ON TWO EXTREMUM PROPERTIES OF POLYNOMIALS

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Introduction

This paper is concerned with certain extremum properties of the *measure* $M(f)$ of an arbitrary polynomial

$$f(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m$$

with real or complex coefficients.

In the theory of Diophantine approximations, and in particular in that of transcendental numbers, it has been customary to express the *size* of such a polynomial either by its *height*

$$H(f) = \max(|a_0|, |a_1|, \cdots, |a_m|),$$

or by its *length*

$$L(f) = |a_0| + |a_1| + \cdots + |a_m|.$$

Here the length has the advantage of being a pseudo-valuation:

$$(1) \qquad L(fg) \leqq L(f)L(g), \qquad L(f \mp g) \leqq L(f) + L(g).$$

There is still another function of the coefficients which is worth considering on account of its simple multiplicative property. This is the *measure* $M(f)$ of $f(x)$ which is defined by

$$(2) \qquad \begin{aligned} M(f) &= 0 && \text{if } f(x) \equiv 0, \\ &= \exp\left\{\int_0^1 \log|f(e^{2\pi it})|\,dt\right\} && \text{otherwise.} \end{aligned}$$

Thus, for any two polynomials,

$$(3) \qquad M(fg) = M(f)M(g).$$

If $f(x)$ has the exact degree m and the zeros $\alpha_1, \cdots, \alpha_m$, then

$$(4) \qquad M(f) = |a_0| \prod_{j=1}^{m} \max(1, |\alpha_j|),$$

as follows at once from Jensen's formula (see e.g. [2]).

The height, the length, and the measure of $f(x)$ are connected by the inequalities

$$(5) \qquad \begin{gathered} \binom{m}{[m/2]}^{-1} H(f) \leqq M(f) \leqq H(f)\sqrt{(m+1)}, \\ 2^{-m}L(f) \leqq M(f) \leqq L(f); \end{gathered}$$

for proofs see e.g. Mahler [2], [4], or G. Pólya and G. Szegö [7, p. 265]. As

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the example $f(x) = (x+1)^m$ shows, the left-hand sides of these two inequalities cannot be improved; and the same is true for the right-hand side of the second inequality as the example $f(x) = x^m$ shows. On the other hand, the right-hand side of the first inequality can always be improved when $m \geqq 1$.

We shall therefore be concerned in the first chapter of this paper with the upper bound

$$c_m = \sup M(f)/H(f)$$

extended over all polynomials $f(x) \not\equiv 0$ at most of degree m, and we shall study certain polynomials for which this upper bound is attained.

The exact evaluation of c_m seems to be quite difficult. In the lowest cases,

$$c_0 = 1, \qquad c_1 = 1, \qquad c_2 = (1 + \surd 5)/2,$$

and it is clear that c_n is a nondecreasing function of n. From the inequality above, c_m is always finite, viz.

$$c_m \leqq \surd(m+1).$$

In the opposite direction, a result by J. Clunie [1] implies that

$$c_m \geqq A\surd(m+1),$$

where A is a positive absolute constant. In some unpublished work, by means of an entirely different method, C. B. Haselgrove has shown that $\liminf\,(c_m/\surd(m+1)) \geqq e^{-\gamma/4}$ where γ is Euler's constant.

The main result of the first chapter may be expressed as follows.

THEOREM 1. *For every degree m there exists a polynomial*

$$F(x) = A_0 x^m + A_1 x^{m-1} + \cdots + A_m$$

such that $M(F)/H(F) = c_m$, *and that moreover*

$$|A_0| = |A_1| = \cdots = |A_m| = H(F) = 1.$$

One may conjecture that, in fact, $F(x)$ can be chosen as a *real* polynomial so that all its coefficients are equal to $+1$ or -1; however, I have not succeeded in proving this.[1]

By (3), the measure $M(f)$ has a much simpler multiplicative behaviour than $L(f)$. To make up for this, its behaviour under addition is less simple and will be studied in the second chapter.

From the second formulae in (1) and (5), it follows that

$$M(f \mp g) \leqq L(f \mp g) \leqq L(f) + L(g) \leqq 2^m\{M(f) + M(g)\}$$

[1] *Added in proof.* This conjecture has been disproved numerically by Mr. Michael Comenetz and Mr. Stephen Grant, students in the 1963 Undergraduate Summer Program of the University of Illinois Digital Computer Laboratory. They used Illiac II to study the cases $m = 6, 7, 8$ and found that the conjecture is false in each of these cases. The conjecture appears to be true for $m \leqq 5$.

if both $f(x)$ and $g(x)$ are at most of degree m. More generally, let $f_1(x), \cdots, f_n(x)$ be finitely many polynomials of degrees not exceeding m. By the same kind of computation, we find now that

$$\begin{aligned}\sum_{h=1}^{n}\sum_{k=1}^{n} M(f_h - f_k) &\leqq \sum_{h=1}^{n}\sum_{k=1}^{n} L(f_h - f_k)\\ &\leqq \sum_{\substack{h=1\\ h\neq k}}^{n}\sum_{k=1}^{n} \{L(f_h) + L(f_k)\} \leqq 2(n-1)\sum_{h=1}^{n} L(f_h)\\ &\leqq 2^{m+1}(n-1)\sum_{h=1}^{n} M(f_h).\end{aligned}$$

Hence the upper bound

$$C_{mn} = \sup\left(\sum_{h=1}^{n}\sum_{k=1}^{n} M(f_h - f_k)\right)/\left(\sum_{h=1}^{n} M(f_h)\right)$$

extended over all sets of n polynomials, all at most of degree m, is finite and not greater than $2^{m+1}(n-1)$.

We shall again study systems of such n polynomials for which the upper bound is attained, and our main result may be expressed as follows.

THEOREM 2. *For every degree m and for every positive integer n, there exist n polynomials $F_1(x), \cdots, F_n(x)$ which are not all identically zero, with the following properties*:

(a) $\left(\sum_{h=1}^{n}\sum_{k=1}^{n} M(F_h - F_k)\right)/\left(\sum_{h=1}^{n} M(F_h)\right) = C_{mn}$.

(b) *Those of the polynomials $F_1(x), \cdots, F_n(x)$ that are not identically zero, all have the exact degree m, and their zeros lie on the unit circle.*

I conjecture that, in fact, none of the extremum polynomials $F_1(x), \cdots, F_n(x)$ can be identically zero.

Again it does not seem to be easy to find the exact value of C_{mn} for arbitrary m and n. One can, however, use Theorem 2 to replace the trivial upper estimate $2^{m+1}(n-1)$ for C_{mn} by a better one, and it is also easy to establish a nontrivial lower estimate.

Chapter I. The ratio $M(f)/H(f)$

1. If $f(x)$ is any polynomial not identically zero, put

$$P(f) = M(f)/H(f)$$

so that $P(f)$ is a positive number. If $c \neq 0$ is a constant, evidently

$$H(cf) = |c|\,H(f) \quad \text{and} \quad M(cf) = |c|\,M(f),$$

and hence

$$P(cf) = P(f).$$

Thus, for the study of $P(f)$ it suffices to consider polynomials of height

$$H(f) = 1.$$

From now on let S_m be the set of all polynomials at most of degree m and of height 1. The quantity of the introduction,

(6) $$c_m = \sup M(f)/H(f) = \sup P(f),$$

where the upper bound extends over all polynomials $f(x) \not\equiv 0$ at most of degree m, may therefore also be defined by

(7) $$c_m = \sup_{f(x)\epsilon S_m} M(f).$$

2. From this definition of c_m, there exists an infinite sequence of polynomials,

$$\Sigma = \{f_k(x)\},$$

all at most of degree m, such that

(8) $$H(f_k) = 1 \text{ for all } k, \quad \text{and} \quad \lim_{k\to\infty} M(f_k) = c_m .$$

Now the polynomials at most of degree m and of height 1 form a compact set. It follows then from Weierstrass's theorem that there exists an infinite subsequence

$$\Sigma' = \{f_{k_r}(x)\} \qquad (k_1 < k_2 < k_3 < \cdots)$$

of Σ which tends to a limit polynomial

$$\lim_{r\to\infty} f_{k_r}(x) = G(x)$$

in the sense that

$$\lim_{r\to\infty} H(f_{k_r} - G) = 0.$$

Also the polynomial $G(x)$ has at most the degree m, and furthermore

$$H(G) = \lim_{r\to\infty} H(f_{k_r}) = \lim_{k\to\infty} H(f_k) = 1.$$

In addition, by Lemma 1 of my paper [3], $G(x)$ has the measure

$$M(G) = \lim_{r\to\infty} M(f_{k_r}) = \lim_{k\to\infty} M(f_k) = c_m .$$

It is quite possible that $G(x)$ is of lower degree than m, say of the exact degree $m - n$ where $0 \leqq n \leqq m$. Then the new polynomial

$$F(x) = x^n G(x)$$

has the exact degree m, and it is obvious that

$$H(F) = H(G) = 1, \qquad M(F) = M(G) = c_m .$$

We have thus obtained a result which we may express as follows.

LEMMA 1. *There exists at least one polynomial $F(x)$ of the exact degree m such that*

(9) $$H(F) = 1, \qquad M(F) = c_m ,$$

and that moreover

(10) $$M(f) \leqq M(F) \quad \textit{for every polynomial } f(x) \textit{ in } S_m .$$

3. From now on denote by Φ the set of all polynomials

$$F(x) = A_0 x^m + A_1 x^{m-1} + \cdots + A_m \qquad (A_0 \neq 0)$$

of exact degree m with the properties (9) and (10); by Lemma 1, this set is not empty. Our aim is to prove Theorem 1 of the Introduction which, in the new notation, may be formulated as follows.

THEOREM 1. *The set Φ contains at least one polynomial $F(x)$ such that*

$$|A_0| = |A_1| = \cdots = |A_m| = 1.$$

The proof of this theorem is indirect. *It will be assumed that*, on the contrary, *every polynomial $F(x)$ in Φ possesses at least one coefficient A_k of absolute value distinct from* 1 *and hence satisfying*

$$(11) \qquad |A_k| < 1.$$

From now on let $F(x)$ be a fixed polynomial in Φ with the following two properties.

(a) *$F(x)$ has the smallest possible number of coefficients of absolute value less than* 1.

(b) *The suffix k is the largest one for which* (11) *is satisfied.*

Evidently the theorem will be proved if we can show that there exists in Φ still a further polynomial

$$f(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m$$

such that

$$a_h = A_h \quad \text{if} \quad h \neq k \quad \text{and} \quad 0 \leqq h \leqq m, \qquad \text{but} \quad |a_k| = 1.$$

For then we obtain a contradiction to the minimum property (a) of $F(x)$.

4. Let δ and ε be positive constants which will be fixed later. For the present let $f(x)$ yet be any polynomial

$$f(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m$$

which has the following m coefficients in common with $F(x)$:

$$a_h = A_h \quad \text{if} \quad h \neq k \quad \text{and} \quad 0 \leqq h \leqq m,$$

while its remaining coefficient a_k is a complex variable which we restrict to the neighbourhood

$$U : |a_k - A_k| < \delta$$

of A_k.

Denote by $\mathrm{A}_1, \cdots, \mathrm{A}_m$ the zeros of $F(x)$, and by $\alpha_1, \cdots, \alpha_m$ those of $f(x)$. We number the zeros of $F(x)$ so that, say

$$(12) \qquad \begin{aligned} |\mathrm{A}_h| &> 1 \quad \text{if} \quad 1 \leqq h \leqq M, \\ &\leqq 1 \quad \text{if} \quad M + 1 \leqq h \leqq m; \end{aligned}$$

here M is a certain nonnegative integer depending on $F(x)$. This notation implies in particular that

$$M(F) = |A_0 \mathrm{A}_1 \cdots \mathrm{A}_M| = c_m . \tag{13}$$

Let ε be chosen so small that, in particular,

$$2\varepsilon < \min_{1 \leqq h \leqq M} (|\mathrm{A}_h| - 1), \tag{14}$$

and also assume that

$$\delta < \tfrac{1}{2} |A_0|.$$

In the case when $k = 0$, since by hypothesis A_0 does not vanish, it follows that also a_0 is distinct from zero because

$$|a_0| > |A_0| - \tfrac{1}{2}|A_0| = \tfrac{1}{2}|A_0| > 0.$$

Hence, by Hurwitz's theorem on the zeros of a polynomial (see e.g. Marden [5, p. 4]), provided δ is smaller than a certain positive function of ε, the zeros of $f(x)$ can be numbered so that

$$|\alpha_h - \mathrm{A}_h| < \varepsilon \qquad (h = 1, 2, \cdots, m). \tag{15}$$

Thus, from (12) and (14),

$$\begin{aligned} |\alpha_h| &> |\mathrm{A}_h| - \varepsilon > (1 + 2\varepsilon) - \varepsilon = 1 + \varepsilon \quad \text{if} \quad 1 \leqq h \leqq M, \\ &< |\mathrm{A}_h| + \varepsilon \leqq 1 + \varepsilon \quad \text{if} \quad M + 1 \leqq h \leqq m. \end{aligned} \tag{16}$$

It follows then that

$$M(f) \geqq |a_0 \alpha_1 \cdots \alpha_M|, \tag{17}$$

where the sign of equality need not necessarily hold.

5. The m zeros $\alpha_1, \cdots, \alpha_m$ of $f(x)$ are the branches of the algebraic function

$$\alpha = \alpha(a_k)$$

of a_k defined by the equation

$$f(\alpha) = 0.$$

Let now the coefficient a_k vary arbitrarily in U. Then these m branches may be interchanged; but it follows immediately from the inequalities (16) that the first M branches $\alpha_1, \cdots, \alpha_M$ can be permuted only amongst themselves. Hence the product

$$\Pi(a_k) = a_0 \alpha_1 \cdots \alpha_M$$

is a single-valued branch of a certain algebraic function of a_k when this variable is restricted to U. It follows again from Hurwitz's theorem that $\Pi(a_k)$ is continuous and hence also regular for all a_k in U.

Evidently,

$$\Pi(A_k) = A_0 \mathrm{A}_1 \cdots \mathrm{A}_M .$$

Therefore, from the property (10) of $F(x)$ and from the formulae (13) and (17), we deduce that

$$|\Pi(a_k)| \leqq |\Pi(A_k)| \quad \text{if} \quad a_k \in U.$$

This surprising inequality shows that the absolute value of the regular function $\Pi(a_k)$ assumes its maximum in U at the centre $a_k = A_k$. But then, by the maximum modulus principle for regular functions,

$$\Pi(a_k) \quad \textit{must be a constant.}$$

The proof assumes that a_k lies in U; but the assertion remains valid for all complex a_k provided we define $\Pi(a_k)$ outside U by analytic continuation. In this way we are led to the following result.

LEMMA 2. *If a_k is an arbitrary complex number, then $f(x)$ has M zeros $\alpha_1, \cdots, \alpha_M$ such that*

$$a_0 \alpha_1 \cdots \alpha_M = A_0 \mathrm{A}_1 \cdots \mathrm{A}_M .$$

In the exceptional case $M = 0$ this equation simplifies to

$$a_0 = A_0 .$$

6. The proof of Theorem 1 may now be concluded as follows.

Choose for a_k an arbitrary complex number of absolute value 1. By the property (10) of $F(x)$,

$$M(f) \leqq M(F) = c_m .$$

On the other hand, Lemma 2 implies that

$$M(f) = |a_0| \prod_{h=1}^{m} \max(1, |\alpha_h|) \geqq |a_0| \prod_{h=1}^{M} \max(1, |\alpha_h|)$$

$$\geqq |a_0 \alpha_1 \cdots \alpha_M| = |A_0 \mathrm{A}_1 \cdots \mathrm{A}_M| = M(F),$$

and hence that

$$M(f) = M(F) = c_m .$$

However, $f(x)$ has one coefficient less than $F(x)$ that has absolute value smaller than 1, and so a contradiction to the minimum property (a) of $F(x)$ in §3 arises. This concludes the proof of Theorem 1.

Referee's remark

Theorem 1 is directly implied by the following simple proposition, as we readily see if we take

$$A(z) = f(z) - a_k z^{m-k} \quad \text{and} \quad B(z) = z^k,$$

where $f(z) = a_0 z^m + \cdots + a_m$.

PROPOSITION. *Suppose $A(z)$ and $B(z)$ are functions regular in some do-*

main containing the unit circle and not both identically zero. Let

$$F(w) = \int_0^1 \log | A(e^{2\pi it}) + wB(e^{2\pi it}) | \, dt$$

for any complex w. Then

$$\sup_{|w| \leqq 1} F(w) = \sup_{|w|=1} F(w),$$

and the supremum is attained.

Proof. The function $F(w)$ is subharmonic, and the proposition is a direct application of the maximum principle for subharmonic functions. (See, for example, MAURICE HEINS, *Selected Topics in the Classical Theory of Functions of a Complex Variable*, New York, 1962, p. 75.) To prove that F is subharmonic we must show (i) that $-\infty \leqq F(w) < +\infty$ for all w, (ii) that $F(w)$ is not identically $-\infty$ in any open set, (iii) that $F(w)$ is upper semicontinuous, and (iv) that for any positive r and any complex w we have

$$F(w) \leqq \int_0^1 F(w + re^{2\pi iu}) \, du.$$

Assertions (i) and (ii) are immediate. In fact $F(w)$ can be $-\infty$ for at most one value of w. Assertion (iii) follows from the fact that $F(w)$ is the decreasing limit of the sequence of continuous functions

$$F_n(w) = \int_0^1 \max \{-n, \log | A(e^{2\pi it}) + wB(e^{2\pi it}) |\} \, dt.$$

Finally, by Jensen's theorem for the linear function, we have for each z on the unit circle

$$\int_0^1 \log | A(z) + (w + re^{2\pi iu})B(z) | \, du \geqq \log | A(z) + wB(z) |;$$

on putting $z = e^{2\pi it}$ and integrating over $0 \leqq t \leqq 1$, we obtain (iv). Thus the proposition is proved. Obviously the regularity condition on $A(z)$ and $B(z)$ could be weakened considerably.

One could give a direct proof of this proposition without using the notion of a subharmonic function, but the proof would essentially repeat the simple proof of the maximum principle for subharmonic functions. At any rate, only the rudiments of the theory of subharmonic functions were used in the above proof.

The above proposition also implies the following generalization of Theorem 1, as we readily see if we take

$$A(z) = f(z) - a_k z^{m-k} \quad \text{and} \quad B(z) = A_k z^{m-k}.$$

THEOREM. *Let $A_0, A_1, \cdots, A_m$ be given nonnegative real numbers, and let $f(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m$ run over the set S of all polynomials of* *degree at most m with arbitrary real or complex coefficients such that*

$$|a_0| \leqq A_0, \quad |a_1| \leqq A_1, \quad \cdots, \quad |a_m| \leqq A_m.$$

Then the maximum of $M(f)$ is attained for a polynomial $f(x)$ with

$$|a_0| = A_0, \quad |a_1| = A_1, \quad \cdots, \quad |a_m| = A_m.$$

Chapter II. The ratio $N(\mathbf{f})/M(\mathbf{f})$

7. If $f(x)$ is any polynomial not identically zero, we denote its exact degree by the symbol $\delta(f)$. Throughout this chapter m is a fixed nonnegative integer, and apart from the polynomial 0 only polynomials satisfying the inequality $\delta(f) \leqq m$ will occur. If $f(x) \not\equiv 0$ has the property $\delta(f) < m$, then we adopt the convention of ascribing to $f(x)$ in addition to its $\delta(f)$ finite zeros further $m - \delta(f)$ zeros ∞. In this way all polynomials with $\delta(f) \leqq m$ have then exactly m zeros.

If $f(x) \not\equiv 0$, the symbol $d(f)$ will be used to denote the number of those zeros of $f(x)$ (including possible zeros ∞) that do *not* have the absolute value 1.

We shall be concerned with sets of n polynomials where $n \geqq 2$, and for convenience write such sets of polynomials as vectors

$$\mathbf{f}(x) = (f_1(x), \cdots, f_n(x)),$$

which we call polynomial vectors. In particular, $\mathbf{0} = (0, \cdots, 0)$ is the zero vector, and $\mathbf{f}(x)$ is distinct from $\mathbf{0}$ when at least one of its components $f_h(x)$ does not vanish identically.

For polynomial vectors we put

$$M(\mathbf{f}) = \textstyle\sum_{h=1}^{n} M(f_h), \qquad N(\mathbf{f}) = \sum_{h=1}^{n} \sum_{j=1}^{n} M(f_h - f_j);$$

$$Q(\mathbf{f}) = N(\mathbf{f})/M(\mathbf{f}) \quad \text{if} \quad \mathbf{f}(x) \not\equiv \mathbf{0};$$

$$\delta(\mathbf{f}) = \max_{1 \leqq h \leqq n, f_h(x) \not\equiv 0} \delta(f_h); \qquad d(\mathbf{f}) = \textstyle\sum_{1 \leqq h \leqq n, f_h(x) \not\equiv 0} d(f_h) \quad \text{if} \quad \mathbf{f}(x) \not\equiv \mathbf{0}.$$

If further $a(x)$ is an arbitrary polynomial, we write

$$a(x)\mathbf{f}(x) = (a(x)f_1(x), \cdots, a(x)f_n(x)).$$

For such scalar products,

$$M(a\mathbf{f}) = M(a)M(\mathbf{f}) \quad \text{and} \quad N(a\mathbf{f}) = M(a)N(\mathbf{f}),$$

and hence

$$Q(a\mathbf{f}) = Q(\mathbf{f}) \quad \text{if} \quad a(x) \not\equiv 0, \quad \mathbf{f}(x) \not\equiv \mathbf{0}. \tag{18}$$

8. We shall mostly be concerned with a pair of polynomial vectors

$$\mathbf{f}(x) = (f_1(x), \cdots, f_n(x)) \quad \text{and} \quad \mathbf{F}(x) = (F_1(x), \cdots, F_n(x))$$

and with the two derived sets of polynomials

$$\{g_{hj}(x) \mid h, j = 1, 2, \cdots, n\} \quad \text{and} \quad \{G_{hj}(x) \mid h, j = 1, 2, \cdots, n\},$$

where

$$g_{hj}(x) = f_h(x) - f_j(x) \quad \text{and} \quad G_{hj}(x) = F_h(x) - F_j(x).$$

Since for all suffixes h and j

$$g_{hh}(x) \equiv 0 \quad \text{and} \quad g_{hj}(x) \equiv -g_{jh}(x),$$

and similarly for $G_{hj}(x)$, both sets cannot contain more than $(n^2 - n)/2$ essentially distinct elements that do not vanish. It would suffice to consider only the polynomials that correspond to pairs of suffixes (h, j) for which

$$1 \leqq h < j \leqq n.$$

Some of the polynomials $f_h(x)$, $F_h(x)$, $g_{hj}(x)$, $G_{hj}(x)$ may vanish identically. For the remaining polynomials we denote

by $\alpha_{h1}, \cdots, \alpha_{hm}$ the zeros of $f_h(x) \not\equiv 0$,

by $A_{h1}, \cdots, A_{hm}$ the zeros of $F_h(x) \not\equiv 0$,

by $\beta_{hj1}, \cdots, \beta_{hjm}$ the zeros of $g_{hj}(x) \not\equiv 0$, and

by $B_{hj1}, \cdots, B_{hjm}$ the zeros of $G_{hj}(x) \not\equiv 0$,

respectively. Some of these zeros may of course be equal to ∞.

A zero is said to be of the *first*, *second*, or *third kind* according as its absolute value is greater than 1, less than 1, or equal to 1, respectively. The symbol $d(f)$ equals thus the total number of zeros of $f(x) \not\equiv 0$ that are of the first or second kind. In addition to these, the polynomial has $m - d(f)$ zeros of the third kind.

With any zero α of the first kind we consider also its reciprocal α^{-1}; if $\alpha = \infty$, this reciprocal is put equal to 0.

9. The proof of Theorem 2 depends essentially on a generalisation of the maximum modulus principle for regular functions. This principle may be formulated as follows.

LEMMA 3. *Let $\phi_1(z), \cdots, \phi_l(z)$ be finitely many functions of the complex variable z which are regular in a closed region R. If z_0 is an interior point of R, and if*

$$\sum_{\lambda=1}^{l} |\phi_\lambda(z)| \leqq \sum_{\lambda=1}^{l} |\phi_\lambda(z_0)| \qquad \text{for all } z \text{ in } R,$$

then all functions $\phi_1(z), \cdots, \phi_l(z)$ are constants.

A proof of this lemma can be found in G. Pólya and G. Szegö [6, pp. 327–328].

10. Denote by S_{mn} the set of all polynomial vectors $\mathbf{f}(x)$ satisfying

$$\mathbf{f}(x) \neq \mathbf{0}, \qquad \delta(\mathbf{f}) \leqq m, \qquad M(\mathbf{f}) = 1.$$

The least upper bound

$$C_{mn} = \sup N(\mathbf{f})/M(\mathbf{f}) = \sup Q(\mathbf{f})$$

extended over all polynomial vectors $\mathbf{f}(x) \not\equiv 0$ with $\delta(\mathbf{f}) \leqq m$ has already been introduced in the Introduction, and it was then proved that

$$C_{mn} \leqq 2^{m+1}(n-1).$$

The homogeneity property (18) of $Q(\mathbf{f})$, applied with a constant $a \neq 0$, shows that C_{mn} may also be defined as the least upper bound

$$C_{mn} = \sup_{f(x) \epsilon S_{mn}} N(\mathbf{f}).$$

By Lemma 1 of my paper [3], $M(f)$ is a continuous function of the coefficients of $f(x)$. Hence, from their definitions, $M(\mathbf{f})$ and $N(\mathbf{f})$ are continuous functions of the coefficients of the components of $\mathbf{f}(x)$. By the finiteness of C_{mn} and by Weierstrass's theorem a proof similar to that in §2 leads then to the following result.

LEMMA 4. *There exists a polynomial vector*

$$\mathbf{F}(x) = (F_1(x), \cdots, F_n(x))$$

such that

$$\delta(\mathbf{F}) = m, \qquad M(\mathbf{F}) = 1, \qquad N(\mathbf{F}) = C_{mn} \tag{19}$$

and that, moreover,

(20) $N(\mathbf{f}) \leqq N(\mathbf{F}) = C_{mn}$ *for every polynomial vector* $\mathbf{f}(x)$ *in* S_{mn}.

The vector $\mathbf{F}(x)$ in this lemma is not in general unique since, e.g., its components may always be permuted in any way. Denote then by Φ the set of all those polynomial vectors $\mathbf{F}(x)$ satisfying (19) of which the components have already been numbered in such a manner that

$$\begin{aligned} F_h(x) &\not\equiv 0 \quad \text{if} \quad 1 \leqq h \leqq n(\mathbf{F}), \\ &\equiv 0 \quad \text{if} \quad n(\mathbf{F}) + 1 \leqq h \leqq n. \end{aligned}$$

Here, by $\mathbf{F}(x) \not\equiv \mathbf{0}$, $n(\mathbf{F})$ is a certain integer depending on $\mathbf{F}(x)$ for which

$$1 \leqq n(\mathbf{F}) \leqq n.$$

11. From its definition, $n(\mathbf{F})$ has only n possible values. Hence, as $\mathbf{F}(x)$ runs over the elements of Φ, $n(\mathbf{F})$ attains a certain maximum, $n_0 \geqq 1$ say.

Similarly, the nonnegative integer $d(\mathbf{F})$ may be written as the sum

$$d(\mathbf{F}) = \sum_{h=1}^{n(\mathbf{F})} d(F_h).$$

Therefore, by $\delta(\mathbf{F}) \leqq m$ and $n(\mathbf{F}) \leqq n$, $d(\mathbf{F})$ has not more than $mn + 1$

possible values. Hence in the subset of those $\mathbf{F}(x)$ in Φ which satisfy $n(\mathbf{F}) = n_0$, $d(\mathbf{F})$ assumes a certain minimum, d_0 say.

Denote by Ψ the subset of all those $\mathbf{F}(x)$ in Φ which satisfy both extremum conditions

$$n(\mathbf{F}) = n_0 \quad \text{and} \quad d(\mathbf{F}) = d_0 .$$

Our aim is to prove Theorem 2 of the Introduction; in the new notation, it may now be formulated as follows.

THEOREM 2. *The minimum d_0 is zero. Thus there exists a polynomial vector $\mathbf{F}(x)$ in Ψ of which the first n_0 components all have the exact degree m and possess only zeros of absolute value* 1, *while the remaining components vanish identically.*

Again the proof is indirect. It will be assumed that, on the contrary,

$$d_0 \geqq 1.$$

Then, starting from any element $\mathbf{F}(x)$ of Ψ, we shall construct a polynomial vector $\mathbf{f}(x)$ in Φ with the property that

$$n(\mathbf{f}) = n_0 , \quad \text{but} \quad d(\mathbf{f}) < d_0 .$$

Since this is contrary to the minimum definition of d_0, the assumption is false, and hence the theorem is true.

12. Choose for $\mathbf{F}(x)$ an arbitrary, but once for all fixed, polynomial vector in Ψ, so that

$$n(\mathbf{F}) = n_0 \quad \text{and} \quad d(\mathbf{F}) = d_0 \geqq 1.$$

The second formula implies that at least one of the integers $d(F_1), \cdots, d(F_{n_0})$ is positive; denote by k a suffix such that

$$d(F_k) \geqq 1 \quad \text{and} \quad 1 \leqq k \leqq n_0 .$$

It follows that the component $F_k(x)$ of $\mathbf{F}(x)$ has at least one zero of the first or second kind where this zero possibly is equal to ∞.

Denote by

$$\mathrm{A}_{k1}, \quad \mathrm{A}_{k2}, \quad \cdots, \quad \mathrm{A}_{kq}$$

the finite or infinite zeros of $F_k(x)$ that are of the first kind; by

$$\mathrm{A}_{ki} \quad (q + 1 \leqq i \leqq r)$$

its zeros of the second kind, and by

$$\mathrm{A}_{ki} \quad (r + 1 \leqq i \leqq m)$$

its zeros of the third kind; here

$$d(F_k) = r \geqq 1.$$

In terms of these zeros, $F_k(x)$ may be written as

$$F_k(x) = A_k \prod_{l=1}^{q} (\mathrm{A}_{kl}^{-1} x - 1) \cdot \prod_{l=q+1}^{m} (x - \mathrm{A}_{kl}). \tag{21}$$

Here empty products mean 1; $\mathrm{A}_{kl}^{-1} x - 1$ becomes -1 when $\mathrm{A}_{kl} = \infty$; and the coefficient A_k is a constant which does not vanish such that

$$M(F_k) = |A_k|. \tag{22}$$

For all factors $\mathrm{A}_{kl}^{-1} x - 1$ in the first product and all factors $x - \mathrm{A}_{kl}$ in the second product have the measure 1, as follows immediately from (4).

13. Next denote by T the set of all those pairs of suffixes (h, j) for which

$$1 \leqq h \leqq n, \qquad 1 \leqq j \leqq n, \qquad G_{hj}(x) \not\equiv 0.$$

Since C_{mn} obviously is positive, and since further

$$N(F) = \sum_{(h,j)\in T} M(F_h - F_j) = C_{mn}, \tag{23}$$

T cannot be the null set.

In analogy to the notation for $F_k(x)$, denote for each pair (h, j) in T by

$$\mathrm{B}_{hj1}, \quad \mathrm{B}_{hj2}, \quad \cdots, \quad \mathrm{B}_{hju(h,j)}$$

the finite or infinite zeros of $G_{hj}(x)$ that are of the first kind, and by

$$\mathrm{B}_{hji} \quad (u(h, j) + 1 \leqq i \leqq m)$$

its zeros of the second or third kind. Here $u(h, j)$ is a certain integer which may assume any one of the values $0, 1, \cdots, m$.

In analogy to $F_k(x)$, $G_{hj}(x)$ may be written in the form

$$G_{hj}(x) = B_{hj} \prod_{l=1}^{u(h,j)} (\mathrm{B}_{hjl}^{-1} x - 1) \cdot \prod_{l=u(h,j)+1}^{m} (x - \mathrm{B}_{hjl}). \tag{24}$$

Again $\mathrm{B}_{hjl}^{-1} x - 1$ becomes -1 when $\mathrm{B}_{hjl} = \infty$; and the coefficient B_{hj} does not vanish and is such that

$$M(G_{hj}) = |B_{hj}|. \tag{25}$$

The equation (23) is therefore equivalent to

$$N(F) = \sum_{(h,j)\in T} M(G_{hj}) = \sum_{(h,j)\in T} |B_{hj}| = C_{mn}. \tag{26}$$

14. Let r be defined as in §12, and let R be the complex r-dimensional space consisting of all points

$$z = (z_1, \cdots, z_r)$$

with arbitrary complex coordinates $z_1, \cdots, z_r$. If

$$z' = (z_1', \cdots, z_r')$$

is a second point in R, the *distance* $|z - z'|$ of these two points is defined by the formula

$$|z - z'| = \max_{1 \leqq l \leqq r} |z_l - z'_l|.$$

In particular, the point z has the distance

$$|z| = \max_{1 \leqq l \leqq r} |z_l|$$

from the *origin* $0 = (0, \cdots, 0)$.

We shall consider the points in R also as vectors and apply to these vectors the usual notation for the sum, difference, and scalar product.

15. Let $Z = (Z_1, \cdots, Z_r)$ be the special point in R with the coordinates

$$\begin{aligned} Z_l &= A_{kl}^{-1} \quad \text{if} \quad 1 \leqq l \leqq q, \\ &= A_{kl} \quad \text{if} \quad q+1 \leqq l \leqq r, \end{aligned} \tag{27}$$

where A_{kl} and q have the same meaning as in §12. Hence the formula (21) for $F_k(x)$ may now be written as

$$F_k(x) = A_k \prod_{l=1}^{q} (Z_l x - 1) \cdot \prod_{l=q+1}^{r} (x - Z_l) \cdot \prod_{l=r+1}^{m} (x - \mathrm{A}_{kl}). \tag{28}$$

From the definition of the zeros of $F_k(x)$ it is obvious that

$$|Z| < 1.$$

Hence, if δ is any sufficiently small positive constant,

$$|Z| < 1 - 2\delta. \tag{29}$$

From now on let z be an arbitrary point in the neighbourhood

$$U : |z - Z| < \delta \tag{30}$$

of Z. We associate with this point z a variable polynomial vector

$$\mathbf{f}(x) = (f_1(x), \cdots, f_n(x))$$

defined as follows:

(a) *For* $1 \leqq h \leqq n$ *and* $h \neq k$, *let*

$$f_h(x) = F_h(x).$$

(b) *In the remaining case* $h = k$ *put*

$$f_k(x) = A_k \prod_{l=1}^{q} (z_l x - 1) \cdot \prod_{l=q+1}^{r} (x - z_l) \cdot \prod_{l=r+1}^{m} (x - \mathrm{A}_{kl}), \tag{31}$$

or, what is equivalent to this,

$$f_k(x) = A_k \prod_{l=1}^{q} (\alpha_{kl}^{-1} x - 1) \cdot \prod_{l=q+1}^{m} (x - \alpha_{kl}).$$

Here, similar to (27),

$$\begin{aligned} z_l &= \alpha_{kl}^{-1} \quad \text{if} \quad 1 \leqq l \leqq q, \\ &= \alpha_{kl} \quad \text{if} \quad q+1 \leqq l \leqq r, \end{aligned} \tag{32}$$

while

$$\alpha_{kl} = \mathrm{A}_{kl} \quad \text{if} \quad r + 1 \leqq l \leqq m. \tag{33}$$

Again the zeros α_{kl} with $1 \leqq l \leqq q$ are of the first kind, those with $q + 1 \leqq l \leqq r$ are of the second kind, and those with $r + 1 \leqq l \leqq m$ are of the third kind. This follows immediately from (32) and (33) because, by (29) and (30),

$$| z | = | (z - Z) + Z | < \delta + (1 - 2\delta) = 1 - \delta < 1. \tag{34}$$

From the definition, $\mathbf{f}(x) = \mathbf{F}(x)$ when $z = Z$. More generally, as z tends to Z, thus when $| z - Z |$ tends to zero, all coefficients of the component $f_k(x)$ of $\mathbf{f}(x)$ tend to the corresponding coefficients of the component $F_k(x)$ of $\mathbf{F}(x)$. Here $f_k(x)$ has the exact degree m as long as z is a variable point in U because $A_k \neq 0$, and hence, except at most for a subset of U of lower dimension,

$$A_k z_1 \cdots z_q \neq 0.$$

16. Denote by T_k the subset of those pairs (h, j) in T for which both h and j are distinct from k; by T_k^* the set of all pairs (h, k) and (k, h) in T; and finally by T_k^{**} the set of all pairs (h, k) and (k, h) with $h \neq k$ that are *not* in T. The polynomials

$$g_{hj}(x) = f_h(x) - f_j(x)$$

have already been introduced in §8. From the definitions, it is clear that

$$\begin{aligned} g_{hj}(x) &\equiv G_{hj}(x) && \text{if} \quad (h, j) \in T_k ; \\ g_{hk}(x) = -g_{kh}(x) &= F_h(x) - f_k(x) && \text{if} \quad (h, k) \in T_k^* ; \\ g_{hk}(x) = g_{kh}(x) &= F_k(x) - f_k(x) && \text{if} \quad (h, k) \in T_k^{**}. \end{aligned}$$

Thus, when (h, k) lies in either T_k^* or T_k^{**}, $g_{hk}(x)$ does not vanish identically when z is a variable point in U, because then at least one coefficient of $f_k(x)$ is variable since, by hypothesis, $r = d(F_k) \geqq 1$. It is also evident that, for the same pairs (h, k), the coefficients of $g_{hk}(x)$ are continuous functions of z, and that they converge to the corresponding coefficients of $G_{hk}(x)$ as z tends to Z. In the limiting case when $z = Z$, the polynomials $g_{hk}(x)$ become identical with the corresponding polynomials $G_{hk}(x)$.

Since T is the union of T_k and T_k^*, and since trivially

$$M(g_{hk}) = M(g_{kh}) \geqq 0 \quad \text{if} \quad (h, k) \in T_k^{**},$$

we find, in analogy to (26), that

$$N(f) \geqq \textstyle\sum_{(h,j)\in T} M(g_{hj}).$$

Further, by the inequality (20) of Lemma 4,

$$N(f) \leqq N(F) \quad \text{if} \quad z \in U.$$

Hence, by (26),

$$\sum_{(h,j)\epsilon T} M(g_{hj}) \leqq \sum_{(h,j)\epsilon T} M(G_{hj}) \quad \text{if} \quad z \epsilon U.$$

Here the terms that correspond to the pairs (h, j) in T_k are the same on both sides of the inequality and may therefore be omitted. Thus we arrive at the *basic inequality*,

$$\sum_{(h,k)\epsilon T_k^*} M(g_{hk}) \leqq \sum_{(h,k)\epsilon T_k^*} M(G_{hk}) \quad \text{if} \quad z \epsilon U. \tag{35}$$

In this inequality there is naturally no need to add also the contributions from the pairs (k, h) in T_k^*.

17. Written as polynomials in x, the components of $\mathbf{F}(x)$ and $\mathbf{f}(x)$ have the form

$$F_h(x) = A_{h0}x^m + A_{h1}x^{m-1} + \cdots + A_{hm} \qquad (h = 1, 2, \cdots, n)$$

and

$$f_h(x) = a_{h0}x^m + a_{h1}x^{m-1} + \cdots + a_{hm} \qquad (h = 1, 2, \cdots, n).$$

Here, trivially,

$$a_{hl} = A_{hl} \quad \text{if} \quad h \neq k, \quad 0 \leqq l \leqq m,$$

and the coefficients a_{kl} are polynomials in the coordinates of z.

Let now (h, k) be any pair in T_k^*. We obtain then for $g_{hk}(x)$ the explicit formula

$$\begin{aligned} g_{hk}(x) &= F_h(x) - f_k(x) \\ &= (A_{h0} - a_{k0})x^m + (A_{h1} - a_{k1})x^{m-1} + \cdots + (A_{hm} - a_{km}). \end{aligned} \tag{36}$$

Since the coefficients a_{kl} of $f_k(x)$ are polynomials in the coordinates of z, the same is true of the coefficients of $g_{hk}(x)$. The latter are then continuous functions of z. Hence $L(g_{hk})$ and by Lemma 1 of [3] also $M(g_{hk})$ are likewise continuous functions of z. It follows that $M(g_{hk})$ is bounded in every bounded set of R, and that

$$\lim_{z\to Z} M(g_{hk}) = M(G_{hk}). \tag{37}$$

From now on we distinguish the three cases,

$$\text{(I)} \quad q \geqq 1,$$

$$\text{(II)} \quad q = 0 \quad \text{and} \quad A_{h0} \neq A_k,$$

$$\text{(III)} \quad q = 0 \quad \text{and} \quad A_{h0} = A_k,$$

and we put for shortness,

$$\begin{aligned} c_h(z) &= A_{h0} - a_{k0} = A_{h0} - A_k z_1 z_2 \cdots z_q && \text{in case (I)}, \\ &= A_{h0} - a_{k0} = A_{h0} - A_k && \text{in case (II)}, \\ &= A_{h1} - a_{k1} = A_{h1} + A_k\left(\sum_{l=1}^{r} z_l + \sum_{l=r+1}^{m} \mathrm{A}_{kl}\right) && \text{in case (III)}. \end{aligned} \tag{38}$$

Then $c_h(z)$ in none of these three cases can vanish identically in z. More exactly, in cases (I) and (II), $g_{hk}(x)$ has the highest term

$$c_h(z)x^m$$

and is of exact degree m except when $c_h(z) = 0$; and in the remaining case (III) it has the highest term

$$c_h(z)x^{m-1}$$

and is of exact degree $m - 1$ except again when $c_h(z) = 0$.

In this last case (III), $g_{hk}(x)$ has thus always at least one zero ∞, so that the same is true for $G_{hk}(x)$. Without loss of generality we may assume that the zeros B_{hkl} of $G_{hk}(x)$ have been numbered so that

$$\mathrm{B}_{hkl} = \infty \quad \text{for} \quad l = u(h, k).$$

Naturally also the remaining $u(h, k) - 1$ zeros of the first kind of $G_{hk}(x)$ need not all be finite.

18. If $(h, k) \in T_k^*$, let

$$\beta_{hk1}, \quad \beta_{hk2}, \quad \cdots, \quad \beta_{hkm}$$

be all the zeros of $g_{hk}(x)$, and

$$\mathrm{B}_{hk1}, \quad \mathrm{B}_{hk2}, \quad \cdots, \quad \mathrm{B}_{hkm}$$

all those of $G_{hk}(x)$. Our notation in §13 was chosen so that B_{hkl} is of the first kind if $1 \leqq l \leqq u(h, k)$ and is otherwise of the second or the third kind. In other words,

$$(39) \qquad \begin{aligned} &| \mathrm{B}_{hkl}^{-1} | < 1 \quad \text{if} \quad 1 \leqq l \leqq u(h, k), \\ &| \mathrm{B}_{hkl} | \leqq 1 \quad \text{if} \quad u(h, k) + 1 \leqq l \leqq m. \end{aligned}$$

By the first $u(h, k)$ of these inequalities, a positive constant ε can be found so small that

$$(40) \qquad | \mathrm{B}_{hkl}^{-1} | < 1 - 2\varepsilon \quad \text{if} \quad (h, k) \,\varepsilon\, T_k^* \text{ and } 1 \leqq l \leqq u(h, k).$$

Since $g_{hk}(x)$ tends to $G_{hk}(x)$ as $z \to Z$, it is by Hurwitz's theorem possible to number the zeros β_{hkl} of $g_{hk}(x)$ in such a manner that

$$(41) \qquad \left.\begin{cases} \beta_{hkl}^{-1} \to \mathrm{B}_{hkl}^{-1} & \text{if} \quad 1 \leqq l \leqq u(h, k) \\ \beta_{hkl} \to \mathrm{B}_{hkl} & \text{if} \quad u(h, k) + 1 \leqq l \leqq m \end{cases}\right\} \qquad \text{as } z \to Z.$$

Provided then that the constant $\delta > 0$ of §15 has already been chosen sufficiently small, it follows that

$$(42) \qquad \left.\begin{cases} | \beta_{hkl}^{-1} - \mathrm{B}_{hkl}^{-1} | < \varepsilon & \text{if} \quad 1 \leqq l \leqq u(h, k) \\ | \beta_{hkl} - \mathrm{B}_{hkl} | < \varepsilon & \text{if} \quad u(h, k) + 1 \leqq l \leqq m \end{cases}\right\} \qquad \text{where } z \in U.$$

On combining the formulae (39), (40), and (42), we find in particular that

$$(43)\qquad \left.\begin{cases} |\beta_{hkl}^{-1}| < (1-2\varepsilon)+\varepsilon = 1-\varepsilon & \text{if} \quad 1 \leqq l \leqq u(h,k) \\ |\beta_{hkl}| < 1+\varepsilon & \text{if} \quad u(h,k)+1 \leqq l \leqq m \end{cases}\right\}$$

where $z \epsilon U$.

Hence the first $u(h, k)$ zeros of $g_{hk}(x)$ lie in the open set

$$S_1 : |\beta| > (1-\varepsilon)^{-1} > 1+\varepsilon$$

and so are of the first kind, while its remaining zeros lie in the second open set

$$S_2 : |\beta| < 1+\varepsilon$$

and so may, or may not, be of the first kind. It is important to note that *not only S_1 and S_2, but also their closures, are disjoint.*

In case (III), we choose the notation so that

$$\beta_{hkl} = \infty \quad \text{for} \quad l = u(h,k),$$

which is evidently permissible. The $u(h, k) - 1$ further zeros

$$\beta_{hkl}, \quad \text{where} \quad 1 \leqq l \leqq u(h,k) - 1,$$

are then finite whenever $c_h(z)$ does not vanish.

19. Let (h, k) be again an arbitrary pair in T_k^*, and let z be any point in U. We associate with $g_{hk}(x)$ a certain function $\phi_h(z)$ of z which is defined as follows.

Put

$$(44)\qquad \begin{aligned} \phi_h(z) &= c_h(z) \textstyle\prod_{l=1}^{u(h,k)} \beta_{hkl} && \text{in cases (I) and (II),} \\ &= c_h(z) \textstyle\prod_{l=1}^{u(h,k)-1} \beta_{hkl} && \text{in case (III).} \end{aligned}$$

As long as $c_h(z)$ does not vanish, all factors β_{hkl} of $\phi_h(z)$ are finite, so that this function has a good meaning. By Hurwitz's theorem it is a continuous function of z and hence can be defined by continuity in the points where $c_h(z) = 0$. From its definition, in all three cases

$$(45)\qquad M(g_{hk}) = |\phi_h(z)| \textstyle\prod_{l=u(h,k)+1}^{m} \max(1, |\beta_{hkl}|).$$

Here the factor

$$P(z) = \textstyle\prod_{l=u(h,k)+1}^{m} \max(1, |\beta_{hkl}|)$$

is never less than 1; hence

$$(46)\qquad |\phi_h(z)| \leqq M(g_{hk}).$$

Further, in the limit as $z \to Z$, $P(Z) = 1$, and hence

$$(47)\qquad |\phi_h(Z)| = M(G_{hk}).$$

The m zeros β_{hkl} of $g_{hk}(x)$ are branches of algebraic functions of z. Therefore they will in general be multivalued, so that, when z describes any closed curve Γ in U, they will suffer a certain permutation. We saw in §18 that the first $u(h, k)$ zeros β_{hkl} lie in S_1, and the remaining zeros in S_2; moreover, in case (III), the zero $\beta_{hku(h,k)}$ has the constant value ∞ for all z. It follows that, when z describes Γ, the factors β_{hkl} of $\phi_h(z)$ are permuted only amongst themselves, and that therefore $\phi_h(z)$ remains unchanged. Hence, for $z \in U$, $\phi_h(z)$ is a *single-valued* branch of an algebraic function of z. Since this function is, moreover, continuous in U, it is regular in this region.

The basic inequality (35), together with the formulae (46) and (47), imply that

$$\sum_{(h,k)\in T_k^*} | \phi_h(z) | \leqq \sum_{(h,k)\in T_k^*} | \phi_h(Z) | \quad \text{if} \quad z \in U. \tag{48}$$

From this inequality we can now deduce the important fact that, for all pairs $(h, k) \in T_k^*$,

$$\phi_h(z) \equiv \phi_h(Z) \quad \textit{identically in } z \textit{ for } z \in U. \tag{49}$$

For let this assertion be false. Then there exist a pair (h^*, k) in T_k^* and a point z^* in U such that

$$\phi_{h^*}(z^*) \neq \phi_{h^*}(Z). \tag{50}$$

Denote by γ a constant such that

$$| z^* - Z | \leqq \gamma, \qquad 0 < \gamma < \delta,$$

and by V the closed subset

$$V : | z - Z | \leqq \gamma$$

of U; evidently z^* lies in V. Next let t be a complex variable for which

$$| t | \leqq 1.$$

The variable point

$$z = Z + t(z^* - Z)$$

then lies in V because

$$| z - Z | \leqq | t | | z^* - Z | \leqq \gamma.$$

Not all the functions of t defined by

$$\phi_h(z) = \phi_h(Z + t(z^* - Z)), \quad = \psi_h(t) \text{ say},$$

are constants because, by (50),

$$\psi_{h^*}(0) \neq \psi_{h^*}(1). \tag{51}$$

The functions $\psi_h(t)$ are single-valued and regular in the circle $| t | \leqq 1$. By (48), they satisfy the inequality

$$\sum_{(h,k)\in T_k^*} | \psi_h(t) | \leqq \sum_{(h,k)\in T_k^*} | \psi_h(0) | \quad \text{if} \quad | t | \leqq 1.$$

Lemma 3 therefore implies that all functions $\psi_h(t)$ are constants, contrary to (51).

This proves that all the functions $\phi_h(z)$ are constants in V. They are then also constants in U as we may allow γ to tend to δ.

20. The identities (49) have so far been proved only under the hypothesis that z lies in U. Since, however, the functions $\phi_h(z)$ are algebraic, these identities remain true for all z if the functions are defined outside U by analytic continuation.

The result so proved may be formulated as follows.

LEMMA 5. *Let (h, k) be any pair in T_k^*, and let z be an arbitrary point in R. Then the polynomial $g_{hk}(x)$ has a set of $u(h, k)$ zeros*

$$\beta_{hkl} \quad (1 \leqq l \leqq u(h, k))$$

in cases (I) *and* (II), *and a set of $u(h, k) - 1$ zeros*

$$\beta_{hkl} \quad (1 \leqq l \leqq u(h, k) - 1)$$

in case (III), *such that these zeros satisfy the identity*

$$\phi_h(z) \equiv \phi_h(Z).$$

In this lemma, the zero factors β_{hkl} of $\phi_h(z)$ will in general no longer be *zeros of $g_{hk}(x)$ of the first kind* when z is a general point in R.

21. Theorem 2 can now easily be deduced from Lemma 5.

From the definition (38) of $c_h(z)$ it is possible to choose r real numbers $\theta_1, \cdots, \theta_r$ such that the complex numbers

$$z_1 = e^{\theta_1 i}, \quad \cdots, \quad z_r = e^{\theta_r i}$$

satisfy the inequality

$$c_h(z) \neq 0.$$

Let $\mathbf{f}(x)$ be the polynomial vector that belongs to this choice of z, and for this special vector let $f_h(x)$ and $g_{hj}(x)$ be defined as before. Then, by (38), all zeros of $f_k(x)$ lie on the unit circle, thus are of the third kind, and hence

$$d(f_k) = 0 < d(F_k) = r, \qquad d(\mathbf{f}) < d(\mathbf{F}). \tag{52}$$

From the definition of $\mathbf{f}(x)$,

$$M(\mathbf{f}) = M(\mathbf{F}) = 1.$$

Next, trivially,

$$M(g_{hj}) = M(G_{hj}) \quad \text{if} \quad (h, j) \in T_k, \tag{53}$$

and

$$M(g_{hk}) \geqq 0 = M(G_{hk}) \quad \text{if} \quad (h, k) \in T_k^{**}. \tag{54}$$

Let, finally, (h, k) be any pair in T_k^* so that

$$M(g_{hk}) = | c_h(z) | \prod_{l=1}^{m^*} \max(1, | \beta_{hkl} |);$$

here m^* is equal to m in cases (I) and (II) and equal to $m - 1$ in case (III). In case (III) the fixed zero $\beta_{hku(h,k)} = \infty$ has been omitted as a factor from the product on the right-hand side. By the definition of $\phi_h(z)$ and by Lemma 5, this equation for $M(g_{hk})$ implies immediately that

$$(55) \quad M(g_{hk}) \geqq |\phi_h(z)| = |\phi_h(Z)| = M(G_{hk}) \quad \text{if} \quad (h, k) \in T_k^*.$$

On adding all the equations and inequalities (53), (54), and (55), we find that

$$N(\mathbf{f}) \geqq N(\mathbf{F}).$$

By (20), this implies that

$$N(\mathbf{f}) = N(\mathbf{F}).$$

Since $f(x)$ is a polynomial vector in Φ for which

$$n(f) = n_0 \quad \text{and} \quad d(f) < d_0,$$

we obtain a contradiction with respect to the definition of d_0.

The hypothesis in §11 was therefore false, and Theorem 2 is true.

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