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A REMARK ON ALMOST EVERYWHERE CONVERGENCE OF CONVOLUTION POWERS

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ABSTRACT. We answer a question of [BJR] about pointwise convergence a.e. for convolution powers of measures with expectation zero and finite moments of all orders $s < 2$. We compare the conditions that appeared in the study of the strong sweeping out property for convolution powers.

Let $(X, \mathcal{B}, m)$ be a non–atomic probability measure space, τ an invertible measure preserving transformation on X. Given a probability measure μ on $\mathbb{Z}$ and $f \in L^p(X)$ $(p \geq 1)$, put $(\mu f)(x) = \sum_{k=-\infty}^{\infty} \mu(\{k\}) f(\tau^k x)$. In [BJR], the question of pointwise convergence a.e. of the convolution powers $\mu^n f$ has been studied (see also [B] and [BC] for this and related matters). An important rôle is played by the quotient $\frac{|\hat{\mu}(t)-1|}{1-|\hat{\mu}(t)|}$, the *angular ratio* ($\hat{\mu}(t) = \sum \mu(\{k\}) e^{-2\pi i k t}$ denotes the Fourier transform of μ, $t \in \mathbb{R}$). In [BJR], Th. 1.6, it was shown that $\lim \mu^n f(x)$ exists a.e. for $f \in L^p(X)$ $(p > 1)$, if $|\hat{\mu}(t)| < 1$ for $t \notin \mathbb{Z}$ and the angular ratio is bounded. If μ has a finite second moment $m_2(\mu) = \sum k^2 \mu(\{k\})$ and the expectation $E(\mu) = \sum k \mu(\{k\})$ is zero, then μ has bounded angular ratio ([BJR], Prop. 1.9). In this special case (if in addition $|\hat{\mu}(t)| < 1$ for $t \notin \mathbb{Z}$), it was shown in [B], Th. 5 that pointwise convergence a.e. holds even on L^1. On the other hand, for μ with $E(\mu) \neq 0$, $m_2(\mu) < \infty$, it was shown in [BJR], Th. 2.1, that the *strong sweeping out property* holds, i.e., rather drastic divergence (see also [AB]). In [BJR] the question was raised about what happens with measures having finite moments $m_s(\mu)$ for all $s < 2$ and $E(\mu) = 0$.

In the course of the investigations about divergence three conditions appeared. First came

(AR) $$\lim_{t \to 0} \frac{|\hat{\mu}(t) - 1|}{1 - |\hat{\mu}(t)|} = \infty.$$

This is completely sufficient to describe the possible cases for measures with finite second moments. (By [BJR] Prop.1.9 and Lemma 1.7, $m_1(\mu) < \infty$ together with $E(\mu) \neq 0$ implies (AR)).

Then a weaker condition (we call it (BJR)) was introduced in [BJR], Th. 2.2 (the precise definition is given before Prop.3). In [AB], Th. 6.1, it was shown that (BJR) is also sufficient to get the strong sweeping out property.

Finally, [BJR] suggested considering the condition

(UR) $$\limsup_{t \to 0} \frac{|\hat{\mu}(t) - 1|}{1 - |\hat{\mu}(t)|} = \infty.$$

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By [BJR], Prop. 2.3, (AR) implies (BJR) and it is easily seen that (BJR) implies (UR).

We consider the partial sums for $E(\mu)$ and those for $\sum \mu(\{k\})$ and $m_2(\mu)$. We characterize (AR) (Proposition 1) and (BJR) (Propositions 3 and 4) in terms of the asymptotic behaviour of these sums, thus avoiding the use of Fourier transforms and facilitating the construction of various sorts of examples and counterexamples. In particular, this is used to construct examples of measures μ with $E(\mu) = 0$, $m_s(\mu) < \infty$ for all $s < 2$ and satisfying (BJR) (consequently, the strong sweeping out property holds—answering the question of [BJR]). Examples of this type have also been given by Chistyakov [C].

Although (UR) appears to be the decisive condition, (AR) is certainly the easiest to work with (e.g., [C] mentions examples arising from distributions belonging to the domain of attraction of certain stable laws). On the other hand, we show in Proposition 2 that for measures μ with $E(\mu) = 0$, $m_s(\mu) < \infty$ for some $s < 2$, condition (AR) can never be fulfilled (i.e., examples as mentioned above are impossible if (AR) is required, in particular, the measures of Examples 1,2 satisfy (BJR) but not (AR), so (AR) is strictly stronger than (BJR)). Thus (AR) becomes rather insufficient to get a complete picture for measures with $m_2(\mu) = \infty$.

Finally, in Example 3 we give examples showing that (BJR) is strictly stronger than (UR): measures μ with $E(\mu) = 0$, $m_s(\mu) < \infty$ for some (arbitrarily given) $s < 2$ that have unbounded asymptotic ratio but do not satisfy (BJR) (existence of such examples was already mentioned in [BJR], p. 428).

For measures μ with unbounded asymptotic ratio that do not satisfy (AR) the Fourier transform $\hat{\mu}(t)$ oscillates near $t = 0$. (with phases of tangential and phases of non-tangential approach). If μ does not even satisfy (BJR) these oscillations get very sharp.

Notation. $a_n = \mu(\{n\})$ for $n \in \mathbb{Z}$, i.e., $a_n \geq 0$, $\sum_{n=-\infty}^{\infty} a_n = 1$, and we will always assume that $a_n > 0$ for infinitely many n. Then $z(t) = \hat{\mu}(t) = \sum a_n\, e^{-2\pi i n t} = x(t) + i\, y(t)$, with $x(t) = \sum a_n \cos(2\pi n t)$, $y(t) = -\sum a_n \sin(2\pi n t)$. We consider the sums

$$r_N = \sum_{|n|>N} a_n, \quad s_N = \sum_{|n|\leq N} n\, a_n, \quad t_N = \sum_{|n|\leq N} n^2 a_n, \quad N = 0, 1, 2, \ldots.$$

For $s \in \mathbb{R}$, we denote by $\langle s\rangle$ the unique element from $]-\frac{1}{2}, \frac{1}{2}]$ such that $s - \langle s\rangle \in \mathbb{Z}$ (i.e., $\langle s\rangle = \frac{1}{2} - \{\frac{1}{2} - s\}$, where $\{t\}$ denotes the fractional part of t).

LEMMA 1. *Take $z = x + iy$, with $|z| < 1$. Then the following inequalities hold*:

(a) $\frac{|z-1|}{1-|z|} \geq \frac{|y|}{1-x}$.

(b) *If* $y^2 \leq 1 - x < 1$ *then* $\frac{|z-1|}{1-|z|} \leq \frac{2}{x}\left(1 + \frac{|y|}{1-x}\right)$.

Proof. (a) We use $|z-1| \geq |y|$, $1-|z| \leq 1-x$.
(b) Follows from $\frac{1}{1-|z|} = \frac{1+|z|}{1-|z|^2}$ and $|z-1| \leq 1-x+|y|$.

Since $1-|z|^2 = 1-x^2-y^2 \geq x-x^2 = x(1-x)$, $x>0$ and $1+|z| \leq 2$, our claim follows.

COROLLARY.

(*a*) $\frac{|y|}{1-x} \geq \min\left(\frac{x|z-1|}{2(1-|z|)} - 1, \frac{1}{|y|}\right)$.

(*b*) $\lim_{t\to 0} \frac{|\hat{\mu}(t)-1|}{1-|\hat{\mu}(t)|} = \infty$ *if and only if* $\lim_{t\to 0} \frac{|y(t)|}{1-x(t)} = \infty$.

Proof. (a) If $y^2 \geq 1-x$ then $\frac{|y|}{1-x} \geq \frac{1}{|y|}$; now combine with Lemma 1b.
(b) Since $\lim_{t\to 0} \hat{\mu}(t) = 1$, we get $\lim_{t\to 0} x(t) = 1$, $\lim_{t\to 0} y(t) = 0$. Now apply (a) and Lemma 1(a).

LEMMA 2. $\lim_{t\to 0} \frac{|y(t)|}{1-x(t)} = \infty$ *is equivalent to* $\lim_{t\to 0} \frac{|\sum a_n \langle nt\rangle|}{\sum a_n \langle nt\rangle^2} = \infty$.

Remark. In the same way, the equivalences in (b) of the corollary and Lemma 2 hold along sequences (t_n) with $\lim_{n\to\infty} t_n = 0$. (The notation $\langle s\rangle$ is explained before Lemma 1.)

Proof. We use these estimates:

$$1-\cos x \leq \frac{x^2}{2} \quad \text{for all } x$$

$$1-\cos x \geq \left(\frac{2}{\pi}\right)^2 \frac{x^2}{2} \text{for } |x| \leq \pi,$$

$$0 \leq x - \sin x \leq \frac{x^3}{6} \text{for } x \geq 0$$

and

$$\frac{x^3}{6} \leq \frac{\pi}{6} x^2 \text{ for } 0 \leq x \leq \pi.$$

Since $y(t) = -\sum a_n \sin(2\pi\langle nt\rangle)$, we have

$$\left|y(t) - 2\pi \sum a_n \langle nt\rangle\right| \leq \frac{2\pi^3}{3} \sum a_n \langle nt\rangle^2. \tag{1}$$

From $x(t) = \sum a_n \cos(2\pi\langle nt\rangle)$, we get $1 - x(t) = \sum a_n(1 - \cos(2\pi\langle nt\rangle))$, hence

$$8\sum a_n\langle nt\rangle^2 \le 1 - x(t) \le 2\pi^2\sum a_n\langle nt\rangle^2. \tag{2}$$

Thus $$\frac{1}{\pi}\frac{|\sum a_n\langle nt\rangle|}{\sum a_n\langle nt\rangle^2} - \frac{\pi}{3} \le \frac{|y(t)|}{1-x(t)} \le \frac{\pi}{4}\frac{|\sum a_n\langle nt\rangle|}{\sum a_n\langle nt\rangle^2} + \frac{\pi^3}{12}.$$

LEMMA 3. *For* $|t| < \frac{1}{2N}$ *we have*

$$\frac{|ts_N|}{t^2 t_N + \frac{1}{4}r_N} - 2 \le \frac{|\sum a_n\langle nt\rangle|}{\sum a_n\langle nt\rangle^2} \le \frac{|ts_N| + \frac{1}{2}r_N}{t^2 t_N}.$$

Proof. If $|n| \le N$, then $\langle nt\rangle = nt$, hence $\sum a_n\langle nt\rangle = t s_N + \sum_{|n|>N} a_n\langle nt\rangle$. Consequently

$$\left|\sum a_n\langle nt\rangle - ts_N\right| \le \frac{1}{2}r_N. \tag{3}$$

Similarly

$$t^2 t_N \le \sum a_n\langle nt\rangle^2 \le t^2 t_N + \frac{1}{4}r_N. \tag{4}$$

This gives

$$\frac{|\sum a_n\langle nt\rangle|}{\sum a_n\langle nt\rangle^2} \ge \frac{|ts_N| - \frac{1}{2}r_N}{t^2 t_N + \frac{1}{4}r_N},$$

from which the first estimate follows. The second one is obtained similarly.

PROPOSITION 1. *The following statements are equivalent*:

(i) $\lim_{t\to 0}\frac{|1-\hat{\mu}(t)|}{1-|\hat{\mu}(t)|} = \infty$ $\quad(AR)$.

(ii) $\lim_{t\to 0}\frac{|y(t)|}{1-x(t)} = \infty$.

(iii) $\lim_{N\to\infty}\frac{|Ns_N|}{t_N+N^2 r_N} = \infty$.

Proof. (i) $\Leftrightarrow$ (ii). By the corollary to Lemma 1.
(iii)$\Rightarrow$ (ii) By Lemma 3 and Lemma 2.
(ii) $\Rightarrow$ (iii) We have $\int_a^b \cos(2\pi nt)dt = \frac{1}{2\pi n}(\sin(2\pi nb) - \sin(2\pi na)) \le \frac{1}{\pi|n|}$.

In the special case that $N \le |n| \le 2N$, $a = \frac{1}{4N}$, $b = \frac{1}{2N}(= 2a)$, we can say (putting $s = \frac{\pi n}{2N} = 2\pi na$) that $\sin(2\pi nb) - \sin(2\pi na) = \sin(2s) - \sin(s)$. This is not greater than zero for $n \ge 0$, since $\frac{\pi}{2} \le |s| \le \pi$, and nonnegative for $n \le 0$. Consequently

$$\int_{\frac{1}{4N}}^{\frac{1}{2N}} \sum_{|n|>N} a_n(1 - \cos(2\pi nt))\,dt \ge \sum_{2N\ge|n|>N} a_n\frac{1}{4N} + \sum_{|n|>2N} a_n\left(\frac{1}{4N} - \frac{1}{\pi|n|}\right).$$

Since $\frac{1}{4N} - \frac{1}{\pi|n|} \leq \frac{1}{12N}$, the integral is at least $\frac{r_N}{12N}$. It follows that there exists $t \in]\frac{1}{4N}, \frac{1}{2N}[$ such that $\sum_{|n|\leq N} a_n(1-\cos(2\pi nt)) \geq \frac{r_N}{3}$. For such a value t we get

$$\begin{aligned} 1 - x(t) &\underset{\text{(by (2))}}{\geq} 8\sum_{|n|\leq N} a_n\langle nt\rangle^2 + \frac{r_N}{3} \underset{\text{(by (4))}}{=} 8t^2 t_N + \frac{r_N}{3} \\ &\underset{(\text{since } t\geq\frac{1}{4N})}{\geq} \frac{t_N}{2N^2} + \frac{r_N}{3} \\ &\geq \frac{1}{3N^2}(t_N + N^2 r_N). \end{aligned}$$

Furthermore

$$\begin{aligned} |y(t)| &\underset{\text{(by (1))}}{\leq} 2\pi\left|\sum a_n\langle nt\rangle\right| + \frac{2\pi^3}{3}\sum a_n\langle nt\rangle^2 \\ &\underset{\text{(by (3) and (4))}}{\leq} 2\pi\left(|s_N t| + \frac{1}{2}r_N\right) + \frac{2\pi^3}{3}\left(t_N t^2 + \frac{1}{4}r_N\right) \\ &\underset{(\text{since } t\leq\frac{1}{2N})}{\leq} \pi\frac{|s_N|}{N} + O\left(\frac{t_N}{N^2} + r_N\right). \end{aligned}$$

This gives

$$\frac{|y(t)|}{1-x(t)} \leq 3\pi\frac{N|s_N|}{t_N + N^2 r_N} + O(1).$$

PROPOSITION 2. *Fix p with $p > 1$ and assume that μ satisfies $\sum_{n=-\infty}^{\infty}|n|^p a_n < \infty$, $\sum_{n=-\infty}^{\infty} n\,a_n = 0$.*
Then μ cannot satisfy the condition (AR).

Proof. Recall the formula for partial summation: putting $\beta_n = B_{n-1} - B_n$, we have

$$\sum_{n=M}^{N}\gamma_n\beta_n = \sum_{n=M}^{N}(\gamma_{n+1}-\gamma_n)B_n - \gamma_{N+1}B_N + \gamma_M B_{M-1}. \tag{5}$$

Put $\alpha_0 = a_0$, $\alpha_n = a_n + a_{-n}$ for $n > 0$ and $s'_N = \sum_{|n|>N}|n|a_n$. Then for $n > 0$ we get

$$\alpha_n = \frac{1}{n}(s'_{n-1} - s'_n) = \frac{1}{n^2}(t_n - t_{n-1}) = r_{n-1} - r_n. \tag{6}$$

Now assume that property (iii) of Proposition 1 holds. Since $\sum n\,a_n = 0$ implies that $s_N = -\sum_{|n|>N} n\,a_n$, it follows that

$$\lim_{N\to\infty}\frac{s'_N}{N r_N} = \infty. \tag{7}$$

We will show that (7) is incompatible with a moment-condition $\sum |n|^p a_n < \infty$. For $c > 0$ there exists M such that

$$s'_n \geq c\,n\,r_n \qquad \text{for} \qquad n \geq M-1. \tag{8}$$

We assume that $c > \frac{2p}{p-1}$ and $M > p$. Put $\sigma_{MN} = \sum_{n=M}^{N} \alpha_n n^p$. Then by (5) and (6),

$$\sigma_{MN} = \sum_{n=M}^{N} \left((n+1)^{p-1} - n^{p-1}\right) s'_n - (N+1)^{p-1} s'_N + M^{p-1} s'_{M-1}.$$

For $n \geq M$, we have

$$((n+1))^{p-1} - n^{p-1})\, s'_n \geq c((n+1)^{p-1} - n^{p-1})\, n\, r_n.$$

Since $(1+x)^{p-1} - 1 \geq \frac{p-1}{2p}((1+x)^p - 1)$ for $0 \leq x \leq 1$, it follows that for $x = \frac{1}{n}$,

$$((n+1)^{p-1} - n^{p-1})\, n \geq \frac{p-1}{2p}((n+1)^p - n^p).$$

Hence,

$$\begin{aligned}
\sum_{n=M}^{N} \quad ((n+1)^{p-1} - n^{p-1})\, s'_n \;&\geq \frac{p-1}{2p} c \sum_{n=M}^{N} ((n+1)^p - n^p)\, r_n \\
\underset{(\text{by } (5))}{=} \quad & \frac{p-1}{2p} c \left(- \sum_{n=M}^{N} (r_{n+1} - r_n)(n+1)^p + (N+1)^p r_{N+1} - M^p r_M \right) \\
= \quad & \frac{p-1}{2p} c \left(- \sum_{n=M-1}^{N-1} (n+1)^p (r_{n+1} - r_n) + (N+1)^p r_N - M^p r_{M-1} \right) \\
\underset{(\text{by } (6))}{=} \quad & \frac{p-1}{2p} c \left(\sum_{n=M}^{N} n^p \alpha_n + (N+1)^p r_N - M^p r_{M-1} \right).
\end{aligned}$$

Thus,

$$\sigma_{MN} \geq \frac{p-1}{2p} c \left(\sigma_{MN} + (N+1)^p r_N - M^p r_{M-1}\right) - (N+1)^{p-1} s'_N + M^{p-1} s'_{M-1}.$$

By assumption, we have $M - 1 - \frac{p-1}{p} M = \frac{M}{p} - 1 > 0$, hence, by (8),

$$M^{p-1} s'_{M-1} \geq c M^{p-1} (M-1)\, r_{M-1} \geq \frac{p-1}{p} c M^p r_{M-1}.$$

It follows that

$$\sigma_{MN} \geq \frac{p-1}{2p} c \sigma_{MN} - (N+1)^{p-1} s'_N \qquad \text{for all } N > M.$$

Now, observe that

$$(N+1)^{p-1}s'_N = (N+1)^{p-1}\sum_{n>N} n\,\alpha_n \le \sum_{n>N} n^p\alpha_n.$$

By the moment–condition, the right hand side tends to zero for $N \to \infty$ and we see that $\sigma_M = \sum_{n=M}^{\infty} \alpha_n n^p$ satisfies $\sigma_M \ge \frac{p-1}{2p}c\sigma_M$. Since we assumed that $\frac{p-1}{2p}c > 1$, this implies that $\sigma_M = 0$; consequently $\alpha_n = 0$ for $n \ge M$ and we arrive at a contradiction.

Remark. (a) Similarly, it can be shown that (given any probability measure μ on $\mathbb{Z}$) the property $\lim_{N\to\infty} \frac{Ns'_N}{t_N} = \infty$ is incompatible with a moment-condition $\sum |n|^p a_n < \infty$ (for any p with $p > 1$).

(b) In [BJR], 3.5, an example of a probability measure μ on $\mathbb{Z}$ was given, satisfying $m_1(\mu) < \infty$, $E(\mu) = 0$ and (AR). Thus the bound for p in Proposition 2 is sharp.

Recall the notations of [BJR], p. 425: if I is a subinterval of $\mathbb{R}$, then $r_I = \inf\{|\hat{\mu}(t)| : t \in I\}$, if $0 \notin B \subseteq \mathbb{C}$, then $|B|_\rho$ denotes the measure of $B_\rho = \{\frac{z}{|z|} : z \in B\}$ (with respect to normalized Lebesgue measure on the circle). Theorem 6.1 of [AB] says that if there exist intervals $I(k)$ $(k = 1, 2, \ldots)$ such that

$$\text{(BJR)} \qquad \lim_{k\to\infty} \frac{|\hat{\mu}(I(k))|_\rho}{1 - r_{I(k)}} = \infty,$$

then (μ^n) has the strong sweeping out property.

PROPOSITION 3. *If* $\sup_N \frac{|Ns_N|}{t_N + N^2 r_N} = \infty$, *then the condition (BJR) is satisfied, in particular (by [AB], Th. 6.1), it follows that the sequence* (μ^n) *has the strong sweeping out property.*

Proof. We have

$$1 - x(t) \underset{(\text{by } (2))}{\le} 2\pi^2 \sum a_n \langle nt\rangle^2 \underset{(\text{by } (4))}{\le} 2\pi^2 (t^2 t_N + \frac{1}{4} r_N) \text{ for } |t| < \frac{1}{2N}.$$

Now assume that $\frac{|Ns_N|}{t_N + N^2 r_N} \ge c$. Put $I = \left[-\frac{1}{2N}, \frac{1}{2N}\right]$. Clearly, $r_I \ge \inf_{t\in I} x(t)$.

Hence,

$$1 - r_I \le \sup_{t\in I}(1 - x(t)) \le 2\pi^2 \left(\left(\frac{1}{2N}\right)^2 t_N + \frac{1}{4} r_N\right) = \frac{\pi^2}{2N^2}(t_N + N^2 r_N).$$

Furthermore,

$$|\hat{\mu}(I)_\rho| \ge \frac{1}{2\pi} \sup_{t_1, t_2 \in I} |y(t_1) - y(t_2)|.$$

This can be estimated as follows:

$$\begin{aligned}|y(t)-2\pi\, t\, s_N| &\leq |y(t)-2\pi \sum a_n\langle nt\rangle| + 2\pi\, |\sum a_n\langle nt\rangle - t\, s_N| \\ &\underset{(\text{by } (1),(3))}{\leq} \frac{2\pi^3}{3}\sum a_n\langle nt\rangle^2 + 2\pi\cdot\frac{1}{2}r_N \\ &\underset{(\text{by } (4))}{\leq} \frac{2\pi^3}{3}\left(t^2 t_N + \frac{1}{4}r_N\right) + \pi\, r_N .\end{aligned}$$

Consequently,

$$\begin{aligned}|y(\frac{1}{2N}) - y(-\frac{1}{2N})| &\geq 2\pi\,\frac{1}{N}|s_N| - \frac{4\pi^3}{3}\left(\left(\frac{1}{2N}\right)^2 t_N + \frac{1}{4}r_N\right) - 2\pi\, r_N \\ &\geq \frac{2\pi}{N^2}\left(|N s_N| - c_0(t_N + N^2 r_N)\right)\end{aligned}$$

where the constant c_0 does not depend on N.

In combination, this gives

$$\frac{|\hat{\mu}(I)_\rho|}{1-r_I} \geq \frac{2}{\pi^2}\left(\frac{|N s_N|}{t_N + N^2 r_N} - c_0\right) \geq \frac{2}{\pi^2}(c - c_0).$$

PROPOSITION 4. *If μ satisfies the condition (BJR), then* $\sup_N \frac{|N s_N|}{t_N + N^2 r_N} = \infty$.

The proof of Proposition 4 needs more elaborate estimates and is postponed until after the examples.

Remark. There does not seem to be an easy description of (UR) in terms of the asymptotic behaviour of r_n, s_n, t_n. By the remark after Lemma 2, a probability measure μ satisfies (UR) iff $\limsup_{t\to 0} \frac{|y(t)|}{1-x(t)} = \infty$ and iff $\limsup_{t\to 0} \frac{|\sum a_n\langle nt\rangle|}{\sum a_n\langle nt\rangle^2} = \infty$.

If (BJR) does not hold, then $\sup_N \frac{|N s_N|}{t_N + N^2 r_N} < \infty$. From this, one gets as a necessary condition that (UR) (but not (BJR)!) should hold: $\limsup_N \frac{N^2 r_N}{t_N} = \infty$, i.e., the remainder r_N must not go to zero too fast. But this is not sufficient. The outcome depends on the behaviour of the sequences $(\langle nt\rangle)_{n\geq[\frac{1}{2t}]}$. For (UR) there should be t such that $\langle nt\rangle$ gets small, but not too small for sufficiently many n and then the distribution of the signs in $\langle nt\rangle$ (and the relative size of a_n, a_{-n}) comes into play too. See Example 3 below, for how explicit examples of this behaviour can be obtained from this rather sketchy description.

APPLICATION. *Assume that μ is concentrated on $\mathcal{W} = \{\pm w_k : k = 1, 2, \ldots\} \subseteq \mathbb{Z}$ where $0 \leq w_1 < w_2 < \cdots$. Put $\alpha_k = \mu(\{-w_k, w_k\})$. Let $\mathcal{K} = \{k_1 < k_2 <$*

$\cdots\} \subseteq \mathbb{N}$. *Assume the following regularity conditions: there exists* $c, d > 0$ *such that* $r_{w_k} \le c\,\alpha_{k+1}$, $|s_{w_k}| \ge d\,w_k\alpha_k$, $t_{w_k} \le c\,w_k^2\alpha_k$ *for all* $k \in \mathcal{K}$ *(i.e., the size of the sums is governed by the leading term).*

Under these assumptions we will show that $\lim_{k\in\mathcal{K}} \left(\frac{\alpha_{k+1}}{\alpha_k} + \frac{w_k}{w_{k+1}}\right) = 0$ *implies* $\sup\left\{\frac{|Ns_N|}{t_N+N^2r_N} : w_k \le N < w_{k+1}, k \in \mathcal{K}\right\} = \infty$ *(in particular, by Prop.* 3, *property (BJR) holds).*

Proof. For $w_k \le N < w_{k+1}$, we have $r_N = r_{w_k}$, $s_N = s_{w_k}$, $t_N = t_{w_k}$. Hence

$$\frac{|Ns_N|}{t_N + N^2 r_N} \ge \frac{Nd\,w_k\alpha_k}{c\,w_k^2\alpha_k + N^2 c\,\alpha_{k+1}} = \frac{d}{c}\left(\frac{w_k}{N} + \frac{N}{w_k}\cdot\frac{\alpha_{k+1}}{\alpha_k}\right)^{-1}.$$

(a) If $\sqrt{\frac{\alpha_{k+1}}{\alpha_k}} \le \frac{w_k}{w_{k+1}}$, consider $N \ge \frac{w_{k+1}}{2}$ (assuming $w_{k+1} \ge 2$). Then

$$\frac{w_k}{N} + \frac{N}{w_k}\cdot\frac{\alpha_{k+1}}{\alpha_k} \le 2\,\frac{w_k}{w_{k+1}} + \frac{w_{k+1}}{w_k}\left(\frac{w_k}{w_{k+1}}\right)^2 = 3\,\frac{w_k}{w_{k+1}},$$

and the right-hand side tends to zero.

(b) If $\sqrt{\frac{\alpha_{k+1}}{\alpha_k}} > \frac{w_k}{w_{k+1}}$, consider N with $\sqrt{\frac{\alpha_{k+1}}{\alpha_k}} \le \frac{w_k}{N} \le 2\sqrt{\frac{\alpha_{k+1}}{\alpha_k}}$ (this is possible as soon as $\frac{\alpha_{k+1}}{\alpha_k} \le \frac{1}{4}$, $w_k \ge 2$). Then

$$\frac{w_k}{N} + \frac{N}{w_k}\cdot\frac{\alpha_{k+1}}{\alpha_k} \le 2\sqrt{\frac{\alpha_{k+1}}{\alpha_k}} + \sqrt{\frac{\alpha_k}{\alpha_{k+1}}}\cdot\frac{\alpha_{k+1}}{\alpha_k} = 3\sqrt{\frac{\alpha_{k+1}}{\alpha_k}},$$

and the right hand side tends to zero again.

Remark. If $|s_{w_k}| \le c\,w_k\alpha_k\left(\sqrt{\frac{\alpha_{k+1}}{\alpha_k}} + \frac{w_k}{w_{k+1}}\right)$ for $k \in \mathcal{K}$ (and the regularity conditions for r_{w_k} and t_{w_k} stay true), then the converse follows from similar computations, i.e.,

$$\sup\left\{\frac{|Ns_N|}{t_N + N^2 r_N} : w_k \le N < w_{k+1}, \quad k \in \mathcal{K}\right\} < \infty.$$

This is applicable, in particular, if

$$|s_{w_k}| \le c\,w_k\,\alpha_k \text{ for } k \in \mathcal{K} \quad \text{and} \quad \liminf\left(\frac{\alpha_{k+1}}{\alpha_k} + \frac{w_k}{w_{k+1}}\right) > 0.$$

Examples. (1) If $1 < p < 2$ is fixed, there exists μ satisfying the condition of Proposition 3 (i.e., (BJR) holds) and having the properties

$$\sum_{n=-\infty}^{\infty} n\,a_n = 0, \quad \sum_{n=-\infty}^{\infty} |n|^p a_n < \infty.$$

Explicitly: Choose α with $p < \alpha < 2$. Then $u = [\frac{1}{2-\alpha}] + 1 \geq 2$.
Put

$$\begin{aligned} w_m &= m! \qquad (m = 1, 2, \ldots) \\ a_{w_m} &= \frac{\gamma}{w_m^\alpha} \\ &\qquad\qquad \text{for } m \equiv 0 \ (\operatorname{mod} u) \\ a_{-w_{m+1}} &= \frac{a_{w_m}}{m+1} \\ a_n &= 0 \qquad \text{otherwise.} \end{aligned}$$

($\gamma > 0$ is chosen so that $\sum a_n = 1$.) Then $w_m a_{w_m} - w_{m+1} a_{-w_{m+1}} = 0$, hence $\sum n\, a_n = 0$. For $n = w_m$, $m \equiv 0 \,(\operatorname{mod} u)$, we have $n^p a_n = \frac{\gamma}{n^{\alpha-p}}$, and for $n = -w_{m+1}$, we get

$$|n|^p a_n = (m+1)^{p-1} \frac{\gamma}{w_m^{\alpha-p}} \leq \frac{\gamma}{w_m^{\frac{\alpha-p}{2}}} \text{holds for large } m.$$

Hence $\sum |n|^p a_n < \infty$.

Put $\mathcal{K} = \{m : m \equiv 0 \ (\operatorname{mod} u),\ m > 0\}$. We check the conditions for (BJR) specified above. For $m \in \mathcal{K}$, we have the following:

(a) $r_{w_m} = \sum\limits_{|n| \geq w_{m+1}} a_n$, and $a_{w_{m+u}} = \left(\frac{w_m}{w_{m+u}}\right)^\alpha a_{w_m} \leq \frac{1}{m+1} a_{w_m} = -a_{w_{m+1}} \leq a_{w_m}$.

Hence $r_{w_m} \leq a_{-w_{m+1}} \cdot 2 \cdot \sum\limits_{k=0}^{\infty} \frac{1}{k!}$.

(b) $s_{w_m} = w_m\, a_{w_m}$.

(c) $w_m^2 a_{w_m} = \gamma\, w_m^{2-\alpha} \leq \gamma\, \frac{w_{m+u}^{2-\alpha}}{m^{(2-\alpha)u}} \leq \frac{w_{m+u}^2}{m} a_{w_{m+u}}$ since $(2-\alpha)u > 1$.

This gives

$$\begin{aligned} w_{m+1}^2 a_{-w_{m+1}} &= (m+1)^2 w_m^2 \frac{a_{w_m}}{m+1} = \gamma\, (m+1)\, w_m^{2-\alpha} \\ &\leq \gamma\, \frac{m+1}{(m+1)^{u(2-a)}}\, w_{m+u}^{2-\alpha} \leq w_{m+u}^2\, a_{w_{m+u}}. \end{aligned}$$

Hence $t_{w_m} \leq w_m^2 a_{w_m} \cdot 2 \cdot \sum\limits_{k=0}^{\infty} \frac{1}{k!}$.

(d) $\frac{\alpha_{m+1}}{\alpha_m} = \frac{a_{-w_{m+1}}}{a_{w_m}} = \frac{1}{m+1} \to 0$.

(e) $\frac{w_m}{w_{m+1}} = \frac{1}{m+1} \to 0$.

(2) There exists μ satisfying the condition of Proposition 3 (i.e., (BJR) holds) and having the following properties:

$$\sum_{n=-\infty}^{\infty} n\, a_n = 0, \quad \sum_{n=-\infty}^{\infty} |n|^p a_n < \infty \quad \text{for all} \quad 1 < p < 2.$$

Explicitly: Put

$$\begin{aligned} w_m &= m! \\ \mathcal{K} &= \{2^k\colon k = 1, 2, \ldots\} \\ a_{w_m} &= \frac{\gamma}{w_m^{2-\frac{1}{k}}} \\ & && \text{for } m = 2^k \in \mathcal{K} \\ a_{-w_{m+1}} &= \frac{a_{w_m}}{m+1} \\ a_n &= 0 && \text{otherwise.} \end{aligned}$$

The estimates are done as in (1).

(3) If $1 < p < 2$ is fixed, there exists μ with unbounded asymptotic ratio (i.e., $\limsup_{t\to 0} \frac{|\hat{\mu}(t)-1|}{1-|\hat{\mu}(t)|} = \infty$, property (UR)), but not satisfying (BJR) and having the properties

$$\sum_{n=-\infty}^{\infty} n\, a_n = 0, \quad \sum_{n=-\infty}^{\infty} |n|^p a_n < \infty.$$

Construction. Assume that $w_1 \geq 1$, $w_k \mid w_{k+1}$ and $w_{k+1} > 2\, w_k$ for k sufficiently large. Then, taking $t = \frac{1}{w_{k+1}}$, it can be shown by computations similar to those in Lemma 3 that $\sum a_n \langle nt \rangle = t s_{w_k}$ and $\sum a_n \langle nt \rangle^2 = t^2 t_{w_k}$. Hence by the remark to Lemma 2, the asymptotic ratio is unbounded as soon as

$$\sup \frac{|s_{w_k}| w_{k+1}}{t_{w_k}} = \infty.$$

Applying Proposition 4, we have only to show that this is compatible with the first condition given in the remark before Example 1 (with $\mathcal{K} = \mathbb{N}$).

Explicitly: Choose α with $p < \alpha < 2$, put $\alpha_k = \frac{\gamma}{w_k^\alpha}$ for $k \in \mathbb{N}$ and balance a_{-w_k} and a_{w_k} so that $s_{w_k} = \gamma \sqrt{\frac{w_k^{2-\alpha}}{w_{k+1}^\alpha}}$ (this is possible, since the right side is smaller than $\alpha_{k+1} w_{k+1} < \alpha_k w_k$). Then the condition from the remark before Example 1 is fulfilled:

$$s_{w_k} = w_k \alpha_k \sqrt{\frac{\alpha_{k+1}}{\alpha_k}},$$

$$r_{w_k} \leq \gamma\, w_{k+1}^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{2^{n\alpha}} \leq c\, \alpha_{k+1},$$

$$t_{w_k} \leq \gamma\, w_k^{2-\alpha} \sum_{n=0}^{\infty} 2^{n(\alpha-2)} \leq c\, w_k^2 \alpha_k.$$

Thus,

$$\frac{|s_{w_k}| w_{k+1}}{t_{w_k}} \geq \frac{1}{c} \left(\frac{w_{k+1}}{w_k} \right)^{1-\frac{\alpha}{2}}$$

and the asymptotic ratio is unbounded when $\sup \frac{w_{k+1}}{w_k} = \infty$. Hence, for example, one can take, $w_k = k!$.

Before proving Proposition 4, we show an auxiliary result.

LEMMA 4. *If* $m_2(\mu) = \infty$, *then* $\lim_{t \to 0} \frac{y(t)^2}{1-x(t)} = 0$.

Proof. Take $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $\tau = r_N = \sum_{|n|>N} a_n < \epsilon$. Put

$$z_1 = \frac{1}{1-\tau} \sum_{|n| \leq N} a_n e^{-2\pi i n t} \text{and } z_2 = \frac{1}{\tau} \sum_{|n|>N} a_n e^{-2\pi i n t},\ z_j = x_j + i\, y_j$$

(for $j = 1, 2$; everything depends on t which is omitted for brevity). By (2), we can estimate $1 - x_1$ by $\frac{1}{1-\tau} \sum_{|n| \leq N} a_n \langle nt \rangle^2$. For $|t| < \frac{1}{2N}$ this equals $\frac{1}{1-\tau} \sum_{|n| \leq N} a_n n^2 t^2$. For $1 - x_2$, we get $\frac{1}{\tau} \sum_{N<|n|} a_n \langle nt \rangle^2$ and for $|t| < \frac{1}{2M}$ (with $M > N$) this is bigger than $\frac{1}{\tau} \sum_{N<|n| \leq M} a_n n^2 t^2$. Since $m_2(\mu) = \infty$, it follows that there exists $t_o > 0$ such that

$$1 - x_1 < \tau^2 (1 - x_2) \quad \text{for} \quad |t| < t_o.$$

By definition, $y = (1-\tau) y_1 + \tau y_2$ and $1 - x = (1-\tau)(1-x_1) + \tau(1-x_2) \geq \tau(1-x_2)$. Since $|z_j| \leq 1$, we have $|y_j| \leq \sqrt{2\,(1-x_j)}$ and this gives

$$\begin{aligned} y^2 \leq 2\, y_1^2 + 2\, \tau^2 y_2^2 &\leq 4\,(1 - x_1) \ + \ 4\, \tau^2 (1 - x_2) \\ &< \ 8\, \tau^2 (1 - x_2) \leq 8\, \tau\, (1 - x) < 8\, \epsilon\, (1 - x). \end{aligned}$$

Remark. If $m_2(\mu) < \infty$, it is easy to see (e.g., by de l'Hôpital's rule) that

$$\lim_{t\to 0} \frac{y(t)^2}{1-x(t)} = \frac{2\,E(\mu)^2}{m_2(\mu)}.$$

Proof of Proposition 4. Assume that $\sup \frac{|Ns_N|}{t_N+N^2r_N} < \infty$. First, we claim that

$$y(t)^2 = o(1-x(t)) \quad \text{for } t \to 0. \tag{9}$$

If $m_2(\mu) = \infty$, (9) was shown in Lemma 4. Next, we give an argument for the case $m_1(\mu) < \infty$. As before, put $\alpha_n = \mu(\{-n, n\})$. We define $\bar{y}(t) = -\sum_{n=1}^{\infty} \alpha_n \sin(2\pi nt)$, $\bar{s}_N = \sum_{|n|\le N} |n|\, a_n = \sum_{n=1}^{N} n\,\alpha_n$. Then $\lim \bar{s}_N = m_1(\mu)$. By partial summation,

$$t_N = \sum_{n=1}^{N} n^2\alpha_n = -\sum_{n=1}^{N} \bar{s}_n + (N+1)\,\bar{s}_N.$$

Hence $\lim \frac{1}{N}\, t_N = -m_1(\mu) + m_1(\mu) = 0$. Furthermore,

$$N\,r_N = N \sum_{n=N+1}^{\infty} \alpha_n \le \sum_{N+1}^{\infty} n\,\alpha_n \to 0 \text{ for } N \to \infty.$$

This entails

$$\lim_{N\to\infty} \frac{N\bar{s}_N}{t_N + N^2 r_N} = \infty, \tag{10'}$$

hence, by our initial assumption,

$$\lim \frac{\bar{s}_N}{|s_N|} = \infty. \tag{10''}$$

In particular, $E(\mu) = 0$. For $m_2(\mu) < \infty$, (9) now follows from the remark above or from [BJR], Prop. 1.9 (2), but we give yet another argument. Since $x(t) = \sum_{n=0}^{\infty} \alpha_n \cos(2\pi nt)$, we have $x(t)^2 + \bar{y}(t)^2 \le 1$, hence $\bar{y}(t)^2 \le 1 - x(t)^2 \le 2\,(1-x(t))$.

As in the last part of the proof of Proposition 1, for $|t| < \frac{1}{2N}$ we get

$$|y(t)| \le 2\pi\,|s_N t| + O(t^2 t_N + r_N)$$

and similarly

$$|\bar{y}(t)| \ge 2\pi\,|\bar{s}_N t| - O(t^2 t_N + r_N).$$

For $\frac{1}{4N} \le t < \frac{1}{2N}$, it follows from (10′) that $t^2 t_N + r_N = o(\bar{s}_N t)$, hence by (10″)

$$\frac{|y(t)|}{|\bar{y}(t)|} \le \frac{|s_N t| + o(\bar{s}_N t)}{|\bar{s}_N t| - o(\bar{s}_N t)} = o(1) \qquad \text{for } t \to 0 \text{ (which implies } N \to \infty).$$

This gives (9). As a consequence,

$$1-|\hat{\mu}(t)| \sim \frac{1}{2}(1-|\hat{\mu}(t)|^2) \underset{(9)}{\sim} \frac{1}{2}\left(1-x(t)^2\right) \sim 1-x(t) \quad \text{for } t \to 0.$$

Now assume that $I(k)$ are intervals satisfying

$$\lim_{k\to\infty} \frac{|\hat{\mu}(I(k))|_\rho}{1-r_{I(k)}} = \infty.$$

In particular, $1-r_{I(k)} \to 0$. This gives $|I(k)| \to 0$. After passing to a subsequence and translating, we may assume that $I(k)$ tends to $\{0\}$ for $k\to\infty$. Put $r'_I = \inf\{x(t) : t \in I\}$. Then it follows from the above that

$$1-r_{I(k)} \sim 1-r'_{I(k)} \quad \text{for } k\to\infty.$$

If J denotes a subinterval of the circle, it is easy to see that

$$|J|_\rho \sim \frac{1}{2\pi}\sup\{|z_1-z_2| : z_1, z_2 \in J\} \quad \text{for } J \to \{1\}.$$

This gives,

$$|\hat{\mu}(I)|_\rho \sim \frac{1}{2\pi}\sup\{|\hat{\mu}(t_1)-\hat{\mu}(t_2)| : t_1, t_2 \in I\} + O(1-r_I) \quad \text{for } I \to \{0\}.$$

Since $|x(t_1)-x(t_2)| \le 2(1-r'_I)$, we get

$$|\hat{\mu}(t_1)-\hat{\mu}(t_2)| \le |y(t_1)-y(t_2)| + 2(1-r'_I) \quad \text{for } t_1, t_2 \in I.$$

Thus $|\hat{\mu}(I)|_\rho \sim \frac{1}{2\pi}|y(I)| + O(1-r'_I)$. This gives

$$\lim_{k\to\infty} \frac{|y(I(k))|}{1-r'_{I(k)}} = \infty \tag{11}$$

(where $|I|$ denotes the length of a real interval I). By symmetry, we can assume that $I(k) \subseteq [0,\infty[$, hence $I(k) \subseteq [0,\frac{1}{3}[$ for large k.

We proceed with an estimate for $1-r'_I$. We assume that $I=[u,v]$ with $0 \le u < v \le \frac{1}{3}$. Put $N = \left[\frac{1}{|I|}\right]$ (where $[s]$ denotes the integer part of $s \in \mathbb{R}$). Define

$$\rho_N = \frac{1}{N^2}t_N + \sum_{|n|\le N}\langle nv\rangle^2 a_n + r_N \quad \text{and} \quad \beta_n = \frac{1}{|I|}\int_I \langle nt\rangle^2 dt = \frac{1}{|nI|}\int_{nI}\langle s\rangle^2 ds.$$

If $|n|>N$, we have $|nI| \ge 1$, hence (by elementary computations) $\beta_n \ge \frac{1}{16}$. For all n, we have $\beta_n \ge \frac{1}{4}\langle nv\rangle^2$ and for $|n| \le N$ we have the estimate $\beta_n \ge \frac{1}{12}(n\,|I|)^2 \ge \frac{1}{48}\left(\frac{n}{N}\right)^2$ (using $|nI| \le 1$ and $|I| > \frac{1}{2N}$). This gives

$$\frac{1}{|I|}\int_I \sum a_n \langle nt\rangle^2 dt \ge c_1\rho_N$$

(for some $c_1 > 0$ independent of N, I; one could take $c_1 = \frac{1}{48}$). Using (2), we conclude that

$$1 - r'_I \geq 8 \sup_{t \in I} \sum a_n \langle nt \rangle^2 \geq 8 c_1 \rho_N. \tag{12}$$

Next, we want to estimate $|y(I)|$. By (1) and (2), for $t_1, t_2 \in I$ we have

$$|y(t_1) - y(t_2)| = 2\pi \left| \sum a_n (\langle nt_1 \rangle - \langle nt_2 \rangle) \right| + O(1 - r'_I).$$

Put $M = [\frac{N}{3}]$ (observe that $N \geq 3$) and assume $t_1 < t_2$. If $t \mapsto \langle nt \rangle$ has no jump between t_1 and t_2, then $\langle nt_1 \rangle - \langle nt_2 \rangle = n\,(t_1 - t_2)$. If $|n| \leq M$, then $|nI| \leq \frac{|n|}{N} \leq \frac{1}{3}$. Thus, if there is a jump between t_1 and t_2, then $|\langle nv \rangle| \geq \frac{1}{6}$, hence

$$\langle nt_1 \rangle - \langle nt_2 \rangle - n\,(t_1 - t_2) = 1 = O(\langle nv \rangle^2).$$

Since $\sum_{|n| \leq M} \langle nv \rangle^2 a_n \leq \rho_N \underset{(11)}{=} O(1 - r'_I)$, this gives

$$|y(t_1) - y(t_2)| = 2\pi |s_M (t_1 - t_2)| + O(1 - r'_I).$$

Since $|I| \leq \frac{1}{N} \leq \frac{1}{M}$, we get

$$|y(I)| \leq \frac{2\pi}{M} |s_M| + O(1 - r'_I). \tag{13}$$

Now it follows from (11) and (13) that

$$\frac{\frac{1}{M}|s_M|}{1 - r'_{I(k)}} \to \infty \text{ for } k \to \infty,$$

hence by (12), $\frac{1}{M}|s_M| / \rho_N \to \infty$ (where $N = N(k) = [\frac{1}{|I(k)|}]$ and $M = M(k) = [\frac{N(k)}{3}]$). But since $M \geq \frac{N}{4}$, we have $\rho_N \geq \frac{1}{16}(\frac{1}{M2} t_M + r_M)$, and we arrive at a contradiction to our initial assumption.

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