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A NOTE ON THE EQUATION $Y = (I - T)X$ IN L^1

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ABSTRACT. We give a characterization of L^1 coboundaries for a set of bounded operators. This set includes measure preserving transformations.

This short note is a contribution to the problem of characterizing the coboundary functions of a bounded linear operator in L^1 of a measure space $(X, \mathcal{B}, \mu)$. As a corollary we obtain a different proof, for ϕ a measure-preserving transformation on the measure space $(X, \mathcal{B}, \mu)$, of the equivalence of the following two statements:

(a) $$\sup_n \left\| \sum_{k=0}^{n} f \circ \phi^n \right\|_1 .$$

(b) There exists $g \in L^1(\mu)$ such that $g - g \circ \phi = f$

This equivalence follows from the result obtained in [LS] for contractions in L^1, which includes the case of measure preserving transformations.Some additional information for contractions in L^1 has been obtained recently in [FLR].

Our proof also works for operators which are no longer contractions in L^1. For instance if $Tf = f \circ \phi$ where $\sup_n \phi^{-n}(A) \leq M\mu(A)$ for all measurable sets $A \in \mathcal{B}$ (the smallest constant M satifying this inequality being strictly greater than 1) then T is not a contraction of L^1. An example of such a linear operator can be found in [AW] , the constant M is equal to 2. We have not been able to use the method in [LS] for contractions in L^1 to prove the equivalence between (a) and (b) for this example.

We would like to prove the following result.

THEOREM. *Let $(X, \mathcal{B}, \mu)$ be a measure space and T a power bounded operator on $L^1(\mu)$ with the following property:*

For all sequences of L^1 functions h_n such that $\lim_n h_n = 0$ a.e., *then* $\lim_n Th_n = 0$ a.e. *Then the following statements are equivalent for a function $f \in L^1(\mu)$:*

$$\sup_n \left\| \sum_{k=1}^{n} T^k f \right\|_1 < \infty. \tag{1}$$

$$\textit{There exists } g \in L^1 \textit{ such that } f = g - Tg \tag{2}$$

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Proof. The only nontrivial implication is (1) implies (2). We can observe that the condition (1) implies the convergence in L^1 norm of the averages $M_N(f) = \frac{1}{N}\sum_{n=1}^{N} T^n f$ to 0. We can extract a subsequence N_k such that $\lim_k M_{N_k}(f) = 0$ a.e. For this sequence N_k we consider the L^1 bounded sequence of functions $H_{N_k}(f) = \frac{1}{N_k}\sum_{n=1}^{N_k}\sum_{j=0}^{n-1} T^j f$. By Komlos' theorem [K], there exists a subsequence of N_k, that we denote also by N_k, and a function $H \in L^1$ such that the averages

$$\frac{1}{K}\sum_{k=1}^{K}\left(\frac{1}{N_k}\sum_{n=1}^{N_k}\sum_{j=0}^{n-1} T^j f\right)$$

converge a.e. to the function H.

The difference $H - TH$ is equal to the pointwise limit

$$\lim_K \left(\frac{1}{K}\sum_{k=1}^{K} f - \frac{1}{K}\sum_{k=1}^{K}\left(\frac{1}{N_k}\sum_{n=1}^{N_k} T^n f\right)\right) = f$$

because of the assumption made on T. This finishes the proof of our theorem.

References

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