotic formulas*

$$S_3(M) = \log M \, (C_4 \int_S \Phi_2 dV + \bar{o}(1))$$
$$S_m(M) = M^{m-3} \, (C_5 \int_S \Phi_2 dV + \bar{o}(1)) \quad \textit{for} \quad m > 3,$$

In the case that $p = 1$ and m is an odd integer, for the curvature with sign we have

$$\int_{\mathcal{A}(p+1,n+p)} \left(\int_{\pi \cap \{|k| \leq M\} \cap \mathcal{S}} k^3 ds \right) d\pi$$
$$= \log M \left(C_6 \int_S (5\sigma_1^3 - 12\sigma_1\sigma_2 + 8\sigma_3) dV + \bar{o}(1) \right),$$

$$\int_{\mathcal{A}(p+1,n+p)} \left(\int_{\pi \cap \{|k| \leq M\} \cap \mathcal{S}} k^m ds \right) d\pi$$
$$= M^{m-3} \left(C_7 \int_S (5\sigma_1^3 - 12\sigma_1\sigma_2 + 8\sigma_3) dV + \bar{o}(1) \right) \qquad m > 3,$$

where the σ_i's are the corresponding symmetric functions of principal curvatures of $\mathcal{S}^n$.

In the both theorems, the C_i's are constants depending only on dimensions and powers.

Surfaces of contact. Let $\mathcal{F}^n \subset U \subset \mathbb{R}^{n+p}$ be a smooth foliation in a domain U in a Euclidean space $\mathbb{R}^{n+p}$ $(n, p \geq 1)$. For any $(n+p-1)$-dimensional affine hyperplane π in $\mathbb{R}^{n+p}$ consider the set of its contact with $\mathcal{F}^n$. We obtain a submanifold $\mathcal{S}_\pi$ of dimension $p-1$ in π (in general, with singularities).

At each smooth point of $\mathcal{S}_\pi$ one has two n-dimensional subspaces lying in π, namely its normal subspace and the tangent space to the foliation. Let $\{\phi_i\}_{i=1}^n$, $0 \leq \phi_i \leq \pi/2$ be n principal angles between these subspaces [2], [23]. Recall that the total absolute curvature of a submanifold (or of a foliation) at a point is an integral over the unit sphere in the normal space of the absolute value of the determinant of its shape operator with respect to a normal normalized by the volume of that sphere.

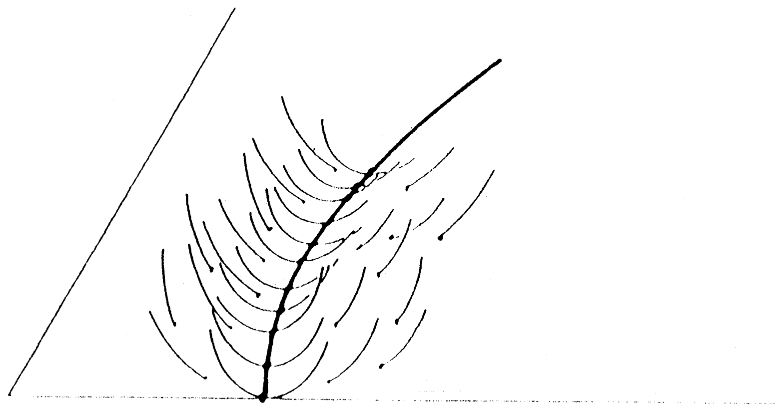

Figure 2. Set of contact of a foliation on curves in $\mathbb{R}^3$ with a plane.

THEOREM 3. *Let $\mathcal{F}^n \subset U \subset \mathbb{R}^{n+p}$ be a smooth foliation in a domain U in a Euclidean space $\mathbb{R}^{n+p}$. Then*

$$\int_U \|K\|(x)dV = C_8 \int_{\mathcal{A}(n+p-1,n+p)} \left(\int_{\mathcal{S}_\pi} \prod_{i=1}^{n} \cos\phi_i \right) d\pi,$$

where $\|K\|(x)$ is the total absolute curvature of the foliation $\mathcal{F}^n$.

In particular, in the case $n = 1$ and $p = 2$, the integral of the curvature of curves of foliation over the domain in $\mathbb{R}^3$ equals the integral over $\mathcal{A}(2, 3)$ of integrals over curves of contact of the sine of the angle between that curve and the leaf.

On the other hand, for the case of a hyperfoliation ($p = 1$), the sets $\mathcal{S}_\pi$ are in general discrete and all the angles ϕ_i vanish. In this case the theorem asserts that the total absolute curvature of a hypersurface equals the integral over $\mathcal{A}(n, n + 1)$ of the numbers of points of contact [12].

Integral of conformal invariant of foliation. Let $\mathcal{S}^2$ be a smooth surface in a three-dimensional *conformal* space, that is in S^3 (or in $\bar{\mathbb{R}}^3$) equipped with a group of conformal transformations keeping the curvature constant. It is well known that the *Willmore integrand* $(k_1 - k_2)^2 dS$ is invariant under this group [5], [21], [22] where k_1, k_2 are principal curvatures of the surface.

Moreover, identify S^3 with a positive light half-cone in a Lorentzian space $\mathbb{R}^{4+1}$, and consider a unit Lorentzian sphere $Q^4 \subset \mathbb{R}^{4+1}$, that is the set of vectors of unit Lorentzian length. Then the group of conformal transformations of S^3 is the group $SO(4, 1)$ of isometries of $\mathbb{R}^{4+1}$ and Q^4 can be canonically identified with a set of 2-spheres in S^3. In [4] a *conformal Gauss map* $\gamma : \mathcal{S}^2 \to Q^4$ was constructed, assigning

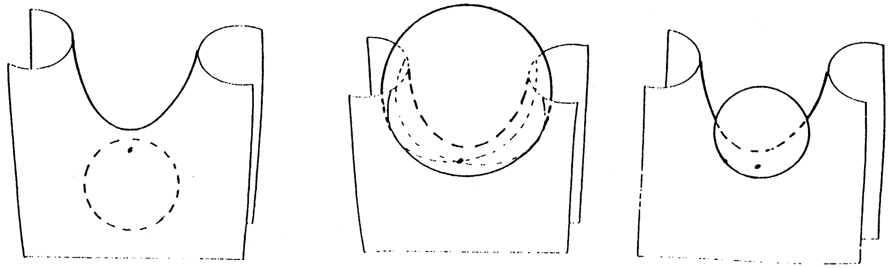

Figure 3. The sphere Σ has a saddle tangency with a leaf of the foliation $\mathcal{F}^2$ if its curvature k is in between the principal curvatures k_1 and k_2 of the leaf at the point of contact.

to every point $x \in \mathcal{S}$ the tangent sphere with the same mean curvature vector. It can be easily seen, that for two surfaces which differ by a conformal transformation the conformal Gauss maps differ by a motion in Q^4.

The unit sphere Q^4 induces a pseudo-Riemannian structure from $\mathbb{R}^{4+1}$ and an invariant volume element μ. Moreover, the conformal Gauss map γ is an immersion outside umbilical points, the induced metric on $\gamma(\mathcal{S})$ is Riemannian and

$$\text{Area}(\gamma(\mathcal{S})) = \frac{1}{4}\int_{\mathcal{S}} (k_1 - k_2)^2 dS,$$

as shown in [4].

We obtain the analogue of this formula for foliations in a three-dimensional conformal space. Though the theorem is formulated for foliations in $\mathbb{R}^3$, it is clear that it remains true for any 3-space of constant curvature by the conformal invariance arguments.

Let $\mathcal{F}^2 \subset U \subset \mathbb{R}^3$ be a smooth foliation in a domain U in a Euclidean space $\mathbb{R}^3$. For any 2-dimensional generalized sphere (that is, a sphere or a plane) Σ in $\mathbb{R}^3$ denote by $N^-(\Sigma)$ the number of its negative contacts with the foliation $\mathcal{F}^2$, i.e., the number of points of a saddle tangency of Σ and $\mathcal{F}^2$ (Fig. 3). It is clear that the number $N^-(\Sigma)$ is conformally well defined.

Identifying $\bar{\mathbb{R}}^3$ with S^3 via stereographic projection, one can identify the set $\mathcal{B}$ of generalized spheres in $\bar{\mathbb{R}}^3$ with Q^4 and equip it with the same conformally invariant measure μ. Then the following theorem holds

THEOREM 4. *Let $\mathcal{F}^2 \subset U \subset \mathbb{R}^3$ be a smooth foliation in a domain U in a*

Euclidean space $\mathbb{R}^3$. *Then*

$$\frac{1}{6}\int_U |k_1 - k_2|^3 dV = \int_{\mathcal{B}} N^-(\Sigma)\, d\mu(\Sigma).$$

Since the right hand side is conformally well defined, one obtains the following:

Corollary. *Let* $\mathcal{F}^2 \subset U \subset \mathbb{R}^3$ *be a smooth foliation in a Euclidean space* $\mathbb{R}^3$. *Then the* 3*-form*

$$|k_1 - k_2|^3 dV,$$

where k_i *are the principal curvatures of leaves and* dV *is the volume element, is a conformal invariant.*

2. Proofs of the theorems

In this section we prove Theorems 1 and 2 (modulo a number of technical lemmas moved to section 3) and also Theorems 3 and 4.

Proof of Theorems 1 *and* 2. In the proofs of both theorems we are going to use the *coarea formula* [8]. Recall that for a Riemannian manifold $\mathcal{M}^l$ and a manifold $\mathcal{N}^q$ $(l \geq q)$ with a measure element dy, a smooth map $P : \mathcal{M} \to \mathcal{N}$ and a measurable function $f\colon \mathcal{M} \to \mathbb{R}$, we have

$$\int_{\mathcal{M}} f(x)\, \|\mathrm{Jac}(P)\|\, dx = \int_{\mathcal{N}} \left(\int_{P^{-1}(y)} f(x)\, dx \right) dy, \tag{1}$$

where the inner integral in the right hand side is computed with respect to the $(l-q)$-dimensional Hausdorff measure induced on $P^{-1}(y)$ from $\mathcal{M}$ and $\|\mathrm{Jac}(P)\|$ is the Radon derivative of the measure on $\mathcal{N}$ with respect to the image under dP of the q-dimensional measure on $\mathcal{M}$. In the case of Riemannian manifolds the measure elements involved are Riemannian volume elements. To evaluate the Jacobian $\|\mathrm{Jac}(P)\|$ at a point $x \in \mathcal{M}$ one can choose orthonormal frames in $T_x\mathcal{M}$ and in $T_{P(x)}\mathcal{N}$ and consider the rectangular $q \times l$-matrix Q of the linear operator dP with respect to these frames. Then $\|\mathrm{Jac}(P)\| = \sqrt{\det Q\,{}^tQ}$, where ^{t} indicates transposition.

For our purposes in the foliation case, we use the following notation.

$\mathcal{M}$ is the Grassmann bundle $G(p+1, n+p, TU)$ of all $(p+1)$-dimensional linear subspaces in the tangent bundle over $U \subset \mathbb{R}^{n+p}$. It can be equipped with a standard Riemannian structure which is in fact just the structure of the Riemannian product $U \times G(p+1, n+p)$ with the homogeneous metric on the Grassmannian $G(p+1, n+p)$ [10].

$\mathcal{N}$ is $\mathcal{A}(p+1, n+p)$ with its standard measure.

P is a map which sends a point (x, π) to the plane through x parallel to π in $\mathbb{R}^{n+p}$.

f is the function such that $f_{|P^{-1}(\pi)} = k^m(\pi \cap \mathcal{F})\,\chi(|k| \leq M)$, where χ is the characteristic function.

In the case of a submanifold we just change U and $\mathcal{F}$ to $\mathcal{S}$ in the above paragraph.

Now the right hand side of the formula (1) is exactly the integral $F_m(M)$ (respectively $S_m(M)$) and our goal is to evaluate the left hand side. Moreover, all the measures involved are just the corresponding Riemannian volume (or length) elements.

First we compute the Jacobian of the map P.

LEMMA 1. 1. *In the case of a foliation,* $\|\mathrm{Jac}(P)\| \equiv 1$.

2. *In the case of a submanifold,* $\|\mathrm{Jac}(P)\|(x, \pi) = \|pr_{T_x\mathcal{S}}\pi^\perp\|$, *where* $\pi^\perp$ *is the subspace orthogonal to* π *at* x, *pr is the projection operator, and its norm is the* $(p+1)$*-dimensional volume of the projection of a unit cube in* $\pi^\perp$ *on* $T_x\mathcal{S}$.

Notice that for the foliation case the norm of the projection of $\pi^\perp$ on T_xU is 1 and so both the cases in the Lemma 1 are basically the same. Moreover, it is clear that $\|pr_{T_x\mathcal{S}}\pi^\perp\| = \|pr_{N_x\mathcal{S}}\pi\|$.

To evaluate the left hand side in the formula (1) express the function f in terms of local invariants. Lemma 2 below in fact states the multidimensional version of the *Meusnier formula* (compare [7], [9], [13]). It can be also derived from the *Meusnier Euler-type Theorem* in [7].

LEMMA 2 (MULTIDIMENSIONAL MEUSNIER FORMULA). *Let* $\mathcal{S}^n \subset \mathbb{R}^{n+p}$ *be a regular submanifold in a Euclidean space* $\mathbb{R}^{n+p}$ *and let* h *be its second fundamental form. Let* $\pi \subset \mathbb{R}^{n+p}$ *be a* $(p+1)$*-dimensional plane passing through a point* $x \in \mathcal{S}$ *and intersecting* $T_x\mathcal{S}$ *in a line. Let* X *be a unit vector on that line and let* $\{u_\alpha\}_{\alpha=1}^p$ *be orthonormal vectors in* π *normal to* X *and such that their projections* $pr_{N_x\mathcal{S}}u_\alpha$ *on the normal space are also mutually orthogonal. Denote unit vectors of* $pr_{N_x\mathcal{S}}u_\alpha$ *by* n_α *and let* $\cos\phi_\alpha = \langle u_\alpha, n_\alpha\rangle$, *where* $\langle\ ,\ \rangle$ *is the Euclidean inner product. Then for* $p > 1$,

$$k(\mathcal{S}^n \cap \pi) = \sqrt{\sum_{\alpha=1}^{p}\langle h(X, X), n_\alpha\rangle^2/\cos^2\phi_\alpha}, \tag{2}$$

and for $p = 1$,

$$k(\mathcal{S}^n \cap \pi) = \langle h(X, X), n\rangle/\cos\phi, \tag{3}$$

where $h(X, X)$ *is a normal curvature vector in the direction* X.

The above formula for the case $p > 1$ has the following geometric meaning (for details see the beginning of the proof of Lemma 4): let L be an n-dimensional plane

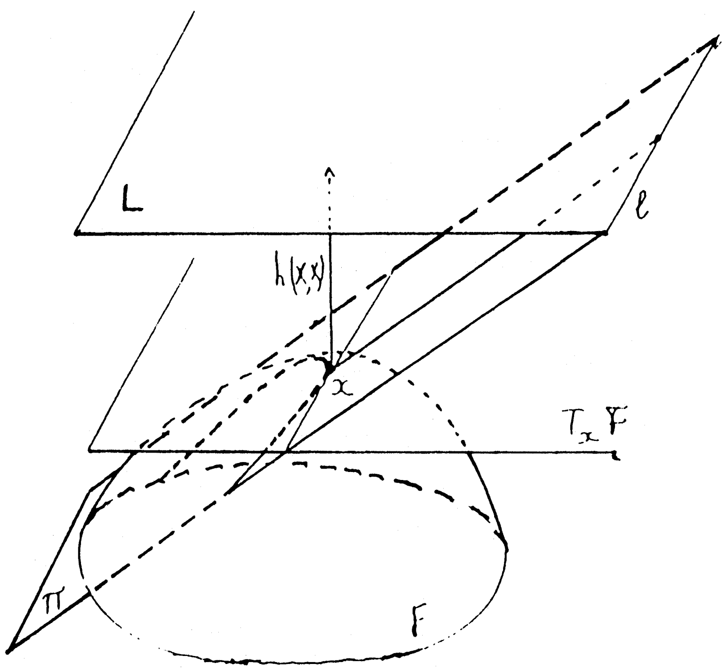

Figure 4. Multidimensional Meusnier formula.

parallel to $T_x\mathcal{S}$ and passing through the endpoint of the vector $h(X, X)$. It intersects π in a line parallel to X. The curvature $k(\mathcal{S}^n \cap \pi)$ is just the distance between this line and the point x (Fig. 4). Also note that with the above notation the Jacobian of the map P for the case of a submanifold equals $|\prod_{\alpha=1}^{p} \cos\phi_\alpha|$.

Taking into account that the domain of integration in the left hand side in (1) is the Riemannian product of U (resp. $\mathcal{S}$) and the Grassmann manifold $G(p+1, n+p)$, one has to compute the following integrals at every point x of U (resp. $\mathcal{S}$):

$$\int_{G(p+1,n+p)} k^m(\pi \cap \mathcal{S})\, \chi(|k| \leq M) d\pi \tag{4}$$

for the foliation case, substituting k from (2) or (3), where $\mathcal{S}$ is a leaf of $\mathcal{F}$ passing through x; and

$$\int_{G(p+1,n+p)} k^m(\pi \cap \mathcal{S})\, \chi(|k| \leq M) \| pr_{N_x\mathcal{S}}\pi \| d\pi \tag{5}$$

for the submanifold case, substituting k from (2) or (3).

To this end we first decompose each of these integrals into repeated integrals over $\mathbb{P}T_x\mathcal{S}$ of the integral over the Grassmann manifold $G(p, n+p-1)$ in the $(n+p-1)$-dimensional subspace $\mathbb{R}^{n+p-1} = X^{\perp}$ orthogonal to $X \in \mathbb{P}T_x\mathcal{S}$ using the following lemma (compare with Chern's linear kinematic formula [7]).

LEMMA 3. *Let $T^n \subset \mathbb{R}^{n+p}$ be a fixed n-dimensional subspace in a Euclidean space $\mathbb{R}^{n+p}$ and let $g : G(p+1, n+p) \to \mathbb{R}$ be a function. Then*

$$\int_{G(p+1,n+p)} g(\pi^{p+1})d\pi^{p+1} = \int_{\mathbb{P}T} dX \int_{\pi^p \in G(p,n+p-1)} g(\pi^p \wedge X)\, \|pr_{T^{\perp}}\pi^p\| d\pi^p,$$

where $\mathbb{P}T$ is the projective space over T, $G(p, n+p-1)$ is the Grassmann manifold of p-dimensional subspaces in $\mathbb{R}^{n+p-1} = X^{\perp}$ and $X \in \mathbb{P}T$.

Notice that in terms of Lemma 3, $\|pr_{T^{\perp}}\pi^p\| = \|pr_{T^{\perp}}\pi^{p+1}\|$ where $\pi^{p+1} = \pi^p \wedge X$.

If $g(\pi) = k^m(\pi \cap \mathcal{S})$, homogenity arguments, like those of Chern in the proof of the linear kinematic formula, will allow us to separate the curvature type contribution and the angle type contribution.

It is clear from (2) or (3) that in our case(s) the integral over $G(p, n+p-1)$ of the function $g(\pi^p \wedge X)$ in Lemma 3 *does not depend of the direction* of the normal vector $h(X, X)$, but only on its length and on M (although, as the referee pointed out, the function $g(\pi^p \wedge X)$ itself may depend on the direction of $h(X, X)$). To be more precise, the integral (4) equals

$$(6) \qquad \int_{\mathbb{P}T_x\mathcal{S}} \left(\int_{G(p,n+p-1)} \mu^m(\pi^p)\, \|h(X, X)\|^m\, \|pr_{N_x\mathcal{S}}\pi^p\| \right.$$
$$\left. \chi(\|h(X, X)\|\, |\mu(\pi^p)| \le M) d\pi^p \right) dX,$$

where the function $\mu(\pi^p)$ in the notation of Lemma 2 equals $\sqrt{\sum_{\alpha=1}^{p} \langle n, n_\alpha\rangle^2 / \cos^2\phi_\alpha}$ and n is an arbitrary fixed unit normal vector. In the case of the curvature with sign, one gets $\mu(\pi^1) = \pm 1/\cos\phi$. The same is true for the integral (5) (that is, in the case of the submanifold) with $\|pr_{N_x\mathcal{S}}\pi^p\|^2$ instead of just $\|pr_{N_x\mathcal{S}}\pi^p\|$ in (6).

Denoting the inner integral in (6) by $I(\|h(X, X)\|, M)$ one can easily see that for a positive λ, x and y, $\quad I(\lambda x, \lambda y) = \lambda^m I(x, y)$. So

$$I(\|h(X, X)\|, M) = \|h(X, X)\|^m I(1, M/\|h(X, X)\|)$$

and it is sufficient to compute the two integrals

$$A_m(M) = \int_{G(p,n+p-1)} \mu^m(\pi^p)\, \|pr_{N_x\mathcal{S}}\pi^p\|\, \chi(|\mu(\pi^p)| \le M) d\pi^p$$

and

$$B_m(M) = \int_{G(p,n+p-1)} \mu^m(\pi^p) \, \|pr_{N_x S} \pi^p\|^2 \, \chi(|\mu(\pi^p)| \le M) d\pi^p$$

for the case of the foliation and for the case of the submanifold respectively. This computation is contained in Lemma 4.

LEMMA 4.

$$A_m(M) = C_9 \int_1^M (x^2 - 1)^{\frac{n-3}{2}} x^{m-n} dx$$

and

$$B_m(M) = C_{10} \int_1^M (x^2 - 1)^{\frac{n-3}{2}} x^{m-n-1} dx$$

where C_9 and C_{10} are certain constants depending on n, p and m.

Naturally, the integrals in Lemma 4 can be computed explicitly (in the case of integer m). To prove the theorems we only need to notice that for $M \to \infty$, $A_m(M)$ converges when $m < 2$, diverges as const $\log M$ when $m = 2$ and as const M^{m-2} when $m > 2$. Similarly, $B_m(M)$ converges when $m < 3$, diverges as const $\log M$ when $m = 3$ and as const M^{m-3} when $m > 2$.

Substituting this in the above formulas we obtain the desired results modulo the following lemma.

LEMMA 5. 1. *Let x be a point of a regular submanifold S^n in a Euclidean space. Then*

$$\int_{\mathbb{P}T_x S} \|h(X, X)\|^2 dX = C_{11}(n) \left(3\|H\|^2 - \frac{2}{n^2} \mathrm{Scal} \right),$$

where h is the second fundamental form, H is the mean curvature vector, and Scal *is the scalar curvature of S at x.*

2. *Let x be a point of a regular hypersurface S^n in a Euclidean space. Then*

$$\int_{\mathbb{P}T_x S} h(X, X)^3 dX = C_{12}(n)(5\sigma_1^3 - 12\sigma_1\sigma_2 + 8\sigma_3),$$

where σ_i are the corresponding symmetric functions of principal curvatures of S at x.

Proof of Theorem 3. To prove the theorem, one applies the coarea formula (1) to the map from the projective normal bundle $\mathbb{P}N\mathcal{F}$ of the foliation to $\mathcal{A}(n+p-1, n+p)$ sending a unit normal to $\mathcal{F}$ at a point to the corresponding hyperplane passing through that point. In other words, the map $P : \mathbb{P}N\mathcal{F} \to \mathcal{A}(n+p-1, n+p)$ acts as follows: $P(x, n) = (\langle x, n\rangle, n)$ for $x \in U$ with the usual identification of $\mathcal{A}(n+p-1, n+p)$ with a tautological bundle over $\mathbb{R}P^{n+p-1}$.

Choose orthonormal frames $\{e_\alpha\}_{\alpha=1}^{n}$ tangent to $\mathcal{F}$ at x and $\{n_i\}_{i=1}^{p-1}$ normal to $\mathcal{F}$ and to n.

The map $dP(x,n)$ sends the vectors e_α tangent to U to the vectors $(\langle x, \tilde{\nabla}_{e_\alpha} n\rangle, \tilde{\nabla}_{e_\alpha} n)$; the vectors n_i tangent to U to the vectors $(\langle x, \tilde{\nabla}_{n_i} n\rangle, \tilde{\nabla}_{n_i} n)$; the vector n tangent to U to the vector $(\langle x, \tilde{\nabla}_n n\rangle + 1, \tilde{\nabla}_n n)$; and the (lifts of) vectors n_i tangent to the fiber of $\mathbb{P}N\mathcal{F}$ to the vectors $(\langle x, n_i\rangle, n_i)$, where $\tilde{\nabla}$ is the Euclidean connection in $\mathbb{R}^{n+p}$.

With respect to the frames chosen, the matrix of the linear operator dP at the point (x,n) is a matrix with $n+2p-1$ columns and $n+p$ rows. By standard orthogonal transformations with rows it can be reduced to the form

$$dP = \begin{pmatrix} 0 & 0 & 1 & 0 \\ A & C & * & 0 \\ B & D & * & I_{p-1} \end{pmatrix}$$

where A is an $n \times n$ matrix with the elements $\langle \tilde{\nabla}_{e_\alpha} n, e_\beta\rangle$; B is a $(p-1)\times n$ matrix with the elements $\langle \tilde{\nabla}_{e_\alpha} n, n_j\rangle$; C is an $n \times (p-1)$ matrix with the elements $\langle \tilde{\nabla}_{n_i} n, e_\beta\rangle$; D is an $(p-1)\times(p-1)$ matrix with the elements $\langle \tilde{\nabla}_{n_i} n, n_j\rangle$; I_{p-1} is the identity matrix of the corresponding dimension, and $*$ is a column of some elements.

The preimage $(\mathcal{S}_\pi, \pi^\perp) = P^{-1}(\pi)$ of the (regular) point $\pi = P(x,n)$ is a $(p-1)$-dimensional submanifold in $\mathbb{P}N\mathcal{F}$, whose tangent space at the point (x,n) is spanned by the column vectors of the matrix

$$R = \begin{pmatrix} M \\ I_{p-1} \\ 0 \\ -BM - D \end{pmatrix}$$

where M is an $n \times (p-1)$ matrix such that $AM + C = 0$ and 0 is a row of zeros. The tangent space of the submanifold $\mathcal{S}_\pi \subset \pi$ at the point $x \in \pi$ is spanned by the column vectors of the $(n+p-1)\times(p-1)$ matrix

$$T = \begin{pmatrix} M \\ I_{p-1} \end{pmatrix}.$$

Therefore the volume element of the submanifold $\mathcal{S}_\pi \subset \pi$ at the point x is equal to $\sqrt{\det {}^t RR / \det {}^t TT}$ times the volume element of $P^{-1}(\pi)$ at (x,n) (here ^{t} denotes the transposition). Since $AM + C = 0$, $\|\mathrm{Jac}(P)\|$ equals $|\det A|$ multiplied by $\sqrt{\det Q\,{}^t Q}$, where Q is the $(n+p-1)\times(2n+p-2)$ matrix

$$\begin{pmatrix} I_n & -M & 0 \\ B & D & I_{p-1} \end{pmatrix}.$$

Subtracting the first "row" of Q multiplied by B from the left from the second "row" one can check that $\sqrt{\det Q\,{}^t Q} = \sqrt{\det {}^t RR}$. Hence

$$\|\mathrm{Jac}(P)\| = |\det A| \sqrt{\det {}^t RR}.$$

Now the coarea formula (1) with the function $f = (\det {}^t RR)^{-1/2}$ gives

$$\int_U \left(\int_{\mathbb{P}N_x\mathcal{F}} |\det A| dn \right) dV = \int_{\mathcal{A}(n+p-1,n+p)} \left(\int_{\mathcal{S}_\pi} (\det {}^t TT)^{-1/2} \right) d\pi.$$

Since the matrix A is just the matrix of the shape operator of $\mathcal{F}$ at the point x with respect to the normal n, the inner integral on the left-hand side is the total absolute curvature at x multiplied by the volume of $\mathbb{R}P^{p-1}$. Moreover, at the point $x \in \pi$ there are two n-dimensional subspaces: the normal space of the submanifold $\mathcal{S}_\pi \subset \pi$ spanned by the column vectors of the $(n + p - 1) \times n$ matrix

$$\tilde{T} = \begin{pmatrix} I_n \\ -{}^t M \end{pmatrix}$$

and the tangent space to the foliation spanned by the column vectors of the $(n + p - 1) \times n$ matrix

$$\begin{pmatrix} I_n \\ 0 \end{pmatrix}.$$

Therefore the squared tangents of the principal angles $\{\phi_i\}_{i=1}^n$ between these subspaces are equal to the eigenvalues of the matrix $M\,{}^t M$ (see [2], for example). Since $\det {}^t TT = \det {}^t \tilde{T}\tilde{T} = \det(I_n + M\,{}^t M)$, we are done. □

Proof of Theorem 4. Apply the coarea formula (1) (in the case of equal dimensions) to the map P from the bundle of generalized spheres tangent to $\mathcal{F}$ to the space $\mathcal{B}$ of generalized spheres. More precisely, consider the normal line bundle $N\mathcal{F}$ and construct a map sending a point (x, t) to the sphere of radius $|t|$ centered at the point $x + tn$, where $t \in \bar{\mathbb{R}}$ and n is a unit normal to $\mathcal{F}$ at x. Define P to be this map with the domain restricted to those values of (x, t), for which the the sphere $P(x, t)$ has a *saddle* contact with the foliation.

Taking into account that the density of the measure μ on $\mathcal{B}$ at the sphere of radius r centered at the point (x_1, x_2, x_3) is equal to $r^{-4}\, dx_1\, dx_2\, dx_3\, dr$ one can easily obtain

$$\|\mathrm{Jac}(P)\| = |(1 - k_1 t)(1 - k_2 t)|\, t^{-4}.$$

The values of $t \in \bar{\mathbb{R}}$ with a saddle contact form an arc between $1/k_1$ and $1/k_2$ which does not contain a sphere of zero radius. Integrating $\|\mathrm{Jac}(P)\|$ along the corresponding set of the fiber one obtains

$$\int_{\mathcal{B}} N^-(\Sigma)\, d\mu(\Sigma) = \int_U \left(\int_{(1-k_1t)(1-k_2t)<0} \frac{|(1 - k_1 t)(1 - k_2 t)| dt}{t^4} \right) dV,$$

which gives the desired formula. □

3. Proofs of lemmas

Proof of Lemma 1. We are going to compute the Jacobian of the map P for both the case of a submanifold and the case of a foliation simultaneously (see the remark after Lemma 1 in the previous section). Let $\mathcal{S}^n$ be a smooth submanifold in a Euclidean space $\mathbb{R}^N$ (we allow the case $n = N$), $G(k, N, T\mathbb{R}^N)_{|\mathcal{S}}$ a Grassmann bundle of k-subspaces tangent to $\mathbb{R}^N$ at the points of $\mathcal{S}$, and $\mathcal{A}(k, N)$ the space of all k-dimensional affine planes in $\mathbb{R}^N$. We assume $n + k \geq N$. The map P sends a point $(x, \pi) \in G(k, N, T\mathbb{R}^N)_{|\mathcal{S}}$ to the plane in $\mathcal{A}(k, N)$ parallel to π and passing through x.

To compute the Jacobian $\|\mathrm{Jac}(P)\|$ identify the bundle $G(k, N, T\mathbb{R}^N)_{|\mathcal{S}}$ with the bundle $G(N-k, N, T\mathbb{R}^N)_{|\mathcal{S}}$ and $\mathcal{A}(k, N)$ with the tautological bundle over $G(N-k, N)$ in the standard fashion. Then P sends a point $(x, \pi^\perp)$ of the Grassmann bundle to the point $(pr_{\pi^\perp} x, \pi^\perp)$ of the corresponding tautological bundle. Choosing an orthonormal frame $\{u_A\}$ in $\mathbb{R}^N$ such that $\pi^\perp = u_{k+1} \wedge \cdots \wedge u_N$, one can rewrite this as follows:

$$P(x, u_{k+1} \wedge \cdots \wedge u_N) = \left(\sum_\alpha \langle x, u_\alpha \rangle u_\alpha, u_{k+1} \wedge \cdots \wedge u_N \right)$$

where $1 \leq i \leq k$, $k+1 \leq \alpha \leq N$, $1 \leq A \leq N$.

The tangent space to $G(N-k, N, T\mathbb{R}^N)_{|\mathcal{S}}$ is spanned by n vectors $\{e_\mu\}_{\mu=1}^n$ tangent to $\mathcal{S}$ which we choose to be orthonormal and $(N-k)k$ vectors $T_{i\alpha} = u_{k+1} \wedge \cdots \wedge u_{\alpha-1} \wedge u_i \wedge u_{\alpha+1} \wedge \cdots \wedge u_N$ tangent to the fibre $G(N-k, N)$ and orthonormal by the choice of the frame $\{u_A\}$. Notice that since the bundle is isometric to the Riemannian product the vectors $\{e_\mu\}$ and $\{T_{i\alpha}\}$ are mutually orthogonal and hence $\{e_\mu, T_{i\alpha}\}$ form an orthonormal frame in the tangent space to the Grassmann bundle.

The tangent space to the tautological bundle over $G(N-k, N)$ is spanned by $N-k$ vectors $\{u_\alpha\}$ tangent to a fibre and $(N-k)k$ vectors $T_{i\alpha}$ tangent to the Grassmann manifold $G(N-k, N)$. These vectors form the parallelohedron of volume 1 in the standard measure on the bundle [17].

The tangent map dP maps the vectors e_μ to the vectors $(\sum_\alpha \langle e_\mu, u_\alpha \rangle u_\alpha, 0)$ and the vectors $T_{i\alpha}$ to the vectors $(\langle x, u_i \rangle u_\alpha, T_{i\alpha})$ respectively. The matrix of the linear operator dP with respect to the frames chosen looks like

$$Q = \begin{pmatrix} A & 0 \\ * & I_{k(N-k)} \end{pmatrix}$$

where A is the matrix of $N-k$ rows and n columns with the elements $\langle e_\mu, u_\alpha \rangle$ in the α-th row and μ-th column (recall that $N-k \leq n$), $I_{k(N-k)}$ is the identity matrix of the corresponding dimension, and $*$ is some matrix. In view of the choice of frames, the Jacobian of the map P just equals $\sqrt{\det Q\,{}^tQ}$, where ^{t} means the transposition. By simple linear algebra, $\sqrt{\det Q\,{}^tQ} = \sqrt{\det A\,{}^tA}$. The rows of the matrix A are exactly the coordinates of projections of vectors u_α on $T_x\mathcal{S}$ with respect to the orthonormal frame $\{e_\mu\}$ in $T_x\mathcal{S}$. Therefore $\sqrt{\det A\,{}^tA}$ is the $(N-k)$-dimensional volume of the projection of $\{u_\alpha\}$ on $T_x\mathcal{S}$. Since the vectors $\{u_\alpha\}$ form the orthonormal frame in $\pi^\perp$ we are done. Notice that in the case $n = N$ the above volume is just 1. □

Proof of Lemma 2 (*multidimensional Meusnier formula*). With the notation of Lemma 2, let X be a unit vector field tangent to the curve $\mathcal{S}^n \cap \pi$. We have to compute the length of the vector $\tilde{\nabla}_X X$ where $\tilde{\nabla}$ is the Euclidean connection in $\mathbb{R}^{n+p}$. In the case $p = 1$ we assign plus or minus to that expression if the angle between $\tilde{\nabla}_X X$ and the chosen normal vector field is less or greater than 90° respectively. Let $u_\alpha = \cos\phi_\alpha n_\alpha + \sin\phi_\alpha Y_\alpha$, where $\{Y_\alpha\}$ are orthogonal vectors in $T_x\mathcal{S}$, some of them are unit and some can be zero if the corresponding angle ϕ_α is 0. By the Weingarten formula one has

$$\tilde{\nabla}_X X = \sum_\alpha \langle \tilde{\nabla}_X X, u_\alpha\rangle u_\alpha = \sum_\alpha (\langle \nabla_X X, Y_\alpha\rangle \sin\phi_\alpha + \langle h(X,X), n_\alpha\rangle \cos\phi_\alpha) u_\alpha$$

where ∇ is the induced connection on $\mathcal{S}$. Taking into account that the vectors $(-\sin\phi_\alpha n_\alpha + \cos\phi_\alpha Y_\alpha)$ are orthogonal to the plane π containing the curve of intersection, one gets $\langle \tilde{\nabla}_X X, -\sin\phi_\alpha n_\alpha + \cos\phi_\alpha Y_\alpha\rangle = 0\ \forall\alpha$. Hence $\langle \nabla_X X, Y_\alpha\rangle = \langle h(X,X), n_\alpha\rangle \tan\phi_\alpha$. Substituting this in the above formula one obtains

$$\tilde{\nabla}_X X = \sum_\alpha \langle h(X,X), n_\alpha\rangle / \cos\phi_\alpha u_\alpha.$$

Since the vectors $\{u_\alpha\}$ are orthonormal, the lemma is proved. □

The last formula in particular confirms the invariant geometric description of k given in the previous Chapter (Fig. 4). We will use (and prove) that description in the proof of Lemma 4 below.

Proof of Lemma 3. Let X be a unit vector of the line in the intersection of π^{p+1} and T^n. Choose an arbitrary orthonormal frame $\{u_\alpha\}_{\alpha=1}^p$ in the subspace π^p orthogonal to X in π^{p+1} and the orthonormal frame $\{u_i\}_{i=1}^{n-1}$ in the orthogonal complement of π^{p+1} in $\mathbb{R}^{n+p}$ such that the projections of its vectors on T^n are mutually orthogonal. So $u_i = \cos\phi_i Y_i + \sin\phi_i n_i$ for some angles $\{\phi_i\}$, some vectors $\{n_i\}$ normal to T^n and for vectors $\{Y_i\}_{i=1}^{n-1}$ which form an orthonormal frame in T^n together with X. Let $\omega_{i\alpha} = \langle du_i, u_\alpha\rangle$, $\omega_{Xi} = \langle dX, u_i\rangle$. Then the volume element of the Grassmann manifold $G(p+1, n+p)$ is the exterior product $\wedge_i \omega_{Xi} \wedge_{i\alpha} \omega_{i\alpha}$. The second product is actually the volume element of the Grassmann manifold $G(p, n+p-1)$ of p-dimensional subspaces in the Euclidean space $\mathbb{R}^{n+p-1} = X^\perp$. Consider the first product. By the choice of the frame $\{u_i\}$, one obtains $\omega_{Xi} = \langle dX, u_i\rangle = \langle dX, \cos\phi_i Y_i + \sin\phi_i n_i\rangle = \cos\phi_i \langle dX, Y_i\rangle$. Now the exterior product of 1-forms $\langle dX, Y_i\rangle$ is the volume element on the projective space $\mathbb{P}T$, while the product of cosines is the volume of the projection of a unit cube in

$(\pi^{p+1})^{\perp}$ on T^n. This volume equals the volume of the projection of a unit cube in π^{p+1} on $T^{\perp}$ which is the same as that for π^p since the projection of the vector X vanishes. □

Another proof can be obtained applying the coarea formula to the map $G(p+1, n+p) \to \mathbb{P}T$ sending π^{p+1} to X.

Proof of Lemma 4. We are going to compute the integrals

$$A_m(M) = \int_{G(p,n+p-1)} \mu^m(\pi^p)\, \|pr_{N_x S}\pi^p\|\, \chi(|\mu(\pi^p)| \le M) d\pi^p$$

and

$$B_m(M) = \int_{G(p,n+p-1)} \mu^m(\pi^p)\, \|pr_{N_x S}\pi^p\|^2\, \chi(|\mu(\pi^p)| \le M) d\pi^p,$$

where the function $\mu(\pi^p)$ is defined as follows (see Lemma 2): choose the orthonormal frame $\{u_\alpha\}_{\alpha=1}^p$ in $\pi = \pi^p$ such that the projections of its vectors on the p-dimensional space $N = N_x S$ are orthogonal. Let $\{n_\alpha\}_{\alpha=1}^p$ be the unit vectors of these projections and $\phi_\alpha = \angle(u_\alpha, n_\alpha)$. Let n be an arbitrary unit vector in N. Then $\mu(\pi) = \sqrt{\sum_{\alpha=1}^p \langle n, n_\alpha\rangle^2/\cos^2\phi_\alpha}$ if $p > 1$ and $\mu(\pi^1) = \pm 1/\cos\phi$ if $p = 1$. The last expression is obviously the same up to a sign and we will deal with the general one. Notice that $\|pr_N \pi\| = \prod_{\alpha=1}^p \cos\phi_\alpha$.

Let $\tilde{L}$ be an $(n-1)$-dimensional plane in $\mathbb{R}^{n+p-1}$ passing through the endpoint of n and orthogonal to N and $y = \pi \cap \tilde{L}$. Then the length of the vector y is precisely $\mu(\pi)$. This holds because decomposing y by the frame $\{u_\alpha\}_{\alpha=1}^p$ in π one gets $y = \sum_\alpha \xi_\alpha u_\alpha$ and since $\forall\alpha\;\; y - n \perp n_\alpha$ one easily obtains $\xi_\alpha \cos\phi_\alpha = \langle n, n_\alpha\rangle$. Hence $\|y\| = \mu(\pi)$.

Introduce Cartesian coordinates $\{x_1, \dots, x_{n+p-1}\}$ in $\mathbb{R}^{n+p-1}$ such that n is the unit vector of the axis x_1 and N is the coordinate subspace spanned by $\{x_1, \dots, x_p\}$. Almost all subspaces π^p in $\mathbb{R}^{n+p-1}$ (except those projecting on N with degeneracy) can be defined by $n-1$ explicit equations, $\{x_{p+i} = \sum_{\alpha=1}^p z_i^\alpha x_\alpha\}_{i=1}^{n-1}$, and the $p \times (n-1)$ matrix $Z = \{z_i^\alpha\}_{i=1}^{n-1}{}_{\alpha=1}^{p}$ can be taken as the local parametrization of the Grassmann manifold $G(p, n+p-1)$. Actually it parametrizes precisely the subset in $G(p, n+p-1)$ we are computing the integrals over.

In the coordinates $\{z_i^\alpha\}$ the Riemannian metric element on $G(p, n+p-1)$ looks as follows [23]:

$$\begin{aligned} ds^2 &= \operatorname{Tr}((I_p + {}^t ZZ)^{-1} d\,{}^t Z (I_{n-1} + Z\,{}^t Z)^{-1} dZ) \\ &= \sum_{ij\alpha\beta} (I_p + {}^t ZZ)^{-1}_{\alpha\beta} (I_{n-1} + Z\,{}^t Z)^{-1}_{ij} dz_i^\alpha dz_j^\beta, \end{aligned}$$

where Tr is the trace of a matrix, ^{t} is the transposition and I_q is the identity $q \times q$ matrix. The volume element of the Grassmann manifold $G(p, n+p-1)$ is the square root of the determinant of the $p(n-1)\times p(n-1)$ matrix of the metric tensor. This matrix is in fact the Kronecker product of the matrices $(I_p + {}^tZZ)^{-1}$ and $(I_{n-1} + Z\,{}^tZ)^{-1}$. It is known from linear algebra that its determinant equals $(\det(I_p + {}^tZZ)^{-1})^{n-1}(\det(I_{n-1} + Z\,{}^tZ)^{-1})^p = \det(I_p + {}^tZZ)^{1-n-p}$.

The function $\mu(\pi)$ is the distance between the origin and the point of intersection of π with the plane $\tilde{L} = \{x_1 = 1,\ x_2 = \cdots = x_p = 0\}$. So $\mu(\pi) = \sqrt{1 + \sum_i (z_i^1)^2}$.

Moreover considering the polyhedron in π projecting onto the unit cube in N, one easily derives $\| pr_N \pi \| = \det(I_p + {}^tZZ)^{-1/2}$.

Therefore we have to compute the integral

$$A_m(M) = \int_{\mathbb{R}^{p(n-1)}} (1 + \Sigma_i (z_i^1)^2)^{m/2}\,\chi(\Sigma_i (z_i^1)^2 \le M^2 - 1)\det(I_p + {}^tZZ)^{-(n+p)/2} dZ$$

and the integral $B_m(M)$ whose integrand differs by the factor $\det(I_p + {}^tZZ)^{-1/2}$.

Let $z_i^1 = r\zeta_i$ where $\zeta = \{\zeta_1, \ldots, \zeta_{n-1}\}$ is a unit vector running over a sphere S^{n-2} and denote by $\hat{Z}$ the $(p-1)\times(n-1)$ matrix obtained from Z by crossing out the first column. One has

$$\begin{aligned}\det(I_p + {}^tZZ) &= (r^2+1)\det(I_{p-1} + {}^t\hat{Z}\hat{Z}) - r^2\langle (I_{p-1} + {}^t\hat{Z}\hat{Z})^{\nabla}\,{}^t\hat{Z}\zeta,\ {}^t\hat{Z}\zeta\rangle \\ &= \det(I_{n-1} + \hat{Z}\,{}^t\hat{Z}) + r^2\langle (I_{n-1} + \hat{Z}\,{}^t\hat{Z})^{\nabla}\zeta, \zeta\rangle,\end{aligned}$$

where ∇ is the adjoint matrix and we have used the fact that

$$\hat{Z}(I_{p-1} + {}^t\hat{Z}\hat{Z})^{\nabla}\,{}^t\hat{Z} = \det(I_{n-1} + {}^t\hat{Z}\hat{Z})I_{n-1} - (I_{n-1} + {}^t\hat{Z}\hat{Z})^{\nabla}.$$

Substituting this in the above integral one obtains

$$A_m(M) = \int_0^{\sqrt{M^2-1}} \int_{S^{n-2}} \int_{\mathbb{R}^{(p-1)(n-1)}} \frac{(1+r^2)^{m/2} r^{n-2}\,dr\,d\zeta\,d\hat{Z}}{(\det(I_{n-1} + {}^t\hat{Z}\hat{Z}) + r^2\langle (I_{n-1} + {}^t\hat{Z}\hat{Z})^{\nabla}\zeta, \zeta\rangle)^{(n+p)/2}}.$$

Now the inner integral (over $\mathbb{R}^{(p-1)(n-1)}$) is the same for all $\zeta \in S^{n-2}$. So we can choose $\zeta = \{1, 0, \ldots, 0\}$ and multiply the whole integral by the volume of an $(n-2)$-dimensional unit sphere. For every fixed r consider the inner integral

$$\int_{\mathbb{R}^{(p-1)(n-1)}} \frac{d\hat{Z}}{(\det(I_{n-1} + {}^t\hat{Z}\hat{Z}) + r^2(I_{n-1} + {}^t\hat{Z}\hat{Z})^{\nabla}_{11})^{(n+p)/2}}$$

and make the change of variable $\hat{Z} = W$, where all the columns of the matrix W but the first one are the same as in $\hat{Z}$ and the first column of W equals the first column of $\hat{Z}$ multiplied by $\sqrt{r^2+1}$. Then $d\hat{Z} = (r^2+1)^{(p-1)/2} dW$ and

$$\det(I_{n-1} + {}^t\hat{Z}\hat{Z}) + r^2(I_{n-1} + {}^t\hat{Z}\hat{Z})^{\nabla}_{11} = (r^2+1)\det(I_{n-1} + {}^tWW).$$

Therefore the above integral equals $(r^2+1)^{(-n-1)/2}$ multiplied by a constant. Hence

$$A_m(M) = \text{const} \int_0^{\sqrt{M^2-1}} (1+r^2)^{(m-n-1)/2} r^{n-2} dr.$$

Substituting $x = \sqrt{1+r^2}$ we get the result.

Calculations for $B_m(M)$ are word for word the same. The only difference is that after substituting $\hat{Z} = W$ we get $(r^2+1)^{(-n-2)/2}$ multiplied by a constant. That gives the exponent of x in the final result to be one less than that for $A_m(M)$. □

Proof of Lemma 5. 1. Choose orthonormal frames $\{e_i\}$ in $T_x\mathcal{S}$ and $\{n_\sigma\}$ in $N_x\mathcal{S}$ and denote the elements of the second fundamental form with respect to these frames by $h^\sigma_{ij} = \langle h(e_i, e_j), n_\sigma\rangle$. Following H.Weyl [20] consider the integral

$$\int_{\mathbb{R}^n} e^{-\Sigma x_i^2} \sum_\sigma \left(\sum_{ij} h^\sigma_{ij} x_i x_j\right)^2 .$$

On one hand in spherical coordinates it equals the integral we are computing up to a constant. On the other hand the direct calculation shows

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-\Sigma x_i^2} \sum_\sigma \left(\sum_{ij} h^\sigma_{ij} x_i x_j\right)^2 &= \int_{\mathbb{R}^n} e^{-\Sigma x_i^2} \sum_{\sigma ijkr} h^\sigma_{ij} h^\sigma_{kr} x_i x_j x_k x_r \\
&= \int_{\mathbb{R}^n} e^{-\Sigma x_i^2} \sum_{\sigma i} (h^\sigma_{ii})^2 x_i^4 + 4 \int_{\mathbb{R}^n} e^{-\Sigma x_i^2} \sum_{\sigma, i<j} (h^\sigma_{ij})^2 x_i^2 x_j^2 \\
&\quad + 2 \int_{\mathbb{R}^n} e^{-\Sigma x_i^2} \sum_{\sigma, i<j} h^\sigma_{ii} h^\sigma_{jj} x_i^2 x_j^2
\end{aligned}$$

since all the integrals containing odd powers of x vanish. The last sum equals

$$\text{const} \sum_\sigma \left(2 \sum_{i<j} h^\sigma_{ii} h^\sigma_{jj} + 4 \sum_{i<j} (h^\sigma_{ij})^2 + 3 \sum_i (h^\sigma_{ii})^2\right).$$

Substituting

$$\|nH\| = \sum_\sigma \left(\sum_i (h^\sigma_{ii})^2 + 2 \sum_{i<j} h^\sigma_{ii} h^\sigma_{jj}\right), \quad \text{Scal} = 2 \sum_\sigma \left(\sum_{i<j} h^\sigma_{ii} h^\sigma_{jj} - \sum_{i<j} (h^\sigma_{ij})^2\right)$$

we get the result.

2. Choose the orthonormal frame $\{e_i\}$ of principal directions in $T_x\mathcal{S}$ and let $\lambda_i = h(e_i, e_i)$. Consider the integral

$$\int_{\mathbb{R}^n} e^{-\Sigma x_i^2} \left(\sum_i \lambda_i x_i^2\right)^3 .$$

Again, changing to spherical coordinates we get a constant multiple of the required integral. Computation in Cartesian coordinates as above gives

$$\int_{\mathbb{R}^n} e^{-\Sigma x_i^2} \left(\sum_i \lambda_i x_i^2\right)^3 = \text{const}\left(5\sum_i \lambda_i^3 + 3\sum_{i\neq j} \lambda_i^2\lambda_j + 2\sum_{i<j<k} \lambda_i\lambda_j\lambda_k\right).$$

Expressing this in terms of symmetric functions we get the formula. □

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