PROOF OF A CONJECTURE OF G. PÓLYA CONCERNING GAP SERIES[1]

BY

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1. Statement and proof of Theorem 1

Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \cdots\}$ be an increasing sequence of nonnegative integers satisfying the gap condition

$$(1) \qquad \lambda_n/n \to \infty.$$

Let

$$(2) \qquad f(z) = \sum c_\lambda z^\lambda \qquad (\lambda \in \Lambda)$$

be an entire function of finite order; put

$$M(r, f) = \sup_\theta |f(re^{i\theta})|, \qquad L(r, f) = \inf_\theta |f(re^{i\theta})| \quad (\theta \text{ real}).$$

In a famous paper [3] G. Pólya conjectured that under the above conditions on Λ and $f(z)$

$$\overline{\lim} \frac{\log L(r,f)}{\log M(r,f)} = 1.$$

I shall prove in this note that the conjecture is correct:

THEOREM 1. *If the entire function* (2) *is of finite order, and if the sequence of exponents* λ *satisfies* (1), *then, given* $\varepsilon > 0$

$$\log L(r, f) > (1 - \varepsilon) \log M(r, f)$$

holds outside a set of logarithmic density 0.

The proof is based on the following result of T. Kövari which improves previous theorems by G. Pólya [3], P. Turán [4], and others.

THEOREM A [2, Theorem 1, p. 326]. *Let*

$$M(r, \gamma, \delta, f) = \sup_{\gamma \leqq \theta \leqq \gamma+\delta} |f(re^{i\theta})|.$$

If $f(z)$ *is an entire function of finite order given by* (2), *and if the exponents* λ *satisfy* (1), *then, given* $\delta > 0$, *and* $\eta > 0$,

$$(3) \qquad \log M(r, \gamma, \delta, f) > (1 - \eta) \log M(r, f)$$

holds, uniformly in γ, *outside a set of* r *of logarithmic density* 0.

Another tool in the proof of Theorem 1 is the following lemma which is an adaptation of Lemma 1 of [1].

Received July 1, 1962.

[1] Research supported by a grant from the National Science Foundation.

LEMMA 1. *Let $h(z)$ be a meromorphic function of finite order ρ. Let $T(r, h)$ be the Nevanlinna characteristic function of $h(z)$. Given $\zeta > 0$ and δ, $0 < \delta < \frac{1}{2}$, there is a constant $K(\rho, \zeta)$ such that for all r in a set of lower logarithmic density $> 1 - \zeta$ and for every interval J of length δ*

$$r \int_J \left| \frac{h'(re^{i\theta})}{h(re^{i\theta})} \right| d\theta < K(\rho, \zeta) \left(\delta \log \frac{1}{\delta}\right) T(r, h). \tag{4}$$

The proof of Lemma 1 is given in §2.

Proof of Theorem 1. Choose η, $\frac{1}{2}\varepsilon > \eta > 0$, and $\zeta > 0$. Determine δ, $\frac{1}{2} > \delta > 0$, so that (with the notation of Lemma 1)

$$K(\rho, \zeta)\delta \log(1/\delta) < \tfrac{1}{2}\varepsilon. \tag{5}$$

By Theorem A and Lemma 1, (3) and (4) hold simultaneously for all r in a set E of lower logarithmic density $> 1 - \zeta$. Let $r \in E$. Then, given ϕ, there is a real ψ such that $|\phi - \psi| < \delta$ and

$$\log |f(re^{i\psi})| > (1 - \eta) \log M(r, f).$$

Now

$$\log | f(re^{i\phi}) | = \log | f(re^{i\psi}) | + \int_\psi^\phi \frac{d}{d\theta} \log | f(re^{i\theta}) | \, d\theta$$

$$\geqq (1 - \eta) \log M(r, f) - r \int_\psi^\phi \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| | d\theta |,$$

since

$$\left| \frac{d}{d\theta} \log | f(re^{i\theta}) | \right| = \left| R \frac{d}{d\theta} \log f(re^{i\theta}) \right| \leqq \left| \frac{d}{d\theta} \log f(re^{i\theta}) \right| = r \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|.$$

Hence, by (4)

$$\log |f(re^{i\phi})| > (1 - \eta) \log M(r, f) - K(\rho, \zeta)(\delta \log (1/\delta))T(r, f).$$

But

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \overset{+}{\log} | f(re^{i\theta}) | \, d\theta \leqq \log M(r, f),$$

so that, by (5),

$$\log |f(re^{i\phi})| > (1 - \eta - K(\rho, \zeta)\delta \log(1/\delta)) \log M(r, f)$$

$$> (1 - \varepsilon) \log M(r, f),$$

i.e.,

$$\log L(r, f) > (1 - \varepsilon) \log M(r, f)$$

for all r in a set of lower logarithmic density $> 1 - \zeta$. Since ζ can be chosen arbitrarily small, Theorem 1 is proved.

2. Proof of Lemma 1

LEMMA 2. *If $T(u)$ is a positive, increasing, unbounded function such that*

$$\overline{\lim} \log T(u)/\log u = \rho < \infty \qquad (u \to \infty), \tag{6}$$

and if

$$b > 2\rho,$$

then

$$e^b T(u/e) \geqq T(u) \tag{7}$$

for all u outside a set G of upper logarithmic density

$$D \leqq 2\rho/b.$$

Proof. Put

$$\phi(x) = \log T(e^x).$$

Then $\phi(x) \geqq 0$ for $x \geqq x_0$, say. Let E be the set of x in $x_0 \leqq x \leqq \log U$ for which

$$b + \phi(x-1) < \phi(x).$$

Let g be the largest integer such that it is possible to place g nonoverlapping intervals of length 1 on the interval $x_0 \leqq x \leqq \log U$ so that their right-hand endpoints are in E. Then

$$\phi(\log U) \geqq gb.$$

Hence

$$g \leqq \phi(\log U)/b = \log T(U)/b.$$

Let $I_j : \alpha_j < x \leqq \alpha_j + 1$ $(j = 1, 2, \cdots, g)$ be a maximal set of intervals of the kind just described. Then no point of E can lie outside the union of the intervals

$$x_0 \leqq x < x_0 + 1, \qquad \alpha_j < x \leqq \alpha_j + 2 \quad (j = 1, 2, \cdots, g),$$

and therefore the measure of E satisfies

$$mE \leqq 2g + 1 \leqq 2(\log T(U)/b) + 1. \tag{8}$$

Given $\varepsilon > 0$ we have for $U > U_0(\varepsilon)$

$$mE < (2(\rho + \varepsilon)/b) \log U,$$

by (6) and (8). Under the mapping $u = e^x$ the set E is mapped into the set C of u, $e^{x_0} < u < U$ for which (7) is not satisfied. And

$$mE = \int_E dx = \int_C \frac{dt}{t} < \frac{2(\rho+\varepsilon)}{b} \log U \tag{9}$$

for $U > U_0(\varepsilon)$. The conclusion of the lemma now follows from (9) on letting $\varepsilon \to 0$.

LEMMA 3. *Let $h(z)$ be meromorphic in $|t| \leqq R$, and let $c_1, c_2, \cdots, c_m$ be the set of all zeros and all poles of $h(z)$ in $|t| \leqq R$. Then for $r < R$*

$$r\left|\frac{h'(re^{i\theta})}{h(re^{i\theta})}\right| \leqq \frac{4Rr}{(R-r)^2} T(R, h) + \sum_{k=1}^{m} \frac{2R}{|re^{i\theta} - c_k|}. \tag{10}$$

This lemma is an immediate consequence of the differentiated Poisson-Jensen formula (see [1, §3]).

LEMMA 4 (Special case of Cartan's Lemma). *Let* $\kappa > 0$. *Let* $c_1, \cdots, c_m$ *be m complex numbers. If* $r < R$ *lies outside an exceptional set F of intervals of total length*

$$mF \leqq \kappa R,$$

then there are at most $(j - 1)$ *of the numbers* $|c_k|$ *at a distance* $< j\kappa R/2m$ *from* r $(j = 1, 2, 3, \cdots, m)$.

We suppose now that r lies outside the exceptional set $F = F(R)$ of Lemma 4 and estimate

$$S = r \int_J \left| \frac{h'(re^{i\theta})}{h(re^{i\theta})} \right| d\theta,$$

where J is any interval of length δ. By (10)

$$S < \frac{4Rr}{(R - r)^2} \delta T(R, h) + \sum_{k=1}^{m} \int_J \frac{2R\, d\theta}{|re^{i\theta} - c_k|}. \tag{11}$$

We divide the c's into two classes:

(I) those c for which $|r - |c|| < \frac{1}{2}\delta r$,
(II) the others.

It is easy to see that

$$r \int_J \frac{d\theta}{|re^{i\theta} - c|}$$

attains its largest possible value for given r and c, if J has its midpoint at $\theta = \arg c$. Then

$$H = \sup_J r \int_J \frac{d\theta}{|re^{i\theta} - c|} = 2r \int_0^{\delta/2} \frac{d\theta}{|re^{i\theta} - |c||}.$$

If $c \in$ (II), i.e., $|r - |c|| \geqq \frac{1}{2}\delta r$, we use the estimate

$$|re^{i\theta} - |c|| \geqq |r - |c||;$$

$$H \leqq \delta r / |r - |c|| \qquad (c \in (\text{II})). \tag{12}$$

If $c \in$ (I), let $\gamma = |r - |c||/r < \frac{1}{2}\delta$. Then

$$H = 2r \int_0^{\gamma} + 2r \int_{\gamma}^{\delta/2} \frac{d\theta}{|re^{i\theta} - |c||}$$

$$\leqq \frac{2\gamma r}{|r - |c||} + 2r \int_{\gamma}^{\delta/2} \frac{d\theta}{|\operatorname{Im}(re^{i\theta} - |c|)|}$$

$$\leqq 2 + 2r \int_\gamma^{\delta/2} \frac{d\theta}{r \sin \theta} \leqq 2 + \pi \int_\gamma^{\delta/2} \frac{d\theta}{\theta}$$

$$\leqq 2 + \pi \log \frac{\frac{1}{2}\delta r}{|\, r - |\, c \,|\, |} \qquad (c \in (\mathrm{I})). \tag{13}$$

Since r is outside F, we deduce from Lemma 4 that the class (I) cannot contain more than M elements, where M is the largest integer such that

$$\tfrac{1}{2}\delta r \geqq M\kappa R/2m, \quad \text{i.e.,} \quad M \leqq \delta r m/\kappa R = M_0 . \tag{14}$$

A second application of Lemma 4 now shows that

$$\begin{aligned} \sum_{c \in (\mathrm{I})} \log \frac{\delta r/2}{|\, r - |\, c \,|\, |} &\leqq \sum_{j=1}^{M} \log \frac{\delta r m}{\kappa R j} \\ &= \sum_{j=1}^{M} \log \frac{M_0}{j} < \sum \int_{j-1}^{j} \log \frac{M_0}{x}\, dx < \int_0^{M_0} \log \frac{M_0}{x}\, dx = M_0 . \end{aligned} \tag{15}$$

Again by Lemma 4, there are at most $[M_0]$ c's in class (II) with $|\, r - |\, c \,|\, | = M_0\, \kappa R/2m = \frac{1}{2}\delta r$. There are at most $[M_0] + 1$ c's in class (II) with $|\, r - |\, c \,|\, | \leqq (M_0 + 1)\kappa R/2m, \cdots$. Hence

$$\begin{aligned} \sum_{c \in (\mathrm{II})} \frac{\frac{1}{2}\delta r}{|\, r - |\, c \,|\, |} &< [M_0] + \sum_{j=[M_0]+1}^{m} \frac{\frac{1}{2}\delta r}{(\kappa R j/2m)} \\ &< M_0 + \frac{\delta r m}{\kappa R} \sum_{[M_0]+1}^{m} \frac{1}{j} \\ &< M_0 + M_0 \left\{ \int_{[M_0]+1}^{m} \frac{dx}{x} + \frac{1}{[M_0] + 1} \right\} \\ &< M_0 + M_0 \left(\log \frac{m}{M_0} + \frac{1}{M_0} \right) \\ &< M_0 + M_0 \log \frac{m}{M_0} + 1. \end{aligned} \tag{16}$$

Using (13) and (15), (12) and (16) in (11) we find that for $r < R$, $r \notin F$.

$$\begin{aligned} S < \frac{4Rr}{(R - r)^2}\, \delta T(R, h) &+ \frac{4RM_0}{r} + \frac{2\pi R M_0}{r} \\ &+ \frac{2R}{r} M_0 + \frac{2R}{r} M_0 \log \frac{m}{M_0} + \frac{2R}{r}; \end{aligned}$$

or, substituting from (14) for M_0,

$$S < \frac{4Rr}{(R - r)^2}\, \delta T(R, h) + \left\{ (6 + 2\pi) + 2 \log \frac{\kappa R}{\delta r} \right\} \frac{\delta m}{\kappa} + \frac{2R}{r}. \tag{17}$$

In order to deduce Lemma 1 from (17) we must now describe in more detail the choice of r and R. Let

$$a = e^{1/4}, \qquad b = (2\rho + 1)/\kappa,$$

and consider the intervals

$$A_n : a^n < r \leqq a^{n+1} \qquad (n = 0, 1, 2, \cdots).$$

The characteristic function $T(u, h)$ satisfies all conditions imposed on $T(u)$ in Lemma 2. We shall call an interval A_n unsuitable, if the whole interval

$$a^{n+3} \leqq R' \leqq a^{n+4} = ea^n$$

is contained in the exceptional set G of Lemma 2 (for our choice of b and $T(u)$). If A_n is suitable (i.e., not unsuitable), then we can find an R',

$$a^{n+3} \leqq R' \leqq a^{n+4},$$

such that

$$e^b T(R'/e, h) > T(R', h),$$

and a fortiori, since $R'/e \leqq a^n \leqq r$, and $T(u, h)$ is increasing,

$$e^{(2\rho+1)/\kappa} T(r, h) = e^b T(r, h) > T(R', h) > T(a^{n+2}, h) = T(R, h), \tag{18}$$

where we have chosen

$$R = a^{n+2}.$$

By a well-known argument using Nevanlinna's first fundamental theorem

$$\begin{aligned} m = n(R, h) + n(R, 1/h) &\leqq \frac{1}{\log(R'/R)} \int_R^{R'} \frac{n(u, h) + n(u, 1/h)}{u}\, du \\ &\leqq (1/\log a)(N(R', h) + N(R', 1/h)) \\ &\leqq 4(T(R', h) + T(R', 1/h)) \\ &\leqq 8T(R', h) + O(1). \end{aligned} \tag{19}$$

Using (18) and (19) in (17) gives

$$\begin{aligned} r \int_J \left| \frac{h'(re^{i\theta})}{h(re^{i\theta})} \right| d\theta &< \{A_1(\kappa, \rho)\delta + A_2(\kappa, \rho)\delta \log(1/\delta)\} T(r, h) + A_3(\kappa, \rho) \\ &< A_4(\kappa, \rho)\delta \log(1/\delta) T(r, h) \quad (r > r_0 = r_0(\kappa, \rho, \delta, h)) \end{aligned} \tag{20}$$

for all r in a suitable interval A_n except those which lie inside an exceptional set F_n of measure $< \kappa R = \kappa a^{n+2}$. The logarithmic length of F_n is at most $\kappa a^{n+2} \cdot (1/a^n) = \kappa a^2$, so that the total logarithmic length L_1 of the exceptional sets F_n $(0 \leqq n \leqq N)$ satisfies

$$L_1 < \kappa a^2 (N + 1).$$

An interval A_n (logarithmic length $= \log a$) is unsuitable only if the interval A_{n+3} is contained in the exceptional set G of Lemma 2. The total logarithmic length L_2 of the unsuitable intervals A_n $(n \leqq N)$ is therefore less than the logarithmic length of the portion of G in $r \leqq a^{N+4}$, so that by Lemma 2

$$L_2 < ((2\rho + 1)/b) \cdot (N + 4) \log a \qquad (N > N_0).$$

With our value for b

$$L_2 < \kappa(N + 4) \log a.$$

Hence (20) holds in $r_0 \leqq r \leqq a^{N+1}$ outside a set of total logarithmic length less than

$$L_1 + L_2 < \kappa(a^2 + \log a)N + \kappa a^2 + 4\kappa \log a.$$

The upper logarithmic density of the exceptional set in which (20) is not valid is therefore

$$D \leqq \lim_{N\to\infty} (1/\log(a^N))(\kappa(a^2 + \log a)N + \kappa a^2 + 4\kappa \log a);$$

$$D \leqq \kappa(a^2 + \log a)/\log a.$$

Lemma 2 follows, if we determine κ from the equation

$$\zeta = \kappa(a^2 + \log a)/\log a.$$

References

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