ON THE CESARI-CAVALIERI INEQUALITY

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The present paper concerns the inequality proved by L. Cesari in 1951 [3], [4], [7] relating the Lebesgue area of a continuous surface S (as a continuous image of a simple closed Jordan region) and the generalized length of the related sets of contours. Much research has followed this initial work (see bibliography for some of the references). The inequality (successively denoted as the Eilenberg inequality, the Cavalieri inequality, and finally the Cesari-Cavalieri inequality, see [12]) is based on a detailed study of properties of Carathéodory ends and prime-ends of open plane sets. L. Cesari and J. Cecconi applied this inequality to surface area theory and the calculus of variations. R. E. Fullerton extended the notion of generalized length and the inequality to mappings from any compact two-manifold with or without boundary [14], [15].

In [7] the inequality was given only for mappings from simple closed Jordan regions, while most of the area theory there was developed for mappings from admissible plane sets (including among others all open sets and all finitely connected Jordan regions). In [8] we showed the need of proving the Cesari-Cavalieri inequality for mappings from all admissible sets. It is the purpose of this paper to obtain this extension. To this end we shall use the familiar process of invading the admissible sets with Jordan regions, and make use of results of R. E. Fullerton in [14], [15]. This in turn requires a preliminary and subtle analysis, which is new, of a monotone relationship of Carathéodory ends and prime-ends for sets $\alpha \subset \alpha'$, open relative to finitely connected closed Jordan regions $J \subset J'$. We dedicate Section 2 to this task. In Section 3 we then define generalized length for mappings from an admissible plane set, and in Section 4 we prove the Cesari-Cavalieri inequality for continuous mappings from admissible plane sets.

1. Preliminary discussion

We shall discuss below Carathéodory ends and prime-ends for certain classes of sets. For clarity of exposition we shall proceed by steps. In 1.1, simply connected open sets of the plane E_2 are considered. In 1.2, connected sets which are open in a finitely connected Jordan region J and whose boundary relative to J is connected are discussed. Finally, in 1.3, connected sets open in a finitely connected Jordan region are considered.

If X is a subset of the plane, then X^*, $\bar{X}$, X^0 will denote respectively the boundary, closure, and interior of X.

1.1. *Simply connected open sets in* E_2.

We summarize here some basic concepts as given in [7] and [16].

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Let α be a bounded open set in the plane E_2 with a connected boundary. Then α is a simply connected open set. An arc b is said to be an *end-cut* of α if

$$b \cap \alpha^* = \{w\} \quad \text{and} \quad b \subset \alpha \cup \{w\},$$

where w is an end-point of b. An arc b is said to be a *cross-cut* of α if

$$b \cap \alpha^* = \{w_1, w_2\} \quad \text{and} \quad b \subset \alpha \cup \{w_1, w_2\},$$

where w_1 and w_2 are end-points of b. A point $w_0 \in \alpha^*$ is said to be *accessible* from α if there exists an end-cut b of α such that $b \cap \alpha^* = \{w\}$. From [17, p. 162] it follows that the set of all points of α^* accessible from α forms an uncountable dense set in α^*. For every cross-cut b of α, $\alpha - b$ is open and the union of exactly two components α_1 and α_2 each of which is simply connected.

We shall now define an equivalence relation on the set of all end-cuts of α and call each equivalence class an *end* η of α^*. Two end-cuts b and b' of α are said to be equivalent if

(1) b and b' have the same end-point $w \in \alpha^*$;

(2) either $b \cap b' \cap (V - w) \neq \emptyset$ for every neighborhood V of w, or there exist subarcs b_1 of b and b_1' of b' and a simple arc c such that

$$b_1 \cap b_1' = \{w\}, \qquad c \subset \alpha, \qquad c \cap b_1 = \{w_1\}, \qquad c \cap b_1' = \{w_1'\},$$

and the open Jordan region J whose boundary is $b_1 \cup b_1' \cup c$ is contained in α.

Let $\{\eta\}$ denote the family of all ends η of α^*. Let η_i $(i = 1, 2, 3, 4)$ be four distinct ends, and b_i $(i = 1, 2, 3, 4)$ any four end-cuts in the respective equivalence classes defined by the ends η_i $(i = 1, 2, 3, 4)$. Suppose that $b_i - \{w_i\}$ are all mutually disjoint, where w_i is the end-point $b_i \cap \alpha^*$, and suppose that b_1 and b_3 are connected by an arc c so that $b_1 \cup c \cup b_3$ forms a cross-cut and $c \cap b_2 = \emptyset = c \cap b_4$. The cross-cut $b_1 \cup c \cup b_3$ separates α into two components, and b_2 and b_4 may be in different components or in the same component. This property is independent of the end-cuts we choose in the respective equivalence classes η_i $(i = 1, 2, 3, 4)$ and the arc c above, and hence this is a property of the ends η_i $(i = 1, 2, 3, 4)$. If b_2 and b_4 lie in different components of $\alpha - (b_1 \cup c \cup b_2)$, then we say that η_1, η_3 *separates* η_2, η_4 in $\{\eta\}$ (and then η_2, η_4 separates η_1, η_3 in $\{\eta\}$). Therefore the collection $\{\eta\}$ can be cyclically ordered. If we denote by ∞ any one of the ends η of $\{\eta\}$, then, given any two distinct ends η_1 and η_2 different from ∞, by the *open interval* (η_1, η_2) in $\{\eta\}$ we mean the set of ends $\eta \in \{\eta\}$ such that η and ∞ separates η_1 and η_2 in $\{\eta\}$. By the *closed interval* $[\eta_1, \eta_2]$ we mean $(\eta_1, \eta_2) \cup \{\eta_1, \eta_2\}$.

Let $[\eta_n', \eta_n'']$ $(n = 1, 2, \cdots)$ be a nested sequence of closed intervals of $\{\eta\}$ such that at most one end is contained in all intervals (η_n', η_n'') and $[\eta_{n+1}', \eta_{n+1}''] \subset (\eta_n', \eta_n'')$ $(n = 1, 2, \cdots)$. The collection of all such nested sequences of intervals can be partitioned into equivalence classes in the usual way, and each equivalence class will be called a *prime-end* ω of α^*. Each end

$\eta \in \{\eta\}$ is a prime-end ω, but there may be prime-ends ω which are not ends η. The family $\{\omega\}$ of all prime-ends ω of α^* can be cyclically ordered by using the cyclic ordering on $\{\eta\}$.

For each $\eta \in \{\eta\}$ let w_η be the accessible point determined by η. Let ω be a fixed prime-end of α^*. Then E_ω is the set of all points w which have the following property:

There is a sequence η_k $(k = 1, 2, \cdots)$ of ends of α^* such that $\eta_k \in [\eta'_{n_k}, \eta''_{n_k}]$ $(k = 1, 2, \cdots)$ and $w_{\eta_k} \to w$, $n_k \to \infty$ as $k \to \infty$, where $[\eta'_n, \eta''_n]$ $(n = 1, 2, \cdots)$ is a sequence in the equivalence class determined by ω.

The set E_ω does not depend upon the particular sequence $[\eta'_n, \eta''_n]$ $(n = 1, 2, \cdots)$ of ω and is a subcontinuum of α^* [16, p. 109], [7, (19.3)]. If $\omega = \eta \in \{\eta\}$, then $w_\eta \in E_\omega$ but $E_\omega - \{w_\eta\}$ need not be empty.

Suppose α is an unbounded connected open set such that α^* is compact, connected, and nondegenerate. Then all the above discussion can be carried out with obvious modifications.

1.2. *Connected sets open in a Jordan region with connected boundary.*

In the discussion to follow, we shall adopt the following notation. Let X be a subspace of the plane, and let $A \subset X$. Then $B[A:X]$, $I[A:X]$, and $C[A:X]$ will denote the boundary, interior, and closure of A in the space X. If $J_0, J_1, \cdots, J_\nu$ are simple closed Jordan regions, $J_i \subset J_0^0$, $J_i \cap J_j = \emptyset$, $i \neq j$, $i, j = 1, \cdots, \nu$, we shall denote by $J = (J_0, J_1, \cdots, J_\nu)$ the finitely connected closed Jordan region $J = J_0 - (J_1^0 \cup \cdots \cup J_\nu^0)$. From now on, by a Jordan region we shall mean a finitely connected closed Jordan region.

Let A be a connected subset of a Jordan region $J = (J_0, J_1, \cdots, J_\nu) \subset E_2$ $(0 \leqq \nu < \infty)$ such that A is open in J and $B[A:J]$ is connected and nondegenerate. Clearly, $A^* \supset B[A:J]$, and A^* has only a finite number of components. A^* and $B[A:J]$ are related in two possible ways:

(a) $B[A:J]$ is a component of A^*.
(b) $B[A:J]$ is not a component of A^*.

Consider case (a). Denote $B[A:J]$ by γ. Then $A^* - \gamma$ is a union of components of $J^* = \bigcup_{i=0}^{\nu} J_i^*$. Since the components of A^* are compact and finite in number, we can discuss the ends and prime-ends of γ in exactly the same way as before in 1.1. The collection of ends $\{\eta\}$ and the collection of prime-ends $\{\omega\}$ of γ with respect to A is again cyclically ordered, and the set E_ω associated with each $\omega \in \{\omega\}$ is formed in the same way.

Consider case (b). Denote $B[A:J]$ by γ again, and let M be the component of A^* which contains γ. Then $A^* - M$ is a union of components of J^*. Since the components of A^* are compact and finite in number, we can discuss, as in case (a), ends and prime-ends of M with respect to $A - M$. $M - \gamma$ is a finite collection of open arcs contained in J^* [6, (2.v)], and every point of $M - \gamma$ is accessible from $A - M$. Let us denote by m_i $(i = 1, 2, \cdots, t)$ the finite number of open arcs of $M - \gamma$. It can be shown that the end-points of the arcs m_i are accessible from $A - M$. If $\{\eta\}_M$ is the collection of ends of M with respect to $A - M$, then $\{\eta\}_M$ is cyclically ordered, and to each m_i there corresponds an open interval $\mu_i = (\eta_i', \eta_i'')$ $(i = 1, 2, \cdots, t)$ [6, §3]. Hence $\{\eta\}_M$ is divided into $2t$ linearly ordered intervals, that is, μ_i and σ_i $(i = 1, 2, \cdots, t)$ where σ_i denotes the t closed intervals of $\{\eta\}_M - \bigcup_{i=1}^t \mu_i$. Let us suppose the indices are taken so that $\sigma_i < \mu_i < \sigma_{i+1} < \mu_{i+1}$ $(i = 1, \cdots, t-1)$.

(i) *Suppose* $B[A:J]$ *is of type* (a). *Then there exists a sequence of simple closed curves* l_n $(n = 1, 2, \cdots)$ *with the following properties:*

(1) $l_n \subset A$ *for all* n, *and* l_n *are mutually disjoint;*

(2) l_m *separates* γ *from* l_n, *where* $m > n$;

(3) $\lim_{n\to\infty} l_n = \bigcup E_\omega$, *where the union is taken over all prime-ends* ω *of* γ *with respect to* A.

(ii) *Suppose* $B[A:J]$ *is of type* (b). *Then there exist* t *sequences of arcs* $l_n^{(i)}$ $(n = 1, 2, \cdots)$, $(i = 1, 2, \cdots, t)$ *with the following properties:*

(1) $l_n^{(i)} \subset A$ *for all* n *and all* i, *and* $l_n^{(i)}$ *are all mutually disjoint;*

(2) $l_n^{(i)}$ *is a cross-cut in* $A - M$ *with one of its accessible points in* m_{i-1} *and the other in* m_i $(i = 1, 2, \cdots, t)$. (*We suppose* $m_0 = m_t$.)

(3) $\lim_{n\to\infty} l_n^{(i)} = \bigcup_{\omega\in\sigma_i} E_\omega$ $(i = 1, 2, \cdots, t)$, *where* σ_i *also denotes the collection of prime-ends associated with the interval* σ_i *defined above;*

(4) $l_n^{(i)}$ *separates* $l_m^{(i)}$ *from* $l_k^{(j)}$ *in* A, *where* $m > n$, $i \neq j$, $k = 1, 2, \cdots$, *and from* $l_k^{(i)}$, *where* $k < n < m$.

Proof. The proofs of (i) and (ii) above are established by invading A with Jordan regions J_n such that $J_n \subset I[J_{n+1}:J] \subset J_{n+1} \subset A$ $(n = 1, 2, \cdots)$ and $\bigcup_{n=1}^{\infty} J_n = A$.

1.3. *Connected sets open in a Jordan region.*

Let us consider a more general case than the one considered in 1.2 above. Let α be a connected set open in $J = (J_0, J_1, \cdots, J_\nu)$ $(0 \leqq \nu < +\infty)$. Then $B[\alpha:J]$ is compact but not necessarily connected. Let γ be a non-degenerate component of $B[\alpha:J]$, and let $A(\gamma, \alpha)$ be the component of $J - \gamma$ which contains α. Then $B[A(\gamma, \alpha):J] = \gamma$, and the discussion of 1.2 applies for γ and $A(\gamma, \alpha)$.

An end-cut b of γ with respect to $A(\gamma, \alpha)$ is said to be *admissible* if $b \cap \alpha$ has w as an accumulation point, where w is the accessible point of γ from $A(\gamma, \alpha)$ determined by b.

An end η of γ with respect to $A(\gamma, \alpha)$ is said to be *admissible* if η has an end-cut b_η which is admissible. An interval of ends $[\eta', \eta'']$ of γ with respect to $A(\gamma, \alpha)$ is said to be an *admissible arc* if each end $\eta \in [\eta', \eta'']$ is admissible. A cyclic collection of ends $\{\eta\}$ of γ with respect to $A(\gamma, \alpha)$ is said to be an *admissible cycle* if each end $\eta \in \{\eta\}$ is admissible.

(i) THEOREM. *If* η *is an admissible end of* γ *with respect to* $A(\gamma, \alpha)$, *then every end-cut* b_η *of* η *is an admissible end-cut* [14].

(ii) THEOREM. *Suppose γ is a component of A^*, where $A = A(\gamma, \alpha)$. Then every end η of γ with respect to $A(\gamma, \alpha)$ is admissible, and hence $\zeta = \{\eta\}_\gamma$ is an admissible cycle* [14].

(iii) THEOREM. *Suppose γ is not a component of A^*, where $A = A(\gamma, \alpha)$, and let σ_i $(i = 1, 2, \cdots, t)$ be defined as in 1.2 above. If σ_i has an admissible end η, then σ_i is an admissible arc* [14].

(iv) *Remark.* In 1.2, (i) we can also suppose that $l_n \subset \alpha$ $(n = 1, 2, \cdots)$ and in 1.2, (ii) we can suppose, for each i such that σ_i is an admissible arc, that $l_n^{(i)} \subset \alpha$ $(n = 1, 2, \cdots)$. Then for case (a), that is, γ is a component of A^* where $A = A(\gamma, \alpha)$, we have that l_m and l_n $(m > n)$ form an annular region H_{mn}, and for all n large enough $H_{mn} \subset A$ and $J - H_{mn} \supset \gamma$. If we consider case (b), that is, γ is not a component of A^*, and σ_i is an admissible arc, then for all n large enough $l_n^{(i)}$ and $l_m^{(i)}$ $(m > n)$ bound a simply connected set $H_{mn}^{(i)}$ in J such that $H_{mn}^{(i)} \subset A$, $J - H_{mn}^{(i)} \supset \gamma \cup \bigcup_{k>m} l_k^{(i)} \cup \bigcup_{k<n} l_k^{(i)}$, and $B[H_{mn}^{(i)}:J] = l_n^{(i)} \cup l_m^{(i)} \subset \alpha$.

(v) *Let β be a component of $J - \bar{\alpha}$, and b_η an admissible end-cut of γ with respect to $A(\gamma, \alpha)$. Then $b_\eta \cap \beta$ does not have w_η as an accumulation point.*

Proof. Let us consider case (b), that is, γ is not a component of A^*. Suppose $b_\eta \cap \beta$ has w_η as an accumulation point. Since b_η is an admissible end-cut, there exist m and n $(m > n)$ such that if $H_{nm}^{(i)}$ is the region defined in 1.3, Remark (iv), then $H_{nm}^{(i)} \cap b_\eta \cap \beta \neq \emptyset$, $B[H_{nm}^{(i)}:J] = l_n^{(i)} \cup l_m^{(i)} \subset \alpha$, and $B[H_{mn}^{(i)}:J]$ separates J. Since it is assumed that w_η is an accumulation point of $\beta \cap b_\eta$, we have $\beta \cap b_\eta \cap (J - H_{mn}^{(i)}) \neq \emptyset$. This implies that β is separated, a contradiction since β is a component of $J - \bar{\alpha}$. Hence $\beta \cap b_\eta$ does not have w_η as an accumulation point.

In case (a), (v) is established in a similar manner.

2. A monotone relationship on ends

Let J' and J be two Jordan regions, $J' \supset J$, and let α' be a connected subset of J' and open in J'. Then $J \cap \alpha'$ is open in J, and $J \cap \alpha' = \bigcup \alpha$, where α is a component of $J \cap \alpha'$ and the union is taken over all such components α. For each α and each nondegenerate component γ of $B[\alpha:J]$, the discussion of 1.3 applies. Hence for each α and γ, a nondegenerate component of $B[\alpha:J]$, we have either a finite number of admissible arcs σ_i or an admissible cycle ζ. $B[\alpha':J']$ need not be connected, but there does exist a nondegenerate component γ' of $B[\alpha':J']$ which contains γ. Let $A'(\gamma', \alpha')$ be that component of $J' - \gamma'$ which contains α'. $A'(\gamma', \alpha')$ need not contain $A(\gamma, \alpha)$; hence every end-cut b of γ with respect to $A(\gamma, \alpha)$ need not be an end-cut of γ' with respect to $A'(\gamma', \alpha')$.

(i) LEMMA. *If $\zeta = \{\eta\}_{\gamma,\alpha}$ is an admissible cycle of γ with respect to $A(\gamma, \alpha)$, then $\gamma = \gamma'$, $A(\gamma, \alpha) \subset A'(\gamma', \alpha')$, and the collection of all ends η of γ' with respect to $A'(\gamma', \alpha')$ is the same as ζ and hence an admissible cycle.*

Proof. γ is a component of A^*, where $A = A(\gamma, \alpha)$. The components of $E_2 - \bar{A}$ are of two kinds, those components V with $V^* \subset \gamma$ and those components W with $W^* \cap \gamma = \emptyset$. Clearly $\gamma' \cap V = \emptyset$, for otherwise γ would separate α'. Consequently $(\gamma' - \gamma) \cap (\gamma \cup \bigcup V) = \emptyset$, where $\bigcup V$ is the union of all components V with $V^* \subset \gamma$. Suppose $\gamma' - \gamma \neq \emptyset$. Then there exists a point $w_0 \in \gamma'$ such that $w_0 \notin \gamma \cup \bigcup V$. By 1.3, Remark (iv), there exists a simple closed curve l_n such that w_0 and γ are separated by l_n and $l_n \subset \alpha$. Hence γ' would be disconnected, a contradiction. Therefore $\gamma' = \gamma$. The remaining parts of the lemma now follow easily.

(ii) LEMMA. *Suppose $\sigma_i = [\eta', \eta'']$ is an admissible arc of γ with respect to $A(\gamma, \alpha)$. Suppose $\eta_0 \in (\eta', \eta'')$ and b_{η_0} is an end-cut of η_0 such that no subend-cut is contained in $A'(\gamma', \alpha')$. Then either every end $\eta \in \sigma_i$ with $\eta > \eta_0$ has no end-cut b_η with $b_\eta \subset A'(\gamma', \alpha')$, or every end $\eta \in \sigma_i$ with $\eta < \eta_0$ has no end-cut b_η with $b_\eta \subset A'(\gamma', \alpha')$.*

Proof. Since σ_i is an admissible arc, we have that η_0 is an admissible end, and, by 1.3, Theorem (i), b_{η_0} is an admissible end-cut of γ with respect to $A(\gamma, \alpha)$. Hence $\alpha \cap b_{\eta_0}$ has w_{η_0} as an accumulation point. Also, by hypothesis, b_{η_0} has no subend-cut which is contained in $A'(\gamma', \alpha')$, and hence $b_{\eta_0} \cap \gamma'$ has w_{η_0} as an accumulation point. Suppose the lemma is false. Then there exist two ends η_1 and η_2 in σ_i with $\eta_1 < \eta_0 < \eta_2$ and two corresponding end-cuts b_{η_1} and b_{η_2} such that

$$b_{\eta_1} \cap b_{\eta_2} \cap A(\gamma, \alpha) = \emptyset \quad \text{and} \quad b_{\eta_1} \cup b_{\eta_2} - \{w_{\eta_1}, w_{\eta_2}\} \subset A'(\gamma', \alpha') \cap A(\gamma, \alpha).$$

From 1.3, Remark (iv), there is a simply connected region $H_{mn}^{(i)}$ such that $b_{\eta_0} \cap \gamma'$ has a nonempty intersection with $I[H_{mn}^{(i)}:J]$ and both b_{η_1} and b_{η_2} have nonempty intersection with both $l_m^{(i)}$ and $l_n^{(i)}$, the components of $B[H_{mn}^{(i)}:J]$. Therefore, there is a simply connected region $B \subset J$ such that

$$B^* \subset l_m^{(i)} \cup l_n^{(i)} \cup b_{\eta_1} \cup b_{\eta_2}$$

and B contains a point of $b_{\eta_0} \cap \gamma'$. $B^* \cap \gamma' = \emptyset$ since

$$B^* \subset l_m^{(i)} \cup l_n^{(i)} \cup b_{\eta_1} \cup b_{\eta_2} - \{w_{\eta_1}, w_{\eta_2}\} \subset A'(\gamma', \alpha').$$

Since $w_{\eta_0} \notin B$, we have $\gamma' \cap (J - B) \neq \emptyset$. This implies γ' is disconnected, a contradiction. Hence Lemma (ii) is proved.

(iii) *Remark.* From Lemma (ii) above we see that at most one subarc $\tilde{\sigma}_i$ of an admissible arc σ_i of γ with respect to $A(\gamma, \alpha)$ is contained in an admissible arc σ' or an admissible cycle ζ' with respect to $A'(\gamma', \alpha')$. This subarc may be degenerate. Clearly, the admissible arc σ_i depends on the pair (γ, α). Let (γ_1, α_1) and (γ_2, α_2) be two distinct pairs, and let σ_1 and σ_2 be two admissible arcs of γ_1 and γ_2 with respect to $A(\gamma_1, \alpha_1)$ and $A(\gamma_2, \alpha_2)$, respectively. Suppose $\gamma_1 \cup \gamma_2 \subset \gamma'$, and suppose $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are subarcs of σ_1 and σ_2, respectively, which are contained in the same admissible arc σ' or admissible cycle

ζ' of γ' with respect to $A'(\gamma', \alpha')$. Then, from 1.3, (v) above, we see that $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are nonoverlapping in σ' or ζ'.

Let $\sigma_i = [\eta', \eta'']$ be an admissible arc of γ with respect to $A(\gamma, \alpha)$, and let $\tilde{\sigma}_i$ be the maximal collection of ends $\eta \in \sigma_i$ such that η is also an admissible end of an admissible arc σ' or cycle ζ' of γ' with respect to $A'(\gamma', \alpha')$. Since $\tilde{\sigma}_i$ is a subinterval of σ_i, $\sigma_i - \tilde{\sigma}_i$ will consist of the empty set, or of one proper subinterval of σ_i, or of two proper subintervals of σ_i, or of σ_i itself. Denote by λ_i any one of the nonempty intervals of $\sigma_i - \tilde{\sigma}_i$, and let $\Lambda_i = \{w_\eta : \eta \in \lambda_i\}$.

(iv) LEMMA. *$\Lambda_i \subset E_\omega$ for some prime-end ω of the admissible arc σ' or cycle ζ' of γ' with respect to $A'(\gamma', \alpha')$.*

Proof. We shall prove the lemma for the case $\lambda_i = \sigma_i$ and σ' is an admissible arc. The remaining cases are handled in a similar manner.

Let $\sigma_i = [\eta', \eta'']$, $b_1 = b_{\eta'}$, $b_2 = b_{\eta''}$, $w_1 = w_{\eta'}$, and $w_2 = w_{\eta''}$. As in 1.2, let m_i and m_{i-1} be the open arcs of $M - \gamma$ which correspond to the open arcs of ends μ_i and μ_{i-1}, where M is the component of A^*, $A = A(\gamma, \alpha)$, which contains γ. Since $\lambda_i = \sigma_i$, $\gamma' \cap A(\gamma, \alpha)$ is nonempty, and $\gamma' \cap A(\gamma, \alpha) \cap m_{i-1}$ has w_1 as a limit point, and $\gamma' \cap A(\gamma, \alpha) \cap m_i$ has w_2 as a limit point.

For σ_i and $A(\gamma, \alpha)$, there is, by 1.3, Remark (iv), a sequence of arcs $l_n^{(i)} \subset \alpha$ $(n = 1, 2, \cdots)$ with the properties: (a) $l_n^{(i)}$ has only its end-points in m_{i-1} and m_i $(n = 1, 2, \cdots)$; (b) $\lim_{n\to\infty} l_n^{(i)} = \bigcup_{\omega\in\sigma_i} E_\omega = \gamma$; and (c) $l_n^{(i)}$ separates, in $A(\gamma, \alpha)$, $l_m^{(i)}$ from $l_r^{(i)}$ $(m > n > r)$. Consider now $m_{i-1} \cap A'(\gamma', \alpha')$ and $m_i \cap A'(\gamma', \alpha')$. Each is a collection of open arcs $S^{(i)}$ and $S^{(i-1)}$, and each $l_n^{(i)}$ meets one of these arcs, say $S_n^{(i)}$ and $S_n^{(i-1)}$. Hence the closure of $l_n^{(i)} \cup S_n^{(i)} \cup S_n^{(i-1)}$ contains a cross-cut $b_n^{(i)}$ for the arc σ' of γ' with respect to $A'(\gamma', \alpha')$. Clearly, we may assume $b_n^{(i)} \cap b_m^{(i)} - \gamma' = \emptyset$ for $m \neq n$. Let us denote one of the ends of γ' with respect to $A'(\gamma', \alpha')$ determined by $b_1^{(i)}$ as ∞. Then the cross-cuts $b_n^{(i)}$ $(n > 1)$ form a collection of intervals $[\eta_n', \eta_n'']$ of σ' with the property that one of the following holds: $[\eta_n', \eta_n'']$ is disjoint from $[\eta_m', \eta_m'']$, or $[\eta_n', \eta_n'']$ contains $[\eta_m', \eta_m'']$ properly, or $[\eta_n', \eta_n'']$ is contained in $[\eta_m', \eta_m'']$ properly $(m > n > 1)$. Consequently, by extracting a subsequence if necessary, we may assume that either all the intervals $[\eta_n', \eta_n'']$ are nested or the intervals are all mutually disjoint. In the latter case, we may suppose, again extracting a subsequence if necessary, that

$$[\eta_n', \eta_n''] < [\eta_{n+1}', \eta_{n+1}''] \quad \text{or} \quad [\eta_n', \eta_n''] > [\eta_{n+1}', \eta_{n+1}''] \qquad \text{for all } n.$$

Let us now apply a Carathéodory transformation C to $A'(\gamma', \alpha')$. That is, C maps $A'(\gamma', \alpha')$ homeomorphically into the interior of the unit circle so that the transform of any cross-cut of $A'(\gamma', \alpha')$ is a cross-cut of the unit circle and that the end-points of such transforms of cross-cuts of $A'(\gamma', \alpha')$ are dense on the circumference. (See [7, Chapter VI] for a discussion of Carathéodory transformations.) The problem now reduces to showing that $\limsup_{n\to\infty} C(b_n^{(i)})$ is degenerate. Since $b_n^{(i)} \subset A(\gamma, \alpha)$ and $\lim_{n\to\infty} b_n^{(i)} = \gamma$, we have that $\limsup_{n\to\infty} C(b_n^{(i)})$ is contained in the circumference of the unit circle. Suppose $\limsup_{n\to\infty} C(b_n^{(i)})$ is nondegenerate. Then there exists an end-cut b of γ' with respect to $A'(\gamma', \alpha')$ such that $C(b)$ has its end-point in $\limsup_{n\to\infty} C(b_n^{(i)})$ and $C(b) \cap C(b_n^{(i)}) \neq \emptyset$ for an infinite number of n. Hence $b \cap b_n^{(i)} \neq \emptyset$ for an infinite number of n. This implies that the accessible point determined by b is in γ. Clearly, b is not contained in $A(\gamma, \alpha)$. Therefore, the accessible point determined by b is either w_1 or w_2, say w_1. Consider $\gamma' \cap m_{i-1}$. $S_n^{(i-1)}$ $(n = 1, 2, \cdots)$ partition $\gamma' \cap m_{i-1}$ into disjoint classes which converge to w_1. Consequently, the components of $\gamma' \cap A(\gamma, \alpha)$ which meet m_{i-1} are partitioned into disjoint classes γ_n'. There is a spherical neighborhood U of w_1 such that $\limsup_{n\to\infty} \gamma_n'$ is not contained in U. Otherwise, there would be an end-cut b' of γ with respect to $A(\gamma, \alpha)$ which has w_1 as an accessible point and $b' \subset A'(\gamma', \alpha')$; and this is impossible. Now there is a subinterval b_0 of b such that w_1 is an end-point and $b_0 \subset U$. Let n_1 and n_2 be such that $b_{n_1}^{(i)} \cap b_0 \neq \emptyset \neq b_{n_2}^{(i)} \cap b_0$ and $H_{n_2 n_1}^{(i)}$ contains a γ_{n_3}' with $\gamma_{n_3}' \not\subset U$. Since $l_{n_1}^{(i)}$ and $l_{n_2}^{(i)}$ are contained in α, there exists an arc $C \subset \alpha$ such that $l_{n_1}^{(i)} \cup l_{n_2}^{(i)} \cup C$ is connected. Now $(b_0 \cup b_{n_1}^{(i)} \cup b_{n_2}^{(i)} \cup C) - \{w_1\}$ separates the plane, and there exists a simple closed curve P in $(b_0 \cup b_{n_1}^{(i)} \cup b_{n_2}^{(i)} \cup C) - \{w_1\}$ which separates γ_{n_3}' and γ, and $P \cap \gamma' = \emptyset$. This cannot be since $\gamma' \supset \gamma \cup \gamma_{n_3}'$ and γ' is connected. Hence $\limsup_{n\to\infty} C(b_n^{(i)})$ is degenerate. This completes the proof of Lemma (iv).

3. Definition of generalized length

We first define generalized length for Jordan regions as given by R. E. Fullerton [15]. Let (T, J) be a continuous mapping from a Jordan region $J \subset E_2$ into E_n, and let α be any set open in J. Then $\alpha = \bigcup \alpha_k$, where α_k are the components of α and the union is taken over all such components α_k. With each α_k we have associated the collection $\{\gamma_k\}_{\alpha_k}$ of nondegenerate components γ_k of $B[\alpha_k : J]$. And, with each γ_k, we have either a finite collection $\{\sigma_i^{(k)}\}_{\gamma_k, \alpha_k}$ of admissible arcs $\sigma_i^{(k)}$ or an admissible cycle $\zeta^{(k)}$. Let $\sigma_i^{(k)} = [\eta', \eta'']$, and let $P_i^{(k)} = [\eta_1, \eta_2, \cdots, \eta_r]$ be a partition of $\sigma_i^{(k)}$; that is, $\eta' = \eta_1 < \eta_2 < \cdots < \eta_r = \eta''$. Let

$$S(P_i^{(k)}; J) = \sum_{j=1}^{r-1} | T(w_{\eta_{j+1}}) - T(w_{\eta_j}) |,$$

where w_{η_j} is the accessible point of γ_k from $A(\gamma_k, \alpha_k)$ determined by the end η_j and the absolute value sign denotes the Euclidean distance. Finally, let

$$\lambda(\sigma_i^{(k)}; J) = \sup S(P_i^{(k)}; J),$$

where the supremum is taken over all partitions $P_i^{(k)}$ of $\sigma_i^{(k)}$. In a similar way we define a number $\lambda(\zeta^{(k)}; J)$. Hence for each $\gamma_k \in \{\gamma_k\}_{\alpha_k}$ we have a number $\lambda(\gamma_k; J)$ equal to $\lambda(\zeta^{(k)}; J)$ or $\sum \lambda(\sigma_i^{(k)}; J)$, where the sum $\sum$ ranges over $\{\sigma_i^{(k)}\}_{\gamma_k, \alpha_k}$. If $B[\alpha_k : J]$ has no nondegenerate component, then $\lambda(\gamma_k; J)$ will be defined to be zero. The number

$$l(\alpha; T, J) = \sum_{\alpha_k} \sum_{\gamma_k} \lambda(\gamma_k; J)$$

is called the generalized length of $(T, B[\alpha : J])$ [15].

If J_ν ($\nu = 1, 2, \cdots, m$) is a finite collection of disjoint Jordan regions, $(T, \bigcup_{\nu=1}^m J_\nu)$ is a continuous mapping, and α is open in $\bigcup_{\nu=1}^m J_\nu$, then we define

$$l(\alpha;\, T,\, \textstyle\bigcup_{\nu=1}^m J_\nu) = \sum_{\nu=1}^m l(\alpha \cap J_\nu\,;\, T,\, J_\nu).$$

(i) THEOREM. *Let J', J_1, J_2, $\cdots$, J_m be Jordan regions such that $J' \supset \bigcup_{\nu=1}^m J_\nu$ and $J_\nu \cap J_\mu = \emptyset$ for $\nu \neq \mu$. Let (T, J') be a continuous mapping into E_n, and let α' be a set open in J'. Then*

$$l(\alpha';\, T, J') \geqq l(\alpha' \cap \textstyle\bigcup_{\nu=1}^m J_\nu\,;\, T,\, \bigcup_{\nu=1}^m J_\nu).$$

Proof. We prove the theorem for the case $m = 1$. The case where $m > 1$ follows in a similar manner. Also, it is enough to prove the inequality for the case where α' is connected.

Let $J_1 = J$. By 2, Lemma (i), an admissible cycle ζ of γ with respect to $A(\gamma, \alpha)$, where α is a component of $J \cap \alpha'$ and γ is a nondegenerate component of $B[\alpha{:}J]$, is also an admissible cycle for the larger Jordan region J'. Hence $\lambda(\zeta; J) = \lambda(\zeta; J')$ for each admissible cycle ζ of J.

Let σ_i be an admissible arc of γ with respect to $A(\gamma, \alpha)$, and let $\tilde{\sigma}_i$ be the maximal subarc of σ_i which is contained in an admissible arc σ' or cycle ζ' of γ' with respect to $A'(\gamma', \alpha')$. If Λ_i is defined as in Section 2, then by 2, Lemma (iv), $\Lambda_i \subset E_\omega$ for some prime-end ω of σ' or ζ'. If T is not constant on Λ_i, then, by [7, 20.2, (iii)], $\lambda(\sigma'; J) = +\infty$ or $\lambda(\zeta'; J) = +\infty$. If T is constant on Λ_i, then $\lambda(\sigma_i\,; J) = \lambda(\tilde{\sigma}_i\,; J)$. Hence, by 2, Remark (iii), we have in any case that

$$\lambda(\sigma'; J') \geqq \sum \lambda(\sigma_i\,; J) \quad \text{or} \quad \lambda(\zeta'; J') \geqq \sum \lambda(\sigma_i\,; J),$$

where the sums on the right-hand sides are extended over all admissible arcs σ_i of J which have a subarc $\tilde{\sigma}_i$ contained in σ' or ζ' respectively.

Therefore, we have from the definition of generalized length that

$$l(\alpha';\, T,\, J') \geqq l(\alpha' \cap J;\, T,\, J).$$

This concludes the proof of Theorem (i).

We now define generalized length for mappings from admissible sets. Let us recall the definition of admissible set [7]. A set $A \subset E_2$ is called *admissible* if

(a) A is an open set in E_2;
(b) A is the union of a finite number of disjoint Jordan regions;
(c) A is a set open in the type (b) above.

Let A be an admissible set, and B_ν ($\nu = 1, 2, \cdots$) a sequence of sets with the following properties:

(1) B_ν is the union of a finite number of disjoint Jordan regions ($\nu = 1, 2, \cdots$);
(2) $B_\nu \subset I[B_{\nu+1}{:}A] \subset B_{\nu+1} \subset A \quad (\nu = 1, 2, \cdots)$;
(3) $\bigcup_{\nu=1}^\infty B_\nu = A$.

Let α be a set open in A. Then $\alpha \cap B_\nu$ is open in B_ν, and $l(\alpha \cap B_\nu\,; T, B_\nu)$ is defined. By 3, Theorem (i), we have

$$l(\alpha \cap B_{\nu+1}\,; T, B_{\nu+1}) \geqq l(\alpha \cap B_\nu\,; T, B_\nu) \qquad (\nu = 1, 2, \cdots).$$

Hence, $l(\alpha; T, A) = \lim_{\nu\to\infty} l(\alpha \cap B_\nu\,; T, B_\nu)$ exists and is called the *generalized length* of $(T, B[\alpha\colon A])$. Since $B_\nu \subset I[B_{\nu+1}\colon A]$ $(\nu = 1, 2, \cdots)$, it is clear that the limit is independent of the sequence B_ν $(\nu = 1, 2, \cdots)$. Also, since $\bigcup_{\nu=1}^{\infty} B_\nu = \bigcup_{\nu=1}^{\infty} I[B_\nu\colon A] = A$, if A is compact, then $B_\nu = A$ for all ν sufficiently large. Hence this new definition is clearly an extension of generalized length for Jordan regions given earlier.

4. The Cesari-Cavalieri inequality

Let (T, A) be a continuous mapping from an admissible set $A \subset E_2$ into E_n, and let f be a continuous real-valued function defined on E_n. Then fT is a continuous real-valued function on A. Let t be a real number, and let

$$D^+(t; f) = \{w \in A : fT(w) > t\}, \qquad D^-(t; f) = \{w \in A : fT(w) < t\},$$
$$C(t; f) = \{w \in A : fT(w) = t\}.$$

Since $D^-(t; f) = D^+(-t; -f)$, we need only consider $D^-(t) = D^-(t; f)$. $D^-(t)$ is open in A. Hence $l(D^-(t); T, A)$ is defined (Section 3). Let

$$l(t; T, A, f) = l(D^-(t); T, A).$$

(i) LEMMA. *$l(t; T, A, f)$ is a measurable function of t.*

Proof. Let $\bigcup_{\nu=1}^{m} J_\nu$ be any finite union of disjoint Jordan regions contained in A. Then by [15], $l(\bigcup_{\nu=1}^{m} J_\nu \cap D^-(t); T, \bigcup_{\nu=1}^{m} J_\nu)$ is a measurable function of t. Since $l(t; T, A, f)$ is a limit of such measurable functions, it is also a measurable function of t.

(ii) LEMMA. $\liminf_{\tau\to t-0} l(\tau; T, A, f) \geqq l(t; T, A, f)$.

Proof. Let $\bigcup_{\nu=1}^{m} J_\nu$ be as in the proof of Lemma (i) above. Then by [15], we have

$$l(\textstyle\bigcup_{\nu=1}^{m} J \cap D^-(t); T, \bigcup_{\nu=1}^{m} J_\nu) \leqq \liminf_{\tau\to t-0} l(\bigcup_{\nu=1}^{m} J_\nu \cap D^-(t); T, \bigcup_{\nu=1}^{m} J_\nu)$$
$$\leqq \liminf_{\tau\to t-0} l(\tau; T, A, f).$$

Since $\bigcup_{\nu=1}^{m} J_\nu$ is arbitrary,

$$l(t; T, A, f) \leqq \liminf_{\tau\to t-0} l(\tau; T, A, f),$$

and Lemma (ii) is proved.

It should be noted that, by [7, 20.3, Lemma (iv)], Lemma (ii) above also implies the measurability of $l(t; T, A, f)$.

We are now able to establish the Cesari-Cavalieri inequality.

(iii) THEOREM. *Let (T, A) be a continuous mapping of an admissible set $A \subset E_2$ into E_n, and let f be a real-valued Lipschitzian function of Lipschitz constant $K > 0$ defined on E_n. Then*

$$KL(T, A) \geqq \int_{-\infty}^{+\infty} l(t; T, A, f)\, dt,$$

where $L(T, A)$ is the Lebesgue area of (T, A).

Proof. Let B_ν $(\nu = 1, 2, \cdots)$ be a sequence of sets which satisfy conditions (1), (2), and (3) of Section 3. Then from [15] we have for each ν,

$$KL(T, B_\nu) \geqq \int_{-\infty}^{+\infty} l(B_\nu \cap D^-(t); T, B_\nu)\, dt.$$

Hence by [7, 5.14, (iv)] and the theorem of Beppo Levi,

$$KL(T, A) = \lim_{\nu\to\infty} KL(T, B_\nu) \geqq \lim_{\nu\to\infty} \int_{-\infty}^{+\infty} l(B_\nu \cap D^-(t); T, B_\nu)\, dt$$
$$= \int_{-\infty}^{+\infty} \lim_{\nu\to\infty} l(B_\nu \cap D^-(t); T, B_\nu)\, dt = \int_{-\infty}^{+\infty} l(t; T, A, f)\, dt.$$

Thereby, Theorem (iii) is proved.

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