ILLINOIS JOURNAL OF MATHEMATICS
Volume 44, Number 2, Summer 2000

THE SEARCH FOR TRIVIAL TYPES

TRACEY MCGRAIL

ABSTRACT. In this paper, we look at strongly minimal sets definable in a differentially closed field of characteristic 0. In [3], Hrushovski and Sokolović show that such sets are essentially Zariski geometries. Thus either thre is a definable strongly minimal field nonorthogonal to D, or D is locally modular and nontrivial, or D is trivial. We show that the strongly minimal sets defined by a certain family of differential equations are trivial. We also prove a theorem wich provides a test for the orthogonality of types over an ordinary differential field.

1. Introduction

In this paper, we look at strongly minimal sets in the theory DCF_0. In Section 2, we use a criterion due to Rosenlicht [7] to determine whether or not each member of a family of strongly minimal sets is orthogonal to the constants. A consequence of this result is tat orthogonality to the constants is not definable in a differentially closed field.

Hrushovski and Sokolović [3] show that the strongly minimal sets D definable in differentially closed fields are essentially Zariski geometries. Thus either there is a definable strongly minimal field nonorthogonal t D, or D is locally modular and nontrivial, or D is trivial. In the first case, the field is known to be definably isomorphic to the field of constants. In the second case, there is a definable strongly minimal group G nonorthogonal to D. In Section 3, we use the criterion mentioned above, along with results related to the Manin kernel of a simple abelian variety to produce a family of trivial sets.

In Section 4, we state a generalization of a theorem due to Rosenlicht [7] that gives us a nice test for the orthogonality of types.

All rings and fields in this paper have characteristic 0. See Marker's account of the model theory of differential fields [6] for notation and background. Let $\mathcal{M}$ be a monster model of DCF_0; i.e., $\mathcal{M}$ is a sufficiently saturaed differentially closed field of characteristic 0 and any differential field, k, that we mention may be considered a (small) elementary substructure of $\mathcal{M}$. k may be differentially closed. We denote the constant field of k by C_k and the deivation by δ. For $\bar{a} \in \mathcal{M}^n$, $k\langle\bar{a}\rangle$ denotes the differential field generated by $\bar{a}$ over k.

Whenever we say that p is orthogonal to the constants, we mean that p is orthogonal to the generic type (over $\emptyset$) of the formula $\delta(y) = 0$. By a generic realization or solution of an equation of the form $f(\bar{y}) = 0$ over k, where $f \in k\{\bar{y}\}$, we mean

Received July 27, 1998; received in final form October 2, 1999.
1991 Mathematics Subject Classification. Primary 03C60; Secondary 12H05.

$\bar{a} \in \mathcal{M}^n$ such that $f(\bar{a}) = 0$ and the transcendence degree of $k\langle\bar{a}\rangle/k$ is maximal for such solutions. Similarly, $\bar{a}$ is a generic point of the definable set $X \subseteq \mathcal{M}^n$ over k if $\bar{a} \in X$ and he transcendence degree of $k\langle\bar{a}\rangle/k$ is maximal for any such $\bar{a}$.

2. Orthogonality to the constants

We study a family of differential equations and a test for determining whether or not the constant field of the associated differential extension extends the original constant field.

In particular, we look at types over $\mathcal{C}_k$ which correspond to simple transcendental extensions over k. In other words, we are interested in types over $\mathcal{C}_k$, such that if $a \in \mathcal{M}$ is a realization of p, then the differential field extension $klaa\rangle$ has algebraic transcendence degree one over k and is, in fact, equal to the field extension $k(a)$.

Let us look at some examples of such extensions. More details are in Kolchin [5].

Example 2.1. Let k be a differential field. Let $a \in \mathcal{M}\backslash k$ be such that $\delta a = c \in k$. We say that a is *primitive over* k. Of course, in this case, $k\langle a\rangle = k(a)$. If k is differentially closed, we can choose $b \in k$ such that $\delta b = c$. Then $\delta(a - b) = 0$ and $a - b \in \mathcal{C}_{k\langle\alpha\rangle}\backslash\mathcal{C}_k$. Thus, $\mathcal{C}_k \neq \mathcal{C}_{k\langle a\rangle}$.

Example 2.2. Let k be a differential field. Let $a \in \mathcal{M}\backslash k$ such that $\delta a = ca$ for some $c \in k$. We say that a is *exponential over* k. Again, $k\langle a\rangle = k(a)$. If k is differentially closed, we can choose $b \in k$ such that $\delta = cb$. Then $\delta(\frac{a}{b}) = 0$ and $\frac{a}{b} \in \mathcal{C}_{k\langle\alpha\rangle}\backslash\mathcal{C}_k$. Thus, $\mathcal{C}_k \neq \mathcal{C}_{k\langle a\rangle}$.

Example 2.3. Let k be a differential field. Let $g_2, g_3 \in \mathcal{C}_k$ be such that the polynomial $4y^3 - g_2y - g_3$ has simple roots only and let $c \in k$. Let $a \in \mathcal{M}\backslash k$ be such that $(\delta a)^2 = c(4a^3 - g_2a - g_3)$. We say that a is *Weierstrassan over* k. If k is differentially closed, Kolchin [5, pages 405–407] shows that $\mathcal{C}_{k\langle a\rangle} \neq \mathcal{C}_k$.

We will use the following two lemmas as a test for orthogonality of types.

LEMMA 2.4 [6]. *Let K be a differentially closed field. Let $p, q \in S_1(K)$ be such that $p \perp q$. If $a \in \mathcal{M}$ realizes p, and F is the differential closure of $K\langle a\rangle$, then q is not realized in F.*

LEMMA 2.5 [6]. *Suppose $F \supset K$ are differentially closed, $\varphi(v)$ is a formula with parameters from K and every element of F that satisfies $\varphi(v)$ is already in K. Let $a \in F\backslash K$, let $p = tp(a/K)$ and let $q \in S_1(K)$ be a type containing $\varphi(v)$. Then $p \perp q$.*

We denote the differential closure of K by $\widetilde{K}^{\text{diff}}$ and the algebraic closure of K by $\widetilde{K}^{\text{alg}}$.

PROPOSITION 2.6. *Suppose K is a differentially closed field. Let $a \in \mathcal{M}$ and $p = tp(a/K)$. Then p is orthogonal to the constants if and only if $\mathcal{C}_{K<a>} = \mathcal{C}_K$.*

Proof. First suppose that p is orthogonal to the constants. Then certainly by Lemma 2.4, $\mathcal{C}_{K<a>} = \mathcal{C}_K$.

Now suppose that p is not orthogonal to the constants. Then by Lemma 2.5, there is some $c \in \mathcal{C}_{\widetilde{K\langle a\rangle}^{\text{diff}}} \setminus \mathcal{C}_K$. But $\mathcal{C}_{\widetilde{K\langle a\rangle}^{\text{diff}}} = \mathcal{C}_{\widetilde{K\langle a\rangle}^{\text{alg}}}$. Since $\mathcal{C}_K$ is algebraically closed, c is transcendental over $\mathcal{C}_K$, but algebraic over $\mathcal{C}_{K<a>}$. So $\mathcal{C}_{K<a>} \neq \mathcal{C}_K$. $\square$

We will now describe a family of strongly minimal sets which are nonorthogonal to the field of constants. Examples 2.1 and 2.2 fall into this family.

Remark 2.7. We will use the following representation of rational polynomials, $g(y)$, in one variable over an algebraically closed field k. Using partial fractions, we can write $g(y)$ as a sum of polynomials of the following form:

$$g(y) = \sum_{i=1}^{m} c_i \frac{\frac{\partial u_i}{\partial y}}{u_i} + d\frac{\partial v}{\partial y} \tag{2.1}$$

where the u_i and v are rational polynomials in $k(y)$. Moreover, denoting $\frac{\partial u_i}{\partial y}$ by $u'(y)$ or just u', and $\frac{\partial v}{\partial y}$ by $v'(y)$ or just v', we can put restrictions on the coefficients c_i in (2.1):

$$-\frac{u'}{u} = \frac{\left(\frac{1}{u}\right)'}{\frac{1}{u}} \tag{2.2}$$

$$\frac{u'}{u} + \frac{v'}{v} = \frac{vu' + v'u}{uv} = \frac{(uv)'}{uv}. \tag{2.3}$$

Notice that by (2.2) and (2.3), we can choose to write (2.1) with the c_i linearly independent over $\mathbb{Q}$.

THEOREM 2.8. *Let K be a differentially closed field. Suppose that $f(y)$ is a rational function in $\mathcal{C}_K(y)$ such that $\frac{1}{f(y)}$ is of the form $c\frac{\frac{\partial u}{\partial y}}{u}$ or $c\frac{\partial v}{\partial y}$ where u and v are in $\mathcal{C}_K(y)$and $c \in \mathcal{C}_K$. Let $X = \{a \in \mathcal{M} : \delta a = f(a)\}$. Then X is a strongly minimal set and is nonorthogonal to the field of constants $\mathcal{C}_{\mathcal{M}}$.*

Proof. The equation $\delta y = f(y)$ is the defining formula of X. Since $\mathcal{M}$ is sufficiently saturated, X contains a point, a, not algebraic over K; so X must have Morley rank 1. Moreover, the equation $\delta y = f(y)$ is linear in δy. Therfore the Morley degree of Σ must be 1. Hence, X is strongly minimal.

Now, to show that X is nonorthogonal to the constants $\mathcal{C}_{\mathcal{M}}$, by Proposition 2.6, it is enough to show that $\mathcal{C}_{K\langle a\rangle} \neq \mathcal{C}_K$ for a a generic point of X over K. Let a

be such a point. Since $\delta a = f(a)$, $K\langle a\rangle = K(a)$. Choose $b \in K\backslash \mathcal{C}_K$ satisfying $\delta b = f(b)$. (We can find such a b because K is a differentially closed field.) Consider the following two cases.

Case 1. Suppose $\frac{1}{f(y)} = c\frac{\frac{\partial u}{\partial y}}{u}$ where $u \in \mathcal{C}_K(y)$ and $c \in \mathcal{C}_K$. If $u(b) = 0$, then since $u(y) \in \mathcal{C}_K(y)$ and $\mathcal{C}_K$ is algebraically closed, $b \in \mathcal{C}_K$. But we chose $b \in K\backslash \mathcal{C}_K$, so$u(b) \neq 0$. The same argument shows that $u'(b) \neq 0$. Hence we can consider the derivative of $\frac{u(a)}{u(b)}$:

$$
\begin{aligned}
\delta\left(\frac{u(a)}{u(b)}\right) &= \frac{u'(a)(\delta a)u(b) - u(a)u'(b)(\delta b)}{(u(b))^2} \\
&= \frac{u'(a)\left(\frac{u(a)}{cu'(a)}\right)u(b) - u(a)u'(b))\left(\frac{u(b)}{cu'(b)}\right)}{(u(b))^2} \\
&= \frac{u(a)u(b) - u(a)u(b)}{c\,(u(b))^2} \\
&= 0.
\end{aligned}
$$

Now, $\frac{u(a)}{u(b)} \notin K$ since a is transcendental over K. Hence, $\frac{u(a)}{u(b)} \in \mathcal{C}_{K\langle a\rangle}\backslash \mathcal{C}_K$.

Case 2. Suppose $\frac{1}{f(y)} = c\frac{\partial v}{\partial y}$ where $v \in \mathcal{C}_K(y)$ and $c \in \mathcal{C}_K$. Then

$$
\begin{aligned}
\delta\,(v(a) - v(b)) &= v'(a)(\delta a) - v'(b)(\delta b) \\
&= v'(a)\frac{1}{cv'(a)} - v'(b)\frac{1}{cv'(b)} \\
&= \frac{1}{c} - \frac{1}{c} \\
&= 0.
\end{aligned}
$$

Since a is transcendental over K, $v(a) - v(b) \notin K$. Hence $v(a) - v(b) \in \mathcal{C}_{K\langle a\rangle}\backslash \mathcal{C}_K$. □

The following theorem of Rosenlicht [7] says that if $f \in \mathcal{C}_K(y)$ is not of the form stated in Theorem 2.8, then we do not get new constants in the associated extension field. For a nic treatment of this theorem see [6].

THEOREM 2.9 (Rosenlicht). *Let K be a differential field such that $\mathcal{C}_K$ is algebraically closed. Let $f(y) \in \mathcal{C}_K(y)$ and let a be a solution of the differential equation $\delta(y) = f(y)$, where a is transcendental over k. Suppose that $\frac{1}{f(y)}$ is not of te form $c\frac{\frac{\partial u}{\partial y}}{u}$ or $c\frac{\partial v}{\partial y}$ where $u, v \in \mathcal{C}_K(y)$, $c \in \mathcal{C}_K$. Then $\mathcal{C}_{K\langle a\rangle} = \mathcal{C}_K$.*

By Proposition 2.6, Theorem 2.9 is sufficient to show that the type of a generic realization of such an equation over K is orthogonal to the field of constants.

As another application of Theorems 2.8 and 2.9, Marker noted the following consequence.

COROLLARY 2.10. *Let K be differencially closed. Let $X_c = \{a : \delta a = f(a)\}$ where $\frac{1}{f(y)} = \frac{c}{y} + \frac{1}{y-1}$ and $c \in C_K$. Then X_c is orthogonal to the field of constants if and only if $c \in \mathbb{Q}$. (Orthogonality to the constants is not defiable.)*

Proof. Let X_c be as above. If $c \in \mathbb{Q}$, then, by Remark 2.7, we can write $\frac{1}{f(y)}$ in the form $c' \frac{\frac{\partial u}{\partial y}}{u}$ where $c' \in \mathbb{Q}$. Then by Theorem 2.8, X_c is nonorthogonal to the constants. However, if $c \notin \mathbb{Q}$, then Theorem 2.9 applies and X_c is orthogonal to the constants. So X_c is orthogonal to the constants if and only if $c \notin \mathbb{Q}$. However, DCF_0 is a decidable theory of fields. It is thus impossible to define $\mathbb{Q}$, since the field $\mathbb{Q}$ has an undecidable theory. Hence the notion of orthogonality to the constants is not definable over a differentially closed field. □

In contrast to Corollary 2.10, Hrushovski and Itai [2] have shown that orthogonality to a strongly minimal set X is a definable property of parameters when X lies (in an essential way) on curve of genus ≥ 1.

3. A family of trivial types

Let K be a differentially closed field. We show that an equation of the form $\delta(y) = f(y)$ where $\frac{1}{f(y)}$ is not of the form $c\frac{\frac{\partial u}{\partial y}}{u}$ or $c\frac{\partial v}{\partial y}$ where $u, v \in C_K(y)$, $c \in_{\mathcal{K}}$, defines a trivial strongly minimal set.

Definition 3.1. Let $p(\bar{x})$ be a complete type (in finitely many variables) of finite Morley rank over a differential field k. Let $\bar{a}$ realize p. Then $k\langle\bar{a}\rangle$ has finite transcendence degree over k. Call this number $RD(p)$.

Remark 3.2. For $p(x)$ a 1-type, $RD(p)$ is the order of the minimal polynomial associated to p.

For D a strongly minimal set defined over k there is a unique nonalgebraic complete type $p(\bar{x})$ over k containing the formula $\bar{x} \in D$. In this case, RD(p) coincides with what is called the absolute dimension of D.

THEOREM 3.3. *Let G be a strongly minimal group defined over a differentially closed field K. Let $p(\bar{x})$ be the unique nonalgebraic type containing the formula $\bar{x} \in G$. Then, if G is orthogonal to the constants, $RD(p) > 1$.*

Theorem 3.3 is due to Buium, modulo some facts about differential algebraic groups: The model theory of groups definable in differentially closed fields implies that G definably embeds in a simple abelian variety Awhich does not descend to the contants. Then G is equal to what Buium calls $A^{\#}$. We will give another proof of Theorem 3.3 using a differential Galois-theoretic result of Kolchin.

Definition 3.4. Let $L \supset K$ be differential fields. A *differential automorphism of L over K* is an automorphism of L which commutes with the derivative δ, and fixes the elements of K. The *galois group* of L over K, $\mathrm{gal}(L/K)$, is the setof differential automorphisms of L over K.

THEOREM 3.5 (Kolchin [4]). *Let $L \supset K$ be differential fields. Let L be a function field of transcendence degree* 1 *over K such that K is algebraically closed in L and $C_L = C_K$. Suppose that* $\mathrm{gal}(L/K)$ *is infinite. Then there exists an element $d \in L$ such tha either d is primitive over K and $L = K\langle d\rangle$, or d is exponential over K and $L = K\langle d\rangle$, or d is Weierstrassian over K and L is an algebraic extension of $K\langle d\rangle$. In the last case, if K is algebraically closed the the Weierstrassian element d may be chosen so that $L = K\langle d\rangle$.*

Proof of Theorem 3.3. Let G and p be as stated in the theorem. Suppose that G is orthogonal to the constants. Suppose that $RD(p) = 1$. Let g be a generic point of G over K and let $L = K\langle g\rangle$. By Proposition 2.6, $C_L = C_K$. Then, by Theorem 3.5 and Examples 2.1-2.3, $\mathrm{gal}(L/K)$ is finite. But, on the other hand, we can show that $\mathrm{gal}(L/K)$ must be infinite. Since $K \models$ DCF,$K \cap G$ is infinite. Let $c \in K \cap G$. Then the sum $g + c$ in the operation of the group G is again a generic point over K of G in L. So, for each element of $K \cap G$, there is a corresponding differential automorphism of L over K, making $\mathrm{gal}(L/K)$ infinite. Hence, L cannot have transcendence degree 1 over K. $\square$

THEOREM 3.6. *If $RD(p) = 1$ and p is orthogonal to the constants, then p is trivial.*

Proof. If not, then by the trichotomy theorem, p is locally modular and nontrivial. Hence, p is nonorthogonal to the generic type q of a strongly minimal group. Then q must be orthogonal to the constants (since p is)and $RD(q) = 1$. This contradicts Theorem 3.3. $\square$

We are now ready to show that strongly minimal sets defined over a differentially closed field K by an equation of the form $\delta y = f(y)$ are trivial whenever $f(y) \in C_K(y)$ and $\frac{1}{f(y)}$ is of a special form.

COROLLARY 3.7. *Let K be a differentially closed field. Let $f(y) \in C_K(y)$ be such that $\frac{1}{f(y)}$ is not of the form $\frac{\frac{\partial u}{\partial y}}{u}$ or $\frac{\partial v}{\partial y}$ where u, and $v \in C_K(y)$. Let p be the type of a geeric solution to the equation $\delta y = f(y)$ over K. Then p is trivial.*

Proof. Clearly, $RD(p) = 1$. Proposition 2.6 and Theorem 2.9 imply that p is orthogonal to the constants. So, by Theorem 3.6, p is trivial. □

With Theorem 2.8 and Corollary 3.7 we have completely described the strongly minimal sets defined over a differentially closed field by an equation of the form $\delta y = f(y)$ where $f(y)$ is a rational polynomia in one variable over the constant field.

Hrushovski [1] has observed that a result of Jouanoulou can be extended to show that a strongly minimal sets whose generic type over a differentially closed field has RD 1, is either nonorthogonal to the constants or ω-categoical. It is well known that an ω-categorical strongly minimal set is orthogonal to the constants. See [8].

4. Orthogonality

In this section, we are interested in questions related to the orthogonality of types. We would like to know what other differential equations we are solving when we take a differential field extension. The following theorem gives us a test for orthogonality of types of a very special form.

THEOREM 4.1. *Let $k \subset K$ be differential fields with C_K algebraic over C_k. Let f_1 and $f_2 \in C_k(y)$ be of the form*

$$\frac{1}{f_1(y)} = \sum_{j=1}^{n_1} c_{1,j} \frac{\frac{\partial u_{1,j}}{\partial y}}{u_{1,j}} + \frac{\partial v_1}{\partial y}$$

and

$$\frac{1}{f_2(y)} = \sum_{j=1}^{n_2} c_{2,j} \frac{\frac{\partial u_{2,j}}{\partial y}}{u_{2,j}} + \frac{\partial v_2}{\partial y}$$

with $c_{1,j}$ and $c_{2,j} \in C_k$ and $u_{1,j}$, $u_{2,j}$, v_1, and $v_2 \in C_k(y)$.

Let a_1 and $a_2 \in K$ satisfy $\delta a_i = b_i f_i(a_i)$, with b_1 and $b_2 \in k$. Then, if a_1 and a_2 are algebraically dependent over k, then each a_i is algebraic over k or $b_2\delta(v_1(a_1)) = b_1\delta(v_2(a_2))$.

Remark 4.2. Theorem 4.1 is a generalization of a theorem due to Rosenlicht. The proof of Theorem 4.1 is essentially the same as the proof of Rosenlicht's version, which can be found in [6, page 77]. Simpy replace each f with f_1 or f_2, each u_j with $u_{1,j}$ or $u_{2,j}$, and each v with v_1 or v_2 where appropriate.

LEMMA 4.3. *Let K be a differentially closed field. Let $p, q \in S_1(K)$, and suppose that q is strongly minimal. Suppose $a \in \mathcal{M}$ realizes p, and F is the algebraic closure of $K\langle a\rangle$. Suppose that p is nonorthogonal to q. Then q is realzed in F.*

Example 4.4. This example studies a family $\mathcal{F}$ of types over $\mathcal{C}_k$, where k and K are as in the statement of Theorem 4.1. If $p \in \mathcal{F}$, be the type of a generic realization over k of the differential equation $\delta y = \frac{y}{y+m}$ where $m \in \mathbb{Z}$ and $m \neq 0$. We show that if p and q are distinct types in $\mathcal{F}$, then they must be orthogonal by Theorem 4.1.

Recall that since the types in question are strongly minimal, by Corollary 4.3, if $tp(a_1/k)$ is nonorthogonal to $tp(a_2/k)$, then a_1 and a_2 are algebraically dependent over k.

Let $f_1(y) = \frac{y}{y+m_1}$ and $f_2(y) = \frac{y}{y+m_2}$. Then

$$\frac{1}{f_i(y)} = \frac{\frac{\partial u_i}{\partial y}}{u_i} + \frac{\partial v_i}{\partial y}$$

where $u_i = y^{m_i}$ and $v_i = y$. where $m_i \in \mathbb{Z}$, $m_1 \neq 0$ and $m_2 \neq 0$. Let $a_1 \in K$ be a generic solution over k to the differential equation $\delta y = f_1(y)$, and let $a_2 \in K$ be a generic solution over k to the differential equation $\delta y = f_2(y)$.

Suppose that a_1 and a_2 are algebraically dependent over k but neither is algebraically dependent over k. Hence by Theorem 4.1,

$$\delta(v_1(a_1)) = \delta(v_2(a_2)).$$

Since $\delta(v_1(a_1)) = \delta(a_1)$ and $\delta(v_2(a_2)) = \delta(a_2)$,

$$\begin{aligned} \frac{a_1}{a_1+m_1} &= \frac{a_2}{a_2+m_2} \\ a_1a_2 + m_2a_1 &= a_1a_2 + m_1a_2 \\ m_2a_1 &= m_1a_2 \\ \frac{m_2}{m_1}a_1 &= a_2. \end{aligned}$$

Hence,

$$\delta a_2 = \frac{m_2}{m_1}\delta a_1 = \frac{m_2}{m_1}\left(\frac{a_1}{a_1+m_1}\right) = \frac{m_2a_1}{m_1a_1+m_1^2}.$$

But we also have

$$f_2(a_2) = f_2\left(\frac{m_2}{m_1}a_1\right) = \frac{\frac{m_2}{m_1}a_1}{\frac{m_2}{m_1}a_1 + m_2} = \frac{m_2a_1}{m_2a_1+m_1m_2}.$$

Since $\delta a_2 = f_2(a_2)$,

$$\frac{m_2a_1}{m_1a_1+m_1^2} = \frac{m_2a_1}{m_2a_1+m_1m_2}.$$

Since a_1 is not algebraically dependent over k, it must be the case that $m_1 = m_2$. Therefore, if we assume that $m_1 \neq m_2$, then $tp(a_1/k)$ is orthogonal to $tp(a_2/k)$.

References

[1] Ehud Hrushovski, *ODE's of order* 1 *and a generalization of a theorem of Jouanolou*, preprint.
[2] Ehud Hrushovski and M. Itai, *On model complete differential fields*, preprint, 1997.
[3] Ehud Hrushovski and Željko Sokolović, *Minimal subsets of differentially closed fields*, Trans. Amer. Math. Soc., to appear.
[4] E. R. Kolchin, Galois theory of differential fields. *Amer. J. Math.*, **75** (1953), 753–824.
[5] ________, Differential algebra and algebraic groups, Academic Press, New York, 1973.
[6] David Marker, "Model theory of differential fields" in *Model theory of fields*, Springer-Verlag, New York, 1996, pp. 38–113.
[7] M. Rosenlicht, *The nonminimality of the differential closure*, Pacific J. Math. **52** (1974), 529–537.
[8] Ž. Sokolovic, *Model theory of differential fields*, Ph.D. dissertation, Notre Dame, Indiana, 1992.

Department of Mathematics, Marist College, Poughkeepsie, NY 12601
tracey.mcgrail@marist.edu