LIMIT THEOREMS FOR RANDOM WALKS, BIRTH AND DEATH PROCESSES, AND DIFFUSION PROCESSES[1]

BY
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Introduction

As defined here, the three classes of Markov processes mentioned in the title of this paper have in common the fact that the basic state space is a subset of the reals, and the random trajectories do not jump over points in the state space. They are also regular in that a process starting at any point in the state space (with the possible exception of the left and right end points) can, with positive probability, reach any other point in the state space.

In the discrete-parameter case, any such process has a discrete state space and is a random walk. In the continuous-parameter case, if the state space is an interval, the path functions are continuous, and the process is a diffusion process; if the state space is discrete, the process is a birth and death process. We include both possibilities by allowing the state space to be any closed (except possibly at end points) subset of the reals.

These processes are all very similar in their analytic and probabilistic structure. When put in their "natural scale", they are determined by a speed measure $m(dx)$ and a killing measure $k(dx)$ (and a time unit θ in the case of random walks).

It is fairly obvious that in some sense the processes depend continuously on $m(dx)$ and $k(dx)$. The purpose of this paper is to investigate some of the probabilistic aspects of this continuity. In order to do so, we use a method of Itô and McKean [1] to construct all these processes on a single probability space. This construction involves the use of "local time" for Brownian motion, whose existence and continuity properties were obtained by Trotter [9]. The construction shows that all the processes have local times. If the state space is discrete, the local time is simply the (normalized) occupation time.

In Section 1 we summarize the construction in the continuous-parameter case, and in Section 3 we extend it to the discrete-parameter or random-walk case.

In Section 2 we consider a sequence $X_n(x_n\,; t)$, $n \geqq 0$, of continuous-parameter processes with measures $m_n(dx)$ and $k_n(dx)$ and initial state x_n. We suppose that $m_n(dx)$, $k_n(dx)$, and x_n converge suitably to $m_0(dx)$, $k_0(dx)$, and x_0 respectively, and that certain other conditions are satisfied. It then follows that several classes of functionals of the $X_n(x_n\,; t)$ process converge with probability 1 to the corresponding functional of the $X_0(x_0\,; t)$ process.

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In particular the local times converge (essentially uniformly, in all states and all bounded time intervals), and hence so do such other functionals as occupation times which can be defined in terms of local times. The path functions also converge. In nice cases this convergence is uniform in bounded time intervals. In general, however, we are led to convergence in a weaker sense (similar to the J_1-convergence of Skorokhod [7]).

In Section 4 we replace the $X_n(x_n ; t)$, $n \geqq 1$, by random walks $W_n(x_n ; t)$ and consider the same problems. In general we get convergence in probability, but under a slight further condition we get convergence with probability 1 as before. Perhaps the most interesting results concern the uniformity of the convergence of the local time (normalized occupation time) of the random walks to the local time of the limiting diffusion process (see Examples 1 and 2 at the end of Section 4).

The results on convergence with probability 1 and convergence in probability depend of course on the specific construction used. They yield as immediate corollaries, however, statements concerning convergence in distribution which do not depend on the construction. This method of obtaining weak convergence of functionals has been used by Skorokhod [7], Knight [3], and Itô and McKean [1].

In Section 5 we consider a single process $X(x; t)$ with killing measure $k(dx) = 0$ and speed measure $m(dx) \sim | x |^{\beta+1}L(x)$, where $\beta + 1 > 0$ and $L(x)$ is suitably slowly varying as $| x | \to \infty$. The asymptotic behavior of $X(x; t)$ is investigated by reducing the problem to one involving a sequence of processes such that the results of Sections 2 and 4 are applicable.

Some specific limit theorems for birth and death processes were obtained by Karlin and McGregor [2], who used purely analytic methods such as their representation theorem and the Darling-Kac theorem connecting occupation time laws with the Mittag-Leffler distribution. The original motivation of this paper came from trying to understand and extend these results. The material in Section 5 is closely related to [2] and also to some work by Lamperti [4] and [5].

Knight [3] has given a different construction in which the paths of simple random walks converge uniformly in bounded time intervals to those of Brownian motion. He also obtained convergence of the local times (uniformly in time, but not in space) and extended his results to somewhat more general random walks and diffusion processes.

This paper is closely related to the author's Stanford Ph.D. thesis. I wish to thank Professor Karlin for his encouragement and guidance of my work on that thesis. I also wish to thank Professors K. Itô and D. Ray for several helpful discussions concerning the theory of diffusion processes.

1. Construction of the continuous-parameter processes

In this section we use the method of Itô and McKean of obtaining diffusion processes from Brownian motion. Let $Y(t)$ be a Brownian motion process with continuous paths, $Y(0) = 0$, and infinitesimal generator d^2/dx^2 (this normalization is slightly more convenient than is $\frac{1}{2}\, d^2/dx^2$).

Of course $Y(t) = Y(t, \omega)$ is defined on some probability space $(\Omega, \mathcal{B}, P)$. For simplicity in notation we suppress the dependence on ω. All other processes considered in this paper will be defined in terms of Brownian motion and hence will have the same probability space. We assume that this space is chosen so that there is an exponentially distributed random variable independent of $Y(t)$, $t \geqq 0$. Given any finite or countably infinite number of statements, each of which is true with probability 1, we can remove from Ω a subset of probability 0 such that on the remaining space the statements hold everywhere.

The present construction is based on the following theorem of Trotter [9]. The corollary is an immediate consequence of the theorem (see [1] or [8]). Here we let M denote Lebesgue measure.

THEOREM (Trotter). *With probability* 1, *there is a function* $L(t, y)$ *jointly continuous in* t *and* y, *such that for every Borel set* $A \subset (-\infty, \infty)$ *and all* $t \geqq 0$,

$$M\{\tau \mid 0 \leqq \tau \leqq t \text{ and } Y(\tau) \in A\} = \int_A L(t, y)\, dy.$$

COROLLARY. *For each fixed* $y \in (-\infty, \infty)$,

$$\inf\,[t \mid L(t, y) > 0] = \inf\,[t \mid Y(t) = y]$$

with probability 1. *Also, with probability* 1, $L(t, y) \to \infty$ *as* $t \to \infty$ *uniformly for* y *in bounded sets.*

We may assume that the statement of the theorem, the second statement of the corollary and the first statement of the corollary with $y = 0$ all hold everywhere. It follows that $L(t, y) \geqq 0$; if $t_1 \leqq t_2$, then $L(t_1, y) \leqq L(t_2, y)$; and if $Y(t) \notin (a, b)$ for $t_1 < t < t_2$, then $L(t_1, y) = L(t_2, y)$ for $a \leqq y \leqq b$.

Let $m(x)$, $-\infty < x < \infty$, be a right-continuous, nondecreasing nonconstant extended real-valued function. We call x a point of increase of $m(x)$ if $m(x_2) > m(x_1)$ whenever $x_1 < x < x_2$. Let E be the closed set of points of increase of $m(x)$. Now $m(x)$ defines a measure on $(-\infty, \infty)$ in the usual way. This measure has support in E. Let

$$a = \inf\,[x \mid x \in E] \quad \text{and} \quad b = \sup\,[x \mid x \in E].$$

Let B be the subset of $\{a, b\}$ such that $a \in B$ if and only if $a > -\infty$ and $m(a-) = -\infty$, and $b \in B$ if and only if $b < +\infty$ and $n(b) = +\infty$.

For $x \in E$ and $t \geqq 0$ we define successively

$$\tau(x; B) = \inf\,[t \mid Y(t) + x \in B];$$

$$S(x; B) = \int_{-\infty}^{\infty} L(t, y - x)\, m(dy) \quad \text{if} \quad 0 \leqq t < \tau(x; B),$$

$$S(x; B) = \infty \quad \text{if} \quad t \geqq \tau(x; B);$$

$$S^{-1}(x; t) = \sup [\tau \mid S(x; \tau) \leqq t]; \qquad Z(x; t) = Y(S^{-1}(x; t)) + x;$$

$$L_Z(x; t, y) = L(S^{-1}(x; t), y - x), \qquad y \in E.$$

If $E_1 \subset E$ is a Borel set, we define

$$t(x; E_1) = \inf [t \mid Z(x; t) \in E_1].$$

We now summarize the elementary properties of the random variables so defined, referring the reader to [1] and [8] for more details.

For each fixed $x \in E$ the random function $S(x; t)$ is nondecreasing in t, is continuous in t except possibly at $t = \tau(x; B)$, is strictly positive for $t > 0$, and approaches ∞ as $t \to \infty$. It is an additive functional of the process $Y(t)$. The function $S^{-1}(x; t)$ is a right-continuous, nondecreasing function with $S^{-1}(x; 0) = 0$. It is strictly increasing in t for $0 \leqq t < S(x; \tau(x; B)-)$ and equals $\tau(x; B)$ for $t \geqq S(x; \tau(x; B)-)$. The process $Z(x; t)$ takes on values from the set E. As a function of t, $Z(x; t)$ is continuous from the right and has limits from the left. The process $Z(x; t)$ is a strong Markov process with state space E, initial state x, and stationary transition probabilities. The function $L_Z(x; t, y)$, $t \geqq 0$ and $y \in E$, is jointly continuous in t and y. For each Borel set $A \subset E$ and $0 \leqq t \leqq S(x; \tau(x, B)-)$ (see [1] or [8])

$$M\{\tau \mid 0 \leqq \tau \leqq t \text{ and } Z(x; \tau) \in A\} = \int_A L_Z(x; t, y)\, m(dy).$$

For each fixed $x \in E$, with probability 1 the set of times when $S(x; t)$ is increasing in t coincides with the set of times $t \leqq \tau(x; B)$ when $Y(t) + x \in E$.

We now use strongly, for the first time, the fact that $Y(t)$ is actually continuous. Let $x \in E$ be fixed as before. With probability 1, if $0 \leqq t_1 < t_3$, $Z(x; t_1) = y_1$, $Z(x; t_3) = y_3$, $y_2 \in E$, and either $y_1 < y_2 < y_3$ or $y_1 > y_2 > y_3$, then there is a $t_2 \in (t_1, t_3)$ such that $Z(x; t_2) = y_2$. Thus if E is an interval, then $Z(x; t)$ is a diffusion process in the usual sense of having continuous paths; and if E is discrete, then $Z(x; t)$ is a birth and death process.

Let $k(x)$, $-\infty < x < \infty$, be a right-continuous, nondecreasing, possibly constant, extended real-valued function whose points of increase are contained in E, which is finite for $a < x < b$, and which has no infinite jumps in B. Let C be the subset of $\{a, b\}$ such that $a \in C$ if and only if $a > -\infty$ and $k(a-) = -\infty$, and $b \in C$ if and only if $b < +\infty$ and $k(b) = +\infty$.

Let Δ be an abstract point, and set $\bar{E} = E \cup \{\Delta\}$. Let $e(\Delta)$ be a random variable independent of the process $Y(t)$, hence also of $Z(x; t)$, and having an exponential distribution with mean 1 (such a random variable exists by an earlier assumption).

Set $t(\Delta; \Delta) = 0$, and for $x \in E$ set

$$t(x; \Delta) = \inf \left[t \,\middle|\, \int_{-\infty}^{\infty} L_Z(x; t, y)\, k(dy) \geqq e(\Delta) \right].$$

(Here, if $L_Z(x; t, y) = 0$ and $k(dy) = \infty$, we define $L_Z(x; t, y)k(dy)$ to be zero in the integrand.) For $x \in \bar{E}$ and $y \in E$, we set

$$\begin{aligned} X(x;t) &= Z(x;t), & L_X(x;t,y) &= L_Z(x;t,y) && \text{if} \quad t < t(x;\Delta), \\ &= \Delta, & &= L_Z(x;t(x;\Delta),y) && \text{if} \quad t \geqq t(x;\Delta). \end{aligned}$$

The process $X(x; t)$ is a strong Markov process with state space $\bar{E} - C$, initial state x, and stationary transition probabilities. The measures $m(dx)$ and $k(dx)$ are called the speed and killing measures of the process. If $k(dx) = 0$ (i.e., if $k(x)$ is constant) the process is called conservative. If E is an interval, $X(x; t)$ is a diffusion process in the usual sense; and if E is discrete, $X(x; t)$ is a birth and death process (except that we allow the possibility of killing from all points of E).

2. Convergence of continuous-parameter processes

In this section we first give a definition of convergence of path functions. We then consider a sequence of processes constructed as in Section 1. In Theorem 1 we obtain convergence of several types of functionals of the processes under certain conditions preceding the statement of this theorem.

Set $R = (-\infty, \infty)$, $R' = R \cup \{\Delta\}$, and let K denote the space of all functions $x(t)$, $t \geqq 0$, whose values lie in R', which are such that $x(t) = \Delta$ entails $x(t') = \Delta$ for $t' \geqq t$, and which at every point $t < \inf [\tau \mid x(\tau) = \Delta]$ are continuous from the right and (for $t > 0$) have limits from the left.

Let ρ be a pseudo metric on K (we allow $\rho(x, y) = 0$ for $x \neq y$) such that if $x_n \in R$ for $n \geqq 0$ and $x_n - x_0 \to 0$, then $\rho(x_n, x_0) \to 0$.

In terms of ρ we define J-convergence in K: A sequence $x_n(t)$ is said to be J-convergent to $x_0(t)$ if there is a sequence of continuous one-to-one mappings $\lambda_n(t)$ of $[0, \infty]$ onto itself such that for each $N > 0$

$$\sup_{0 \leqq t \leqq N} |\lambda_n(t) - t| \to 0 \quad \text{and} \quad \sup_{0 \leqq t \leqq N} \rho(x_n(t), x_0(\lambda_n(t))) \to 0$$

as $n \to \infty$. As a special case, if $x_0(t)$ is continuous, then $x_n(t)$ is J-convergent to $x_0(t)$ if and only if for each $N > 0$

$$\sup_{0 \leqq t \leqq N} \rho(x_n(t), x_0(t)) \to 0 \qquad \text{as } n \to \infty.$$

Let f_n, $n \geqq 0$, be a sequence of real-valued (Borel-measurable) functionals on K. We say that the sequence $\{f_n\}$ is J-continuous at a point $x_0(t)$ in K if $f_n(x_n(t)) \to f_0(x_0(t))$ whenever $x_n(t)$ is J-convergent to $x_0(t)$.

Given a stochastic process $X(t)$ whose paths are in K, let $F(X(t))$ be the collection of all such sequences of functionals which are continuous almost everywhere with respect to the measure on K induced by the process $X(t)$.

If $X_n(t)$, $n \geqq 0$, are stochastic processes whose paths lie in K, we say that $X_n(t)$ is weakly convergent to $X_0(t)$ if the distribution of $f_n(X_n(t))$ converges to the distribution of $f_0(X_0(t))$ for all sequences $f_n \in F(X_0(t))$.

Let $X_n(x_n; t)$ be regular diffusion processes as constructed in Section 1. We use the subscript n to denote relationship to the n^{th} such process.

For $n \geqq 0$, let $b_n(x)$, $-\infty < x < \infty$, be a finite, right-continuous, non-decreasing function whose set of points of increase is contained in E_n. Let

$$V_{X_n}(x_n ; t) = \int_{-\infty}^{\infty} L_{X_n}(x_n ; t, y)\, b_n(dy).$$

The random function $V_{X_n}(x_n ; t)$ is an additive functional on the process $X_n(x_n ; t)$. As a special case, if $b_n(dx) = v_n(x)m_n(dx)$ (i.e., if $b_n(dx)$ is absolutely continuous with respect to $m_n(dx)$, then

$$V_{X_n}(x_n ; t) = \int_0^t v_n(X_n(x_n ; \tau))\, d\tau.$$

As a further specialization, if $v_n(x)$ is the characteristic function of a Borel set, then $V_{X_n}(x_n ; t)$ is the time spent in that set until time t.

The convergence theorem will depend on the following conditions (unless otherwise specified, continuity and convergence refer to the usual metric $| x - y |$):

(i) $m_n(x) \to m_0(x)$ at all points of continuity of $m_0(x)$;
(ii) $k_n(x) \to k_0(x)$ at all points of continuity of $k_0(x)$;
(iii) $x_n \in E_n - C_n$ for $n \geqq 0$, and $x_n \to x_0$;
(iv) the sets E_n, $n \geqq 0$, are such that whenever $y_n \in E_n$ for $n \geqq 1$, $y_n \to y_0$, and $a_0 < y_0 < b_0$, then $y_0 \in E_0$;
(v) if $a_0 \in B_0 \cap C_0$, then $\rho(a_0, \Delta) = 0$;
(vi) if $b_0 \in B_0 \cap C_0$, then $\rho(b_0, \Delta) = 0$;
(vii) if $a_0 > -\infty$, $a_0 \notin B_0 \cup C_0$ and $\liminf_n \inf [x \mid x \in E_n] < a_0$, then $\rho(a_0, y) = 0$ for $y < a_0$;
(viii) if $b_0 < \infty$, $b_0 \notin B_0 \cup C_0$, and $\limsup_n \sup [x \mid x \in E_n] > b_0$, then $\rho(b_0, y) = 0$ for $y > b_0$;
(ix) $b_n(x) \to b_0(x)$ at all points of continuity of $b_0(x)$.

Perhaps a brief discussion of these conditions is in order. Conditions (i), (ii), and (iii) are obviously natural ones. Condition (iv) is necessary to have J-convergence of the path functions and could probably be eliminated by changing the definition of J-convergence somewhat. It is vacuous if E_0 is an interval. Conditions (v) and (vi) are necessary because when $X_n(x_n ; t)$ approaches a point in $B_0 \cap C_0$, it can not be determined merely from (i)–(iii) whether $X_n(x_n ; t)$ stays near $B_0 \cap C_0$ or jumps to Δ. Conditions (vii) and (viii), which are similar to (iv), may be relevant in applying Theorem 1 to processes which have been converted from their original to their natural scale. Condition (ix) concerns not the processes themselves, but only the functionals $V_{X_n}(x_n ; t)$.

THEOREM 1. *Suppose conditions* (i)–(ix) *above are satisfied. Then, with probability* 1, *as* $n \to \infty$

(1) $X_n(x_n ; t)$ *is J-convergent to* $X_0(x_0 ; t)$;

(2) *for any* $\varepsilon > 0$ *and* $M > 0$ *there exist* $n_0 > 0$ *and* $\delta > 0$ *such that*

$$| L_{X_n}(x_n ; t, y_n) - L_{X_0}(x_0 ; t, y_0) | < \varepsilon$$

uniformly for $n \geqq n_0$, $t \leqq M$, $y_n \in E_n$, $y_0 \in E_0$, *and* $| y_n - y_0 | \leqq \delta$;

(3) $V_{X_n}(x_n ; t) \to V_{X_0}(x_0 ; t)$ *for all* $t \geqq 0$;

(4) $\sup [\tau \mid V_{X_n}(x_n ; \tau) \leqq t] \to \sup [\tau \mid V_{X_0}(x_0 ; \tau) \leqq t]$ *at all points t of continuity of the latter function.*

Remark. It follows from (1) and (2) that for every $M > 0$

(2′) $\lim_{n\to\infty} \sup_{0\leqq t\leqq M} \sup_{y\in E_n\cap E_0} | L_{X_n}(x_n ; t, y) - L_{X_0}(x_0 ; t, y) | = 0.$

COROLLARY 1. *As* $n \to \infty$

(5) $X_n(x_n ; t)$ *is weakly convergent to* $X_0(x_0 ; t)$;

(6) $V_{X_n}(x_n ; t)$ *converges in distribution to* $V_{X_0}(x_0 ; t)$ *for all* $t \geqq 0$;

(7) $\sup [\tau \mid V_{X_n}(x_n ; \tau) \leqq t]$ *converges in distribution to* $\sup [\tau \mid V_{X_0}(x_0 ; \tau) \leqq t]$ *for all* $t > 0$.

Proof. The function $S_0(x_0 ; t)$ is nondecreasing in t, strictly positive for $t > 0$, continuous for $t \neq \tau_0(x_0 ; B_0)$, approaches ∞ as $t \to \infty$, and equals ∞ for $t \geqq \tau_0(x_0 ; B_0)$. With probability 1, for $n > 0$ the set of times t when $S_n(x_n ; t)$ is increasing in t coincides with the set of times $t \leqq \tau_n(x_n ; B_n)$ when $Y(t) + x_n \in E_n$. It follows from conditions (i) and (iii) and the first statement of the corollary to Trotter's theorem that, with probability 1, $S_n(x_n ; t) \to S_0(x_0 ; t)$ for $t \neq \tau(x_0 ; B_0)$. We may assume that the last two statements hold everywhere.

Let $Y(t)$ be a fixed Brownian motion path. Let $y_n \in E_n$ for $n \geqq 0$ and $y_n \to y_0$. Since $L(t, y)$ is jointly continuous in t and y, we see that as $n \to \infty$

$$L(S_n^{-1}(x_n ; t, y_n - x_n) - L(S_n^{-1}(x_n ; t), y_0 - x_0) \to 0$$

uniformly for t in bounded sets. Since $L(S_0^{-1}(x_0 ; t), y_0 - x_0)$ is continuous in t for $y_0 \in E_0$, $S_n^{-1}(x_n ; t) \to S_0^{-1}(x_0 ; t)$ at all points of continuity of $S_0^{-1}(x_0 ; t)$, and these functions are all nondecreasing in t, it follows that

$$L(S_n^{-1}(x_n ; t), y_0 - x_0) \to L(S_0^{-1}(x_0 ; t), y_0 - x_0)$$

uniformly for t in bounded sets. Therefore

$$L_{Z_n}(x_n ; t, y_n) = L(S_n^{-1}(x_n ; t), y_n - x_n)$$
$$\to L(S_0^{-1}(x_0 ; t), y_0 - x_0) = L_{Z_0}(x_0 ; t, y_0)$$

uniformly for t in bounded sets.

The last statement is equivalent to conclusion (2) if all processes are conservative. In order to verify conclusion (2) in general, we have to investigate convergence of $t_n(x_n ; \Delta)$ to $t_0(x_0 ; \Delta)$.

There are several cases that must be distinguished. By condition (ii)

$$\int_{-\infty}^{\infty} L_{Z_n}(x_n ; t, y)\, k_n(dy) \to \int_{-\infty}^{\infty} L_{Z_0}(x_0 ; t, y)\, k_0(dy)$$

for $t < t_0(x_0\,; C_0)$. Thus if $t_0(x_0\,; \Delta) = t_0(x_0\,; C_0) = +\infty$, then

$$t_n(x_n\,; \Delta) \to t_0(x_0\,; \Delta).$$

Now $Z_0(x_0\,; t)$ is a strong Markov process. Hence, with probability 1, if $t_0(x_0\,; \Delta) < t_0(x_0\,; C_0)$, then

$$\int_{-\infty}^{\infty} L_{Z_0}(x_0\,; t, y)\, k_0(dy) > e(\Delta)$$

for $t > t_0(x_0\,; \Delta)$, and $t_n(x_n\,; \Delta) \to t_0(x_0\,; \Delta)$.

With probability 1, if

$$t_0(x_0\,; C_0 \cap \sim B_0) < \infty \quad \text{and} \quad t_0(x_0\,; \Delta) \geqq t_0(x_0\,; C_0 \cap \sim B_0),$$

then

$$\int_{-\infty}^{\infty} L_{Z_0}(x_0\,; t, y)\, k_0(dy) = +\infty \quad \text{and} \quad \int_{-\infty}^{\infty} L_{Z_n}(x_n\,; t, y)\, k_n(dy) \to +\infty$$

for $t > t_0(x_0\,; C_0 \cap \sim B_0)$; hence

$$t_0(x_0\,; \Delta) = t_0(x_0\,; C_0 \cap \sim B_0) \quad \text{and} \quad t_n(x_n\,; \Delta) \to t_0(x_0\,; \Delta).$$

We may assume that the statements of the above two paragraphs hold everywhere. In any of the cases discussed above, we now have that

$$t_n(x_n\,; \Delta) \to t_0(x_0\,; \Delta),$$

and conclusion (2) holds.

In the remaining case $t_0(x_0\,; C_0 \cap B_0) < \infty = t_0(x_0\,; C_0 \cap \sim B_0)$ and $t_0(x_0\,; \Delta) \geqq t_0(x_0\,; C_0 \cap B_0)$. Except for a set of probability zero (which we remove from Ω), $t_0(x_0\,; \Delta) = +\infty$. It is not necessarily true in this case that $t_n(x_n\,; \Delta) \to +\infty$. It is true, though, that

$$\liminf_n t_n(x_n\,; \Delta) \geqq t_0(x_0\,; C_0 \cap B_0).$$

Since $L_{Z_0}(x_0\,; t, y)$ is constant for $t \geqq t_0(x_0\,; B_0)$, conclusion (2) still holds.

We have now verified that (2) holds in all possible cases. Condition (ix) now implies that (3) holds, and hence also (4) and (6). Since $X_n(x_n\,; t)$ is a strong Markov process, it is clear that at any fixed $t > 0$, $\sup\,[\tau \mid V_{X_0}(x_0\,; \tau) \leqq s]$ is continuous at $s = t$ with probability 1. Therefore (7) is a consequence of (4).

Since (5) follows immediately from (1), the proof of Theorem 1 and Corollary 1 will be complete as soon as we prove (1).

It is easy to show that, with probability 1, if $t_0(x_0\,; \Delta) < \infty$, then $S_0^{-1}(x_0\,; t)$ is continuous at $t = t_0(x_0\,; \Delta)$, and hence $Z_0(x_0\,; t)$ is continuous at $t = t_0(x_0\,; \Delta)$. We assume that this statement holds everywhere on Ω.

In order to verify that (1) holds, we have to construct suitable mappings $\lambda_n(t)$. Choose $N > 0$ and $\varepsilon > 0$. Let $\{t_0^{(k)} \mid 1 \leqq k \leqq m_1\}$ be the set of all times $t_0^{(k)} \in [0, N]$ such that $t_0^{(k)} < t_0(x_0\,;\Delta)$ and

$$S_0^{-1}(x_0\,;\,t_0^{(k)}) - S_0^{-1}(x_0\,;\,t_0^{(k)}-) \geqq \varepsilon,$$

i.e., $S_0^{-1}(x_0\,;t)$ has a jump of at least ε at $t = t_0^{(k)}$. (We set $m_1 = 0$ if this set is empty.) If $t_0(x_0\,;\Delta) < \infty$, we set $m = m_1 + 1$ and $t_0^{(m)} = t_0(x_0\,;\Delta)$. If $t_0(x_0\,;\Delta) = \infty$, we set $m = m_1$. We can assume that $t_0^{(0)} < t_0^{(1)} < \cdots < t_0^{(m)}$, where we set $t_0^{(0)} = 0$. For $n \geqq 1$ set $t_n^{(0)} = 0$ and

$$t_n^{(k)} = S_n(x_n\,;\,\tfrac{1}{2}(S_0^{-1}(x_0\,;\,t_0^{(k)}) + S_0^{-1}(x_0\,;\,t_0^{(k)}-\,))), \qquad 1 \leqq k \leqq m_1,$$

and if $m = m_1 + 1$, set $t_n^{(m)} = t_n(x_n\,;\Delta)$.

It follows from the convergence of $S_n(x_n\,;t)$ to $S_0(x_0\,;t)$ that $t_n^{(k)} \to t_0^{(k)}$ for $0 \leqq k \leqq m_1$, and it follows from the convergence of $t_n(x_n\,;\Delta)$ to $t_0(x_0\,;\Delta)$ that if $m = m_1 + 1$, then $t_n^{(m)} \to t_0^{(m)}$. Thus there is an $n_0 > 0$ such that if $n \geqq n_0$, then $t_n^{(0)} < t_n^{(1)} < \cdots < t_n^{(m)}$.

We construct functions $\lambda_n(t, N, \varepsilon)$ as follows: $\lambda_n(t, N, \varepsilon) = t$ for $1 \leqq n < n_0$ and $t \geqq 0$; if $n \geqq n_0$, then $\lambda_n(t_n^{(k)}, N, \varepsilon) = t_0^{(k)}$ and $\lambda_n(t, N, \varepsilon)$ is extended to all of t $\geqq 0$ by linear interpolation

$$(\lambda_n(t, N, \varepsilon) = t + t_0^{(m)} - t_n^{(m)} \quad \text{for} \quad t \geqq t_n^{(m)}).$$

The functions $\lambda_n(t, N, \varepsilon)$ so defined are continuous one-to-one mappings of $[0, \infty)$ onto itself. Furthermore

$$\sup_{0 \leqq t \leqq N} |\,\lambda_n(t, N, \varepsilon) - t\,| \to 0 \qquad \text{as } n \to \infty.$$

We can choose $N(n)$ and $\varepsilon(n)$ so that $N(n) \to \infty$ and $\varepsilon(n) \to 0$ as $n \to \infty$ and for each fixed $N > 0$

$$\sup_{0 \leqq t \leqq N} |\,\lambda_n(t, N(n), \varepsilon(n)) - t\,| \to 0 \qquad \text{as } n \to \infty.$$

For $n > 0$ set $\lambda_n(t) = \lambda_n(t, N(n), \varepsilon(n))$. Then each $\lambda_n(t)$ is a continuous one-to-one mapping of $[0, \infty)$ onto itself. For each $N > 0$

$$\sup_{0 \leqq t \leqq N} |\,\lambda_n(t) - t\,| \to 0 \qquad \text{as } n \to \infty.$$

In order to prove (5) we need to show that for every $N > 0$

$$\sup_{0 \leqq t \leqq N} \rho(X_n(x_n\,;t), X_0(x_0\,;\lambda_n(t))) \to 0 \qquad \text{as } n \to \infty.$$

If $t_0(x_0\,;\Delta) < \infty$, then $\lambda_n(t_n(x_n\,;\Delta)) = t_0(x_0\,;\Delta)$ for n sufficiently large. If $t_0(x_0\,;\Delta) = \infty$, then $\liminf_n t_n(x_n\,;\Delta) \geqq t_0(x_0\,;B_0 \cap C_0)$. Any point in $B_0 \cap C_0$ is an accumulation point of E_0; hence if $s_0 = t_0(x_0\,;B_0 \cap C_0) < \infty$, then $X_0(x_0\,;t) \to X_0(x_0\,;s_0)$ as $t \to s_0$. Thus by conditions (v) and (vi), in either case it suffices to show that

$$\sup_{0 \leqq t \leqq N} \rho(Z_n(x_n\,;t), Z_0(x_0\,;\lambda_n(t))) \to 0 \qquad \text{as } n \to \infty.$$

Alternatively, it suffices to show that if $t_n \to t_0 < \infty$, then

$$\rho(Z_n(x_n\,;t_n), Z_0(x_0\,;\lambda_n(t_n))) \to 0 \qquad \text{as } n \to \infty.$$

That this is indeed the case now follows from the construction of $\lambda_n(t)$ and conditions (iii), (iv), (vii), and (viii). The details are left to the reader.

3. Construction of random walks

In this section we shall construct regular random walks in terms of birth and death processes, and hence in terms of Brownian motion. In the following section this construction will yield limit theorems for random walks similar to those of Section 2.

Let $Z(x;t)$ and $X(x;t)$ be processes constructed as in Section 1. We change notation slightly and let E' (instead of E) denote the state space of $Z(x;t)$. We suppose that E' is the closure of a set of the form

$$E = \{\alpha_i \mid i_1 - 1 < i < i_2 + 1\},$$

where α_i is a strictly increasing sequence and $i_2 - i_1 \geqq 2$. Then $Z(x;t)$ is a birth and death process, and $X(x;t)$ is a birth and death process with killing allowed. Let $m\{\alpha_i\} = m(\alpha_i) - m(\alpha_i -)$ and $k\{\alpha_i\} = k(\alpha_i) - k(\alpha_i -)$. We shall relate the transition rates of $X(x;t)$ to α_i, $m\{\alpha_i\}$, and $k\{\alpha_i\}$.

Given α_{i-1}, α_i, α_{i+1} all in E, let $Y_1(t)$ be the process obtained from $Y(t)$ by making $\alpha_{i-1} - \alpha_i$ and $\alpha_{i+1} - \alpha_i$ absorbing barriers. Let $p_1(t, y)$ be the density with respect to Lebesgue measure of the distribution of $Y_1(t)$. Then (see McKean [6]),

$$\int_0^\infty p_1(t,0)\,dt = \int_0^\infty e^{-0t}\,p_1(t,0)\,dt = v_1(0)v_2(0),$$

where $v_1(x)$ and $v_2(x)$ satisfy

$$d^2v_1/dx^2 = 0v_1 = 0, \qquad d^2v_2/dx^2 = 0v_2 = 0,$$

$$v_1(\alpha_{i-1} - \alpha_i) = v_2(\alpha_{i+1} - \alpha_i) = 0 \quad \text{and} \quad v_1'(x)v_2(x) - v_1(x)v_2'(x) = 1.$$

These equations have the solution

$$v_1(x) = x - \alpha_{i-1} + \alpha_i \quad \text{and} \quad v_2(x) = (\alpha_{i+1} - \alpha_i - x)/(\alpha_{i+1} - \alpha_{i-1}).$$

It is easily shown that $L_{Y_1}(0;\infty,0)$ has an exponential distribution with mean

$$E[L_{Y_1}(0;\infty,0)] = \int_0^\infty p_1(t,0)\,dt = v_1(0)v_2(0) = \frac{(\alpha_i - \alpha_{i-1})(\alpha_{i+1} - \alpha_i)}{\alpha_{i+1} - \alpha_{i-1}},$$

and

$$P[Y_1(\infty) = \alpha_{i+1} - \alpha_i] = (\alpha_i - \alpha_{i-1})/(\alpha_{i+1} - \alpha_{i-1}).$$

It follows from the construction in Section 1 that the waiting time for $Z(x;t)$ in state α_i is exponentially distributed with mean

$$m\{\alpha_i\}(\alpha_i - \alpha_{i-1})(\alpha_{i+1} - \alpha_i)/(\alpha_{i+1} - \alpha_i).$$

Upon leaving α_i, $Z(x;t)$ enters states α_{i-1} and α_{i+1} with respective probabilities

$$(\alpha_{i+1} - \alpha_i)/(\alpha_{i+1} - \alpha_{i-1}) \quad \text{and} \quad (\alpha_i - \alpha_{i-1})/(\alpha_{i+1} - \alpha_{i-1}).$$

$X(x; t)$ will go directly from α_i to Δ if and only if the waiting time for $Z(x; t)$ in α_i exceeds $e'(\Delta)m\{\alpha_i\}/k\{\alpha_i\}$, where $e'(\Delta)$ is a random variable independent of $Z(x; t)$ and exponentially distributed with mean 1.

On the other hand, the transition rates of $X(x; t)$, as $t \to 0$, are defined as follows:

$$\begin{aligned}
P\{X(\alpha_i\,; t) = \alpha_{i-1}\} &= \delta_i\, t + o(t);\\
P\{X(\alpha_i\,; t) = \alpha_{i+1}\} &= \beta_i\, t + o(t);\\
P\{X(\alpha_i\,; t) = \Delta\} &= \gamma_i\, t + o(t);\\
P\{X(\alpha_i\,; t) = \alpha_i\} &= 1 - (\beta_i + \gamma_i + \delta_i)t + o(t);\\
P\{X(\alpha_i\,; t) = \alpha_j\} &= o(t) \quad \text{for} \quad |j - i| > 1.
\end{aligned}$$

The waiting time of $X(x; t)$ in state α_i has an exponential distribution with mean $(\beta_i + \gamma_i + \delta_i)^{-1}$. Upon leaving α_i, $X(x; t)$ enters states α_{i-1}, α_{i+1}, and Δ with respective probabilities

$$\delta_i(\beta_i + \gamma_i + \delta_i)^{-1}, \qquad \beta_i(\beta_i + \gamma_i + \delta_i)^{-1}, \quad \text{and} \quad \gamma_i(\beta_i + \gamma_i + \delta_i)^{-1}.$$

From the probability interpretation of the two sets of parameters, we may relate them as follows:

$$\begin{aligned}
&0 < \delta_i = ((\alpha_i - \alpha_{i-1})m\{\alpha_i\})^{-1}, && i_1 < i < i_2 + 1,\\
&0 < \beta_i = ((\alpha_{i+1} - \alpha_i)m\{\alpha_i\})^{-1}, && i_1 - 1 < i < i_2,\\
&0 = \delta_{i_1}, && i_1 > -\infty,\\
&0 = \beta_{i_2}, && i_2 < +\infty,\\
&0 \leqq \gamma_i = k\{\alpha_i\}(m\{\alpha_i\})^{-1}, && i_1 - 1 < i < i_2 + 1,\\
&0 = \gamma_{i_1}, && i_1 > -\infty \quad \text{and} \quad \beta_{i_1} = 0,\\
&0 = \gamma_{i_2}, && i_2 < +\infty \quad \text{and} \quad \delta_{i_2} = 0.
\end{aligned}$$

Given α_i, $m\{\alpha_i\}$, and $k\{\alpha_i\}$, these equations uniquely determine β_i, γ_i, and δ_i. Conversely, given the β_i, γ_i, and δ_i satisfying the equalities and inequalities to their left, and given α_i chosen in strictly increasing order for two values of i (say i' and $i' + 1$ with $i_1 < i' < i_2$), these equations uniquely determine α_i, $m\{\alpha_i\}$, and $k\{\alpha_i\}$.

Suppose that the transition rates satisfy the following property: For some $\theta > 0$

$$w_i = (\beta_i + \gamma_i + \delta_i)\theta \leqq 1, \qquad i_1 - 1 < i < i_2 + 1.$$

Then a unique birth and death process $X(x; t)$ is defined.

We shall use this $X(x; t)$ to construct a random walk $W(x; t)$. Let $T_0 = 0$. We define $T_{j+1} = T_{j+1}(x)$ and $t_{j+1} = T_{j+1} - T_j$ by induction:

$$T_{j+1} = \inf\,[t \mid t > T_j \quad \text{and} \quad X(x; t) \neq X(x; T_j)], \qquad j \geqq 0,$$

where the infimum of an empty set is $+\infty$.

If $0 < w_i < 1$, let θ_i be the unique solution of $1 - w_i = \exp(-w_i\,\theta_i/\theta)$. Let $S_0 = 0$. We define S_{j+1} and $s_{j+1} = S_{j+1} - S_j$ in terms of T_j: If i is the unique integer such that $\alpha_i = X(x; T_j)$, we set

$$\begin{aligned} s_{j+1} &= \theta && \text{if} \quad w_i = 1, \\ &= m\theta && \text{if} \quad 0 < w_i < 1 \quad \text{and} \quad (m-1)\theta_i \leqq t_j < m\theta_i, \\ &= +\infty && \text{if} \quad w_i = 0 \quad \text{or} \quad X(x; T_j) = \Delta. \end{aligned}$$

Given $X(x; T_j) = \alpha_i$, $s_{j+1}\theta^{-1}$ is geometrically distributed with mean w_i^{-1}: If $w_i = 0$ or $w_i = 1$, this is obvious. Suppose that $0 < w_i < 1$ and $m > 1$. Then

$$\begin{aligned} P\{s_{j+1} = m\theta\} &= \int_{(m-1)\theta_i}^{m\theta_i} (\beta_i + \gamma_i + \delta_i)\, e^{-(\beta_i+\gamma_i+\delta_i)t}\, dt \\ &= e^{-(m-1)\theta_i w_i/\theta}\,(1 - e^{-\theta_i w_i/\theta}) \\ &= (1 - w_i)^{m-1} w_i, \end{aligned}$$

which corresponds to a geometric distribution with mean w_i^{-1}. We see therefore that $E\{s_{j+1} \mid X(x; T_j)\} = E\{t_{j+1} \mid X(x; T_j)\} \geqq \theta$.

Let

$$W(x; t) = X(x; T_j), \qquad S_j \leqq t < S_{j+1}.$$

Then $W(x; S_j) = X(x; T_j)$. $W(x; t)$ is a random walk whose jumps occur at integral multiples of θ and whose state space is E. Let $p_i = \theta\beta_i$, $q_i = \theta\gamma_i$, and $r_i = \theta\delta_i$. We have

$$\begin{aligned} &P\{W(\alpha_i; \theta) = \alpha_{i-1}\} = r_i; \\ &P\{W(\alpha_i; \theta) = \alpha_{i+1}\} = p_i; \\ &P\{W(\alpha_i; \theta) = \Delta\} = q_i; \\ &P\{W(\alpha_i; \theta) = \alpha_i\} = 1 - (p_i + q_i + r_i) = 1 - w_i; \\ &P\{W(\alpha_i; \theta) = \alpha_j\} = 0 \quad \text{for} \quad |j - i| > 1. \end{aligned}$$

The parameters of the random walks are related as follows:

$$\begin{aligned} &0 < r_i = \theta((\alpha_i - \alpha_{i-1})m\{\alpha_i\})^{-1}, && i_1 < i < i_2 + 1, \\ &0 < p_i = \theta((\alpha_{i+1} - \alpha_i)m\{\alpha_i\})^{-1}, && i_1 - 1 < i < i_2, \\ &0 = r_{i_1}, && i_1 > -\infty, \\ &0 = p_{i_2}, && i_2 < +\infty, \\ &0 \leqq q_i = \theta k\{\alpha_i\}\,(m\{\alpha_i\})^{-1}, && i_1 - 1 < i < i_2 + 1, \\ &0 = q_{i_1}, && i_1 > -\infty \quad \text{and} \quad p_{i_1} = 0, \\ &0 = q_{i_2}, && i_2 < +\infty \quad \text{and} \quad p_{i_2} = 0, \\ &1 \geqq p_i + q_i + r_i, && i_1 - 1 < i < i_2 + 1. \end{aligned}$$

For $x \in E$ let

$$L_W(x; t, \alpha_i) = (m\{\alpha_i\})^{-1} M\{\tau \mid 0 \leqq \tau \leqq t \quad \text{and} \quad W(x; \tau) = \alpha_i\}.$$

We shall need the functions $s_W(x; t)$ defined as follows: Let j be such that $T_j \leqq t < T_{j+1}$. If $X(x; t) = \alpha_i$ and if $w_i = 0$ or $w_i = 1$, set

$s_W(x;t) = S_j$; if $X(x;t) = \alpha_i$, $0 < w_i < 1$, and $(m-1)\theta_i \leqq t - T_j < m\theta_i$, set $s_W(x;t) = S_j^+(m-1)\theta$; and finally, if $X(x;t) = \Delta$, set $s_W(x;t) = \inf\,[t \mid W(x;t) = \Delta]$.

4. Convergence of a sequence of random walks

Let $W_n(x_n;t)$, $n \geqq 1$, be random walks constructed from birth and death processes $X_n(x_n;t)$ in the manner of Section 3, and let $X_0(x_0;t)$ be a regular diffusion process. Set

$$V_{W_n}(x_n;t) = \int_{-\infty}^{\infty} L_{W_n}(x_n;t,y)\,b_n(dy) = \sum_{\alpha_i \in E_n} L_{W_n}(x_n;t,\alpha_i)b_n\{\alpha_i\},$$

$$E_n(N) = \{\alpha_i \mid \alpha_i \in E_n \cap [-N,N] \cap [a_0-1, b_0+1]\},$$

and

$$d(N,\beta) = \textstyle\sum_{n=1}^{\infty} \sum_{\alpha_i \in E_n(N)} \exp\left[-\beta(1/(\alpha_{i+1}-\alpha_i) + 1/(\alpha_i - \alpha_{i-1}))\right]$$

(here if $i = i_1 > -\infty$, we replace $1/(\alpha_i - \alpha_{i-1})$ in the summand by 0, and if $i = i_2 < +\infty$, we replace $1/(\alpha_{i+1}-\alpha_i)$ in the summand by 0).

THEOREM 2. *Suppose that conditions* (i)–(ix) *preceding Theorem* 1 *are satisfied and that* $\theta_n \to 0$ *as* $n \to \infty$. *Let* $X_n(x_n;t)$, $n \geqq 1$, *be replaced by* $W_n(x_n;t)$ *in the statements of Theorem* 1 *and Corollary* 1. *Then Corollary* 1 *is valid. The statements of Theorem* 1 *hold in probability* (*for interpretation see below*). *If* $\sum \theta_n < \infty$, *then* (1) *holds with probability* 1. *If* E_0 *is an interval and* $d(N,\beta) < \infty$ *for each* $N > 0$ *and* $\beta > 0$, *then Theorem* 1 *is valid as stated* (*i.e.*, (1)–(4) *hold with probability* 1).

Proof. It is well known that inequalities such as the Kolmogorov inequality can be sharpened considerably when the random variables are identically distributed and have a nice distribution such as exponential or geometric. The following lemma, an example of such sharpening, will enable us to obtain very simply the uniform convergence of the local times of $W_n(x_n;t)$ under the above condition on $d_n(N,\beta)$.

LEMMA 1. *For every* $\varepsilon > 0$ *and* $M > 0$ *there is a* $\beta > 0$ *with the following property: If* $X_1, X_2, \cdots$ *are independently, identically distributed random variables with mean* $\mu = EX_1$ *and such that either* X_1 *has an exponential distribution or* cX_1 *has a geometric distribution for some* $c > 0$, *then*

$$P\{\exists\, k \ni |X_1 + \cdots + X_k - kEX_1| \geqq \varepsilon \quad \text{and} \quad k \leqq M/\mu\} < 2e^{-\beta/\mu},$$

and

$$P\{\exists\, k \ni |X_1 + \cdots + X_k - kEX_1| \geqq \varepsilon \text{ and } X_1 + \cdots + X_k \leqq M\} < 2e^{-\beta/\mu}.$$

Proof. In proving the second result we may assume that M/μ is a positive integer. Then

$$P\{\exists\, k \ni |X_1 + \cdots + X_k - kEX_1| \geqq \varepsilon \quad \text{and} \quad X_1 + \cdots + X_k \leqq M\}$$

$$\leqq P\{\exists\, k \ni |X_1 + \cdots + X_k - kEX_1| \geqq \varepsilon \quad \text{and} \quad k \leqq 2M/\mu\}$$
$$+ P\{X_1 + \cdots + X_{2M/\mu} \leqq M\}$$
$$= P\{\exists\, k \ni |X_1 + \cdots + X_k - kEX_1| \geqq \varepsilon \quad \text{and} \quad k \leqq 2M/\mu\}$$
$$+ P\{|X_1 + \cdots + X_{2M/\mu} - (2M/\mu)\mu| \geqq M\}.$$

It is now clear that the second conclusion follows from the first.

In proving the first conclusion we can assume that $M = 1$ and that $\mu^{-1} = n$, where n is a positive integer. It now suffices to prove the following statement.

For every $\varepsilon > 0$ there is a positive $\beta_0 < 1$ with the following property: If $X_1, X_2, \cdots$ are independent, identically distributed random variables with $EX_1 = n^{-1}$ and such that either X_1 has an exponential distribution or cX_1 has a geometric distribution for some $c > 0$, then

$$P\{|X_1 + \cdots + X_k - k/n| \geqq \varepsilon\} \leqq 2\beta_0^n \qquad \text{for} \quad k \leqq n.$$

In proving this statement we can assume $\varepsilon \leqq \frac{1}{3}$.

Suppose first that the X_k's have a geometric distribution with mean n^{-1}. This distribution has density ne^{-nx}. Thus

$$Ee^{n\varepsilon X_1} = \int_0^\infty ne^{-nx}\, e^{n\varepsilon x}\, dx = 1/(1 - \varepsilon).$$

Hence

$$P\{X_1 + \cdots + X_k - k/n \geqq \varepsilon\} \leqq e^{-\varepsilon^2 n} E[e^{\varepsilon n(X_1 + \cdots + X_n - k/n)}]$$
$$= e^{-\varepsilon^2 n - \varepsilon k}/(1 - \varepsilon)^k \leqq e^{-\varepsilon^2 n - \varepsilon n}/(1 - \varepsilon)^n = (e^{-\varepsilon^2 - \varepsilon}/(1 - \varepsilon))^n = \beta_1^n,$$

where $\beta_1 = (\exp(-\varepsilon^2 - \varepsilon))/(1 - \varepsilon) < 1$. Similarly,

$$P\{X_1 + \cdots + X_k - k/n \leqq -\varepsilon\} \leqq e^{-\varepsilon^2 n} E[e^{-\varepsilon n(X_1 + \cdots + X_n - k/n)}]$$
$$= e^{-\varepsilon^2 n + \varepsilon n}/(1 + \varepsilon)^k \leqq e^{-\varepsilon^2 n + \varepsilon n}/(1 + \varepsilon)^n = (e^{-\varepsilon^2 + \varepsilon}/(1 + \varepsilon))^n = \beta_2^n,$$

where $\beta_2 = (\exp(-\varepsilon^2 + \varepsilon))/(1 + \varepsilon) < 1$. This completes the proof in the exponential case.

Suppose now that $Y_1 = cX_1$ has a geometric distribution with mean p^{-1}. Then $p\{Y_1 = m\} = p(1 - p)^{m-1}$ and

$$Ee^{\varepsilon p Y_1} = \textstyle\sum_{m=1}^\infty p(1 - p)^{m-1} e^{\varepsilon p m} = pe^{\varepsilon p}/(1 - e^{\varepsilon p}(1 - p)).$$

Hence

$$P\{X_1 + \cdots + X_k - k/n \geqq \varepsilon\}$$
$$= P\{Y_1 + \cdots + Y_k - k/p \geqq \varepsilon n/p\} \leqq e^{-\varepsilon^2 n} E[e^{\varepsilon p(Y_1 + \ldots + Y_k - k/p)}]$$
$$= e^{-\varepsilon^2 n}\left(\frac{pe^{\varepsilon p - \varepsilon}}{1 - e^{\varepsilon p}(1 - p)}\right)^k \leqq \left(\frac{pe^{-\varepsilon^2 - \varepsilon + \varepsilon p}}{1 - e^{\varepsilon p}(1 - p)}\right)^n = \beta_1^n.$$

Here we have used implicitly the inequality

$$pe^{\varepsilon p-\varepsilon} \geqq 1 - e^{\varepsilon p}(1-p).$$

This inequality is easily demonstrated by differentiating with respect to ε.

Now

$$\beta_1 = \beta_1(p) = pe^{-\varepsilon^2-\varepsilon+\varepsilon p}/(1-e^{\varepsilon p}(1-p)),$$

and

$$\lim_{p\to 0} \beta_1(p) = (e^{\varepsilon^2+\varepsilon}(1-\varepsilon))^{-1} < \varepsilon \qquad \text{for } \varepsilon \leqq \tfrac{1}{3}.$$

Thus showing that β_1 is bounded away from 1 uniformly for $0 < p \leqq 1$ is now equivalent to showing that

$$pe^{-\varepsilon^2-\varepsilon+\varepsilon p} < 1 - e^{\varepsilon p}(1-p), \qquad 0 < p \leqq 1,$$

or equivalently that

$$1 - e^{-\varepsilon^2-\varepsilon} > p^{-1} - (pe^{\varepsilon p})^{-1} \qquad \text{for } 0 < p \leqq 1.$$

Let $f(p) = p^{-1} - (pe^{\varepsilon p})^{-1}$. Then $f(0+) = \varepsilon < 1 - e^{-\varepsilon^2-\varepsilon}$. It suffices to show that $f(p)$ is monotonically decreasing for $0 < p \leqq 1$. To do this we need only observe that $f'(0+) = -\varepsilon^2/2$ and $(p^2 f'(p))' = -\varepsilon^2 pe^{-\varepsilon p} < 0$ for $0 < p \leqq 1$. This proves that β_1 is bounded away from zero uniformly in p.

Similarly,

$$P\{X_1 + \cdots + X_k - k/n \leqq -\varepsilon\} \leqq \beta_2^n,$$

where

$$\beta_2 = pe^{-\varepsilon^2+\varepsilon-\varepsilon p}/(1-e^{-\varepsilon p}(1-p)).$$

The argument that β_2 is bounded away from 1 uniformly in p is similar to that for β_1 and will be omitted. This completes the proof of Lemma 1.

Let $X(x;t)$ and $W(x;t)$ be any pair of processes of Section 3. Let

$$R_k = (X(x;0), \cdots, X(x;T_k)) = (W(x;0), \cdots, W(x;S_k)).$$

Conditioned on R_k, $t_1, \cdots, t_{k+1}$ are mutually independent random variables with mean $E\{t_j \mid R_k\} = E\{t_j \mid X(T_j)\}$. A similar statement holds for $s_1, \cdots, s_k$.

LEMMA 2. $E\{(s_{j+1} - t_{j+1})^2 \mid X(T_j)\} \leqq 14\theta^2$. *For every* $t > 0$ *and* $\varepsilon > 0$

$$P\{\max[|S_j - T_j| : S_j \leqq t \text{ or } T_j \leqq t - \varepsilon - \theta] \geqq \varepsilon\} \leqq 14\varepsilon^{-2}\theta t.$$

Proof. Let $\alpha_i = X(T_j)$. If $w_i = 0$, then $s_{j+1} = t_{j+1} = +\infty$; in this sense $s_{j+1} - t_{j+1} = 0$ and $E\{(s_{j+1}-t_{j+1})^2 \mid X(T_j)\} = 0$. Recall the identity $\sum_{m\geqq 1} m^2 w_i(1-w_i)^{m-1} = w_i^{-2}(2-w_i)$. If $\frac{1}{2} \leqq w_i \leqq 1$, then

$$\begin{aligned} E\{(s_{j+1}-t_{j+1})^2 \mid X(T_j)\} &\leqq E\{s_{j+1}^2 \mid X(T_j)\} + E\{t_{j+1}^2 \mid X(T_j)\} \\ &= E\{s_{j+1}^2 \mid X(T_j)\} + 2(E\{t_{j+1} \mid X(T_j)\})^2 \leqq 6\theta^2 + 8\theta^2 = 14\theta^2. \end{aligned}$$

Suppose $0 < w_i \leqq \frac{1}{2}$. Then

$$\theta_i = -\theta w_i^{-1}\log(1-w_i) = \theta(1 + \tfrac{1}{2}w_i + \tfrac{1}{3}w_i^2 + \cdots).$$

Hence $\theta < \theta_i \leqq 2\theta \log 2 < \theta\sqrt{2}$ and

$$\theta_i - \theta = \theta(\tfrac{1}{2}w_i + \tfrac{1}{3}w_i^2 + \cdots) = w_i\theta(\tfrac{1}{2} + \tfrac{1}{3}w_i + \cdots) < w_i\theta.$$

Thus if $s_{j+1} = m\theta$, then $(s_{j+1} - t_{j+1})^2 \leqq \theta_i^2 + m^2(\theta_i - \theta)^2 < \theta^2(2 + m^2w_i^2)$. Therefore

$$\begin{aligned} E\{(s_{j+1} - t_{j+1})^2 \mid X(T_j)\} &< \theta^2 \textstyle\sum_{m=1}^{\infty} (2 + m^2w_i^2)w_i(1 - w_i)^{m-1} \\ &\leqq 2\theta^2 + \theta^2(2 - w_i) < 4\theta^2 < 14\theta^2. \end{aligned}$$

This proves the first statement of Lemma 2. By Kolmogorov's inequality, we have

$$P\{\max_{1 \leqq j \leqq k} | S_j - T_j | \geqq \varepsilon \mid R_{k-1}\} \leqq 14\varepsilon^{-2}\theta^2k,$$

and hence

$$P\{\max_{1 \leqq j \leqq k} | S_j - T_j | \geqq \varepsilon\} \leqq 14\varepsilon^{-2}\theta^2k.$$

Consequently

$$P\{\max_{1 \leqq j \leqq [t/\theta]} | S_j - T_j | \geqq \varepsilon\} \leqq 14\varepsilon^{-2}\theta t.$$

Since $S_{[t/\theta]} \geqq \theta[t/\theta] > t - \theta$ and $S_{[t/\theta]+1} > t$, the second statement of Lemma 2 now follows.

We now return to the processes $W_n(x_n ; t)$ and $X_n(x_n ; t)$. Henceforth we suppose that conditions (i)–(ix) are satisfied and that $\theta_n \to 0$ as $n \to \infty$.

It follows from Lemma 2 that there exists a (random) sequence $\mu_n(t)$ of continuous one-to-one mappings of $[0, \infty)$ onto itself such that for each $\varepsilon > 0$ and $N > 0$, as $n \to \infty$

$$P\{\sup_{0 \leqq t \leqq N} | \mu_n(t) - t | \geqq \varepsilon\} \to 0,$$

and

$$\sup_{0 \leqq t \leqq N} \rho(W_n(x_n ; t), X_n(x_n ; \mu_n(t))) \to 0$$

(the second relation holding everywhere on Ω). By Theorem 1, there exists a sequence $\nu_n(t)$ of continuous one-to-one mappings of $[0, \infty)$ onto itself such that for each $N > 0$, as $n \to \infty$

$$\sup_{0 \leqq t \leqq N} | \nu_n(t) - t | \to 0,$$

and, with probability 1,

$$\sup_{0 \leqq t \leqq N} \rho(X_n(x_n ; t), X_0(x_0 ; \nu_n(t))) \to 0.$$

Let $\lambda_n(t) = \nu_n(\mu_n(t))$. Each $\lambda_n(t)$ is a continuous one-to-one mapping of $[0, \infty)$ onto itself. Also for each $\varepsilon > 0$ and $N > 0$ as $n \to \infty$

$$P\{\sup_{0 \leqq t \leqq N} | \lambda_n(t) - t | \geqq \varepsilon\} \to 0,$$

and

$$P\{\sup_{0 \leqq t \leqq N} \rho(W_n(x_n ; t), X_0(x_0 ; \lambda_n(t))) \geqq \varepsilon\} \to 0.$$

In this sense $W_n(x_n ; t)$ is J-convergent to $X_0(x_0 ; t)$ in probability. As a consequence $W_n(x_n ; t)$ is weakly convergent to $X_0(x_0 ; t)$.

If $\sum_n \theta_n < \infty$, then by Lemma 2 we may further assume that, with probability 1, as $n \to \infty$

$$\sup_{0 \leqq t \leqq N} | \mu_n(t) - t | \to 0,$$

and hence that

$$\sup_{0 \leqq t \leqq N} | \lambda_n(t) - t | \to 0$$

and

$$\sup_{0 \leqq t \leqq N} \rho(W_n(x_n ; t), X_0(x_0 ; \lambda_n(t))) \to 0.$$

Thus if $\sum_n \theta_n < \infty$, then with probability 1, $W_n(x_n ; t)$ is J-convergent to $X_0(x_0 ; t)$.

In order to investigate convergence of the local times we first set

$$T_n(x_n ; M) = \inf [t \mid t = M \quad \text{or} \quad \exists\, y \in E_n \ni L_{X_n}(x_n ; t, y) = M],$$

and let $N_n(x_n ; t, \alpha_i)$ denote the number of visits of $X_n(x_n ; t)$ to α_i by time t. The expected increase μ_i of $L_{X_n}(x_n ; t, \alpha_i)$ per visit to α_i is

$$\mu_i = (m\{\alpha_i\}(\beta_i + \gamma_i + \delta_i))^{-1} = (1/(\alpha_i - \alpha_{i-1}) + 1/(\alpha_{i+1} - \alpha_i) + k\{\alpha_i\})^{-1}.$$

(As before, we replace $1/(\alpha_i - \alpha_{i-1})$ by 0, if $i = i_1 > -\infty$ and $1/(\alpha_{i+1} - \alpha_i)$ by 0 if $i = i_2 < +\infty$.)

Choose $\varepsilon > 0$. By Lemma 1 there is a $\beta > 0$ depending only on M and ε such that for $\alpha_i \in E_n$

$$P\{\exists\, t \leqq T_n(x_n ; M) \ni | L_{X_n}(x_n ; t, \alpha_i) - \mu_i N_n(x_n ; t, \alpha_i) | \geqq \varepsilon\}$$
$$\leqq 2e^{-\beta/\mu_i} \leqq 2 \exp[-\beta(1/(\alpha_{i+1} - \alpha_i) + 1/(\alpha_i - \alpha_{i-1}))].$$

Recall the definition of $s_W(x; t)$ given at the end of Section 3. It now follows from Lemma 1 that for $\alpha_i \in E_n$

$$P\{\exists\, t \leqq T_n(x_n ; M) \ni | L_{X_n}(x_n ; t, \alpha_i) - L_{W_n}(x_n ; s_{W_n}(x_n ; t), \alpha_i) | \geqq 2\varepsilon\}$$
$$\leqq 4 \exp[-\beta(1/(\alpha_{i+1} - \alpha_i) + 1/(\alpha_i - \alpha_{i-1}))].$$

For $x \in E_n$, let

$$\sigma_n(x) = \inf [| y - x | : y \in E_n \quad \text{and} \quad y \neq x].$$

For $\delta > 0$ and $N > 0$ let

$$E_n(\delta, N) = \{x \mid x \in E_n \cap [-N, N] \quad \text{and} \quad \sigma_n(x) < \delta\}.$$

Since $e^{-x} \leqq 2/x^2$ for $x > 0$, it follows that

$$\sum_{\alpha_i \in E_n(\delta, N)} \exp[-\beta(1/(\alpha_{i+1} - \alpha_i) + 1/(\alpha_i - \alpha_{i-1}))]$$
$$\leqq 2\beta^{-2} \sum_{\alpha_i \in E_n(\delta, N)} \sigma^2(\alpha_i) \leqq 2\beta^{-2}\delta(2n + \delta).$$

Thus for fixed ε, M, β, and N, there is a $\delta > 0$ such that for $n \geqq 1$

$$P\{\exists\, t \leqq T_n(x_n ; M) \quad \text{and} \quad \alpha_i \in E_n(\delta, N) \ni$$
$$| L_{X_n}(x_n ; t, \alpha_i) - L_{W_n}(x_n ; s_{W_n}(x_n ; t), \alpha_i) | \geqq 2\varepsilon\} < \varepsilon.$$

Now $L_{X_n}(x_n ; t, y)$ is uniformly bounded in n and y for $t \leqq M$. Thus by making δ somewhat smaller if necessary, we can assume that

$$P\{\exists\, t \leqq M \quad \text{and} \quad \alpha_i \in E_n(\delta, N) \ni \\ | L_{X_n}(x_n ; t, \alpha_i) - L_{W_n}(x_n ; s_{W_n}(x_n ; t), \alpha_i) | \geqq 2\varepsilon\} < \varepsilon.$$

Let

$$E'_n(\delta, N) = \{x \mid x \in E_n \cap [-N, N] \quad \text{and} \quad \sigma_n(x) > \delta\}.$$

Clearly each $E'_n(\delta, N)$ is a finite set. Let $\{x^{(1)}, \cdots, x^{(k)}\} = E'_0(\delta, N)$. By condition (i) for each $x^{(j)} \in E'_0(\delta, N)$ we can find $x_n^{(j)} \in E_n$ such that $x_n^{(j)} \to x^{(j)}$ as $n \to \infty$. Again by condition (i), there is an $n_1 > 0$ such that if $n > n_1$, then $E'_n(\delta, N) \subset \{x_n^{(1)}, \cdots, x_n^{(k)}\}$. For fixed j, $1 \leqq j \leqq k$, let n' be an increasing sequence of positive integers such that $x_{n'}^{(j)} \in E'_{n'}(\delta, N)$. With probability 1, the number of visits of $X_{n'}(x_n ; t)$ to $x_{n'}^{(j)}$ is uniformly bounded in n'. Consequently, with probability 1,

$$\lim_{n' \to \infty} | L_{X_{n'}}(x_{n'} ; t, x_{n'}^{(j)}) - L_{W_{n'}}(x_{n'} ; s_{W_{n'}}(x_{n'} ; t), x_{n'}^{(j)}) | = 0$$

uniformly for $t \leqq M$.

We can choose N large enough so that for all $n \geqq 0$

$$P\{\exists\, t \leqq M \ni | X_n(x_n ; t) | \geqq N\} < \varepsilon.$$

Since $E_n \cap [-N, N] = E_n(\delta, N) \cup E'_n(\delta, N)$, there is an $n_0 > 0$ such that for $n \geqq n_0$

$$P\{\exists\, t \leqq M \quad \text{and} \quad \alpha_i \in E_n \ni \\ | L_{X_n}(x_n ; t, \alpha_i) - L_{W_n}(x_n ; s_{W_n}(x_n ; t), \alpha_i) | \geqq 2\varepsilon\} < 3\varepsilon.$$

Observe that

$$t = \int_{E_n} L_{X_n}(x_n ; t, y)\, m_n(dy)$$

and

$$s_{W_n}(x_n ; t) = \int_{E_n} L_{W_n}(x_n ; s_{W_n}(x_n ; t), y)\, m_n(dy).$$

Thus for every $\varepsilon_1 > 0$

$$\lim_{n \to \infty} P\{\exists\, t \leqq M \ni | s_{W_n}(x_n ; t) - t | \geqq \varepsilon_1 \quad \text{and} \quad t \leqq t_0(x_0 ; B_0) - \varepsilon_1\} = 0.$$

But $L_{X_0}(x_0 ; t, y)$ is uniformly continuous in t and y and constant for $t \geqq t_0(x_0 ; B_0)$. Thus there is an $n_0 > 0$ such that for $n \geqq n_0$

$$P\{\exists\, t \leqq M \quad \text{and} \quad \alpha_i \in E_n \ni | L_{X_n}(x_n ; t, \alpha_i) - L_{W_n}(x_n ; t, \alpha_i) | \geqq 3\varepsilon\} < 3\varepsilon.$$

Together with (2) of Theorem 1, this result yields

LEMMA 3. *For any $\varepsilon > 0$ and $M > 0$ there exist $n_0 > 0$ and $\delta > 0$ such that if $n \geqq n_0$, then with probability greater than $1 - \varepsilon$*

$$| L_{W_n}(x_n ; t, y_n) - L_{X_0}(x_0 ; t, y_0) | < \varepsilon$$

uniformly for $t \leqq M$, $y_n \in E_n$, and $| y_n - y_0 | < \delta$.

From Lemma 3 it is easy to verify that for every $\varepsilon > 0$ and $t > 0$

$$\lim_{n\to\infty} P\{\mid V_{X_n}(x_n\,;t) - V_{X_0}(x_0\,;t)\mid \geqq \varepsilon\} = 0,$$

and

$$\lim_{n\to\infty} P\{\mid \sup\,[\tau \mid V_{W_n}(x_n\,;\tau) \leqq t] - \sup\,[\tau \mid V_{X_0}(x_0\,;\tau) \leqq t]\mid \geqq \varepsilon\} = 0.$$

We have now formulated and verified convergence in probability statements corresponding to (1)–(4) of Theorem 1. (5)–(7) of Corollary 1 are immediate consequences.

If E_0 is an interval and $d(N,\beta) < \infty$ for each $N > 0$ and $B > 0$, then the arguments preceding Lemma 3 can easily be modified to give convergence with probability 1, including J-convergence, of the path functions. This completes the proof of Theorem 2.

A simplified version of Lemma 3 is that for every $\varepsilon > 0$ and $M > 0$

$$\lim_{n\to\infty} P\{\sup_{0\leqq t\leqq M} \sup_{y\in E_n\cap E_0} \mid L_{W_n}(x_n\,;t,y) - L_{X_0}(x_0\,;t,y)\mid \geqq \varepsilon\} = 0.$$

Example 1. Let $W_n(x_n\,;t) = W_n(t)$ be the simple random walk normalized to have state space $E_n = \{i/\sqrt{n} \mid -\infty < i < \infty\}$, time unit $\theta_n = 1/2n$, and initial state $x_n = 0$. Then

$$\tfrac{1}{2} = p_i^{(n)} = r_i^{(n)} = P\{W((k+1)/2n) = (i+1)/\sqrt{n} \mid W(k/2n) = i/\sqrt{n}\}$$
$$= P\{W((k+1)/2n) = (i-1)/\sqrt{n} \mid W(k/2n) = i/\sqrt{n}\},$$
$$q_i^{(n)} = 0, \qquad \alpha_i^{(n)} = i/\sqrt{n}, \qquad m_n\{\alpha_i^{(n)}\} = 1/\sqrt{n}, \quad \text{and} \quad k_n\{\alpha_i^{(n)}\} = 0.$$

In this case we can take $X_0(x_0\,;t) = X_0(t) = Y(t)$ to be the Brownian motion process with state space $-\infty < x < \infty$, speed measure $m_0(dx) = dx$, and killing measure $k_0(dx) = 0$. We can take $\rho(x,y) = \mid x - y\mid$. J-convergence is now simply uniform convergence in bounded time intervals. It is obvious that conditions (i)–(ix) are satisfied, and that $d(N,\beta) < \infty$ for all $N > 0$ and $\beta > 0$. Let $N_n(t, i/\sqrt{n})$ be the number of visits of $W_n(t)$ to $i/\sqrt{n}$ by time t. Then in the above construction we have that, with probability 1, for every $M > 0$

$$\lim_{n\to\infty} \sup_{0\leqq t\leqq M} \mid W_n(t) - Y(t)\mid = 0,$$

and

$$\lim_{n\to\infty} \sup_{0\leqq t\leqq M} \sup_{-\infty<i<\infty} \mid (1/2\sqrt{n})N_n(t, i/\sqrt{n}) - L(t, i/\sqrt{n})\mid = 0.$$

Example 2. Let $W_n(x_n\,;t)$ be the random walk with $E_n = \{i/\sqrt{n} \mid 0 \leqq i < \infty\}$, $\theta_n = 1/2n$, and $x_n \in E_n$. We suppose that $p_i^{(n)} = r_i^{(n)} = \frac{1}{2}$ and $q_i^{(n)} = 0$ for $i \geqq 1$, $r_0^{(n)} = 0$, $p_0^{(n)} \geqq 0$, $q_0^{(n)} \geqq 0$, $p_0^{(n)} + q_0^{(n)} \leqq 1$, and if $p_0^{(n)} = 0$, then $q_0^{(n)} = 0$. We have $\alpha_i^{(n)} = i/\sqrt{n}$, $m_n\{\alpha_i^{(n)}\} = 1/\sqrt{n}$, and $k_n\{\alpha_i^{(n)}\} = 0$ for $i \geqq 1$, $m_n\{0\} = 1/2p_0^{(n)}\sqrt{n}$, $k_n\{0\} = q_0^{(n)}\sqrt{n}/p_0^{(n)}$ if $p_0^{(n)} \neq 0$, and $k_n\{0\} = 0$ if $p_0^{(n)} = 0$.

Let $X_0(x_0\,;t)$ be the diffusion process with $E_0 = \{x \mid 0 \leqq x < \infty\}$, $x_0 \in E_0$,

$m_0(dx) = dx$ and $k_0(dx) = 0$ for $x > 0$, $m_0\{0\} \geqq 0$, $k_0\{0\} \geqq 0$, and if $m_0\{0\} = +\infty$, then $k_0\{0\} = +\infty$. This process has the infinitesimal generator $Gf = f''$ for $x > 0$ with boundary condition

$$f'(0) - m_0\{0\}f''(0) - k_0\{0\}f(0) = 0.$$

In order to have convergence of $W_n(x_n\,; t)$ to $X_0(x_0\,; t)$ we take $\rho(x, y) = |x - y|$, $x_n \to x_0$,

$$1/2p_0^{(n)}\sqrt{n} \to m_0\{0\} \quad \text{and} \quad q_0^{(n)}\sqrt{n}/p_0^{(n)} \to k_0\{0\}.$$

Again conditions (i)–(ix) are satisfied, and $d(N, \beta) < \infty$ for all $N > 0$ and $\beta > 0$. Let $N_n(x_n\,; t, i/\sqrt{n})$ be the number of visits of $W_n(x_n\,; t)$ to $i/\sqrt{n}$ by time t. Then, with probability 1, for every $M > 0$

$$\lim_{n\to\infty} \sup_{0\leqq t\leqq M} \sup_{1\leqq i\leqq\infty} |\,(1/2\sqrt{n})N_n(x_n\,; t, i/\sqrt{n}) - L_{X_0}(x_0\,; t, i/\sqrt{n})\,| = 0$$

and

$$\lim_{n\to\infty} \sup_{0\leqq t\leqq M} |\,(p_0^{(n)}/\sqrt{n})N_n(x_n\,; t, 0) - L_{X_0}(x_0\,; t, 0)\,| = 0.$$

With probability 1, $t_n(x_n\,; \Delta) \to t_0(x_0\,; \Delta)$. If $k_0\{0\} = 0$ or $m_0\{0\} = \infty$, then $t_0(x_0\,; \Delta) = \infty$, and, with probability 1,

$$\lim_{n\to\infty} \sup_{0\leqq t\leqq M} |\,W_n(x_n\,; t) - X_0(x_0\,; t)\,| = 0.$$

If $k_0\{0\} > 0$ and $m_0\{0\} < \infty$, then $t_0(x_0\,; \Delta) < \infty$. With probability 1,

$$\lim_{n\to\infty} \sup\,[|\,W_n(x_n\,; t) - X_0(x_0\,; t)\,| \ni t < t_n(x_n\,; \Delta) \text{ and } t < t_0(x_0\,; \Delta)] = 0.$$

5. Asymptotic limits as $t \to \infty$

Let $X_0(x; t)$ be the conservative diffusion process with speed measure

$$\begin{aligned} m_0(x) &= -\rho_1(\beta + 1)^{-1}(-x)^{\beta+1} && \text{if} \quad x \leqq 0, \\ &= \rho_2(\beta + 1)^{-1}x^{\beta+1} && \text{if} \quad x \geqq 0, \end{aligned}$$

where $\beta + 1$, ρ_1, and ρ_2 are positive constants. $X_0(x; t)$ has the infinitesimal generator $G = D_{m_0}D_x = (m_0'(x))^{-1}d^2/dx^2$ (cf. [1]). Associated with G is the equation $Gv(x) = \mu v(x)$, $\mu > 0$, which can be written explicitly as

$$\begin{aligned} d^2v(x)/dx^2 &= \mu\rho_1(-x)^{\beta}v(x) && \text{if} \quad x \leqq 0, \\ &= \mu\rho_2\, x^{\beta}v(x) && \text{if} \quad x \geqq 0. \end{aligned}$$

The general solution to this equation is

$$v(x; \mu) = A \sum_{n=0}^{\infty} \frac{(\nu^2\mu\rho\sigma\,|\,x\,|^{1/\nu})^n}{\Gamma(n + 1 - \nu)n!} + Bx \sum_{n=0}^{\infty} \frac{(\nu^2\mu\rho\sigma\,|\,x\,|^{1/\nu})^n}{\Gamma(n + 1 + \nu)n!},$$

where A and B are arbitrary constants, $\nu = (\beta + 2)^{-1}$, $\sigma = \sigma(x) = 1$ for $-\infty < x < 0$, and $\sigma = \sigma(x) = 2$ for $0 \leqq x < \infty$.

From the general theory of diffusion equations, we know that there exist constants A_1, A_2, B_1, and B_2 such that if $v_1(x;\mu)$ corresponds to A_1 and B_1, and $v_2(x;\mu)$ corresponds to A_2 and B_2, then $v_1(x,\mu)$ is a positive increasing function and $v_2(x,\mu)$ is a positive decreasing function. To find these constants we first obtain, with the aid of Stirling's formula, the asymptotic expansion valid as $y \to +\infty$

$$\sum_{n=0}^{\infty} y^n/\Gamma(n+1+\nu)n! \sim y^{\mp\nu/2}(1/2\pi)\sum_{n=0}^{\infty}(e^2y)^n/n^{2n+1}.$$

If $v_2(x;\mu)$ is a positive decreasing function, it is necessarily bounded as $x \to +\infty$. Consequently

$$A_2(\nu^2\mu\rho_2 x^{1/\nu})^{\nu/2}/-B_2 x(\nu^2\mu\rho_2 x^{1/\nu})^{-\nu/2} \to 1 \qquad \text{as } x \to +\infty,$$

i.e., $B_2 = -A_2(\nu^2\mu\rho_2)^\nu$. Similarly $B_1 = A_1(\nu^2\mu\rho_1)^\nu$. We set $A_1 = \Gamma(1-\nu)$ and $A_2 = \Gamma(1+\nu)/(\nu^2\mu)^\nu(\rho_1^\nu + \rho_2^\nu)$, so that $v_1' v_2 - v_1 v_2' = 1$.

For future reference, we set $v_3(x,\mu) = 1$ for $x \leqq 0$ and

$$v_3(x,\mu) = \Gamma(1-\nu)\sum_{n=0}^{\infty}(\nu^2\mu\rho_2 \mid x \mid^{1/\nu})^n/\Gamma(n+1-\nu)n!, \quad x \geqq 0,$$

$v_3(x,\mu) = v_1(x,\mu)$ if $\rho_1 = 0$. Also we let

$$E_\nu(\mu) = \sum_{n=0}^{\infty}(-\mu)^n/\Gamma(n-\nu+1).$$

Let $p(t,x,y)$ be the density with respect to $m_0(y)$ of the distribution of $X_0(x;t)$. Let

$$g(\mu,x,y) = \int_0^\infty e^{-\mu t}p(t,x,y)\,dt.$$

Then (cf. McKean [6])

$$\begin{aligned} g(\mu,x,y) &= v_1(x,\mu)v_2(y,\mu) \quad \text{if} \quad x \leqq y, \\ &= v_2(x,\mu)v_1(y,\mu) \quad \text{if} \quad x \geqq y. \end{aligned}$$

In particular $g(\mu,0,0) = \delta\mu^{-\nu}$, where

$$\delta = \Gamma(1+\nu)/\Gamma(1-\nu)\nu^{2\nu}(\rho_1^\nu + \rho_2^\nu).$$

Let $X(x;t)$ be a conservative random walk or conservative general diffusion process, with speed measure $m(x) = \mid x \mid^{\beta+1}L(x)$, $-\infty < x < \infty$, where $L(x)$ is finite and

$$\begin{aligned} \lim_{c\to+\infty} L(cx)/L(x) &= 1 \qquad \text{if} \quad x > 0, \\ &= \rho_1/\rho_2, \quad \text{if} \quad x < 0. \end{aligned}$$

Set $g(c) = \rho_2^{-1}(\beta+1)\,cm(c)$.

Let $b(x)$, $-\infty < x < \infty$, be a distribution function on E, and set

$$V(x;t) = \int_{-\infty}^{\infty} L_X(x;t,y)\,db(x).$$

In the following theorem $\mathfrak{L}$ denotes the Laplace transform.

THEOREM 3. *Let $x \in E$, $cx_c \in E$, $cy_c \in E$, $x_c \to x_0$ and $y_c \to y_0 < x_0$ as $c \to +\infty$. Then as $c \to \infty$*

(1) $c^{-1}X(cx_c\,;g(c)t)$ *is weakly convergent to* $X_0(x_0\,;t)$;
(2) $\mathfrak{L}(c^{-1}(V(cx_c\,;g(c)t)))\to\mathfrak{L}(L_{X_0}(x_0\,;0,t))$ *for* $t\geqq 0$;
(3) $\mathfrak{L}(\sup[\tau\mid c^{-1}V(cx_c\,;g(c)\tau)\leqq t])\to\mathfrak{L}(\sup[\tau\mid L_{X_0}(x_0\,;0,\tau)\leqq t])$ *for* $t>0$;
(4) $\mathfrak{L}(\inf[t\mid X(cx_c\,;g(c)t)\leqq cy_c])\to v_2(x_0,\mu)/v_2(y_0,\mu)$;
(5) $\mathfrak{L}(\inf[t\mid X(cy_c\,;g(c)t)\geqq cx_c])\to v_1(y_0,\mu)/v_1(x_0,\mu)$;
(6) $\mathfrak{L}(\delta^{-1}t^{-\nu}c^{-1}V(x;g(c)t))\to E_\nu(\mu)$ for $t>0$;
(7) $\mathfrak{L}(\delta^{1/\nu}t^{-1/\nu}\sup[\tau\mid c^{-1}V(x;g(c)\tau)\leqq t])\to\exp(-\mu^\nu)$ *for* $t>0$;
if $0\in E$ *and* $0<t_0<t_1$,
(8)
$$P\{\exists\, t\in[t_0,t_1]:X(x;g(c)t)=0\}\to(\Gamma(\nu)\Gamma(1-\nu))^{-1}\int_0^{t_1t_0^{-1}}s^{-\nu}(1+s)^{-1}\,ds;$$
if $\rho_1=\rho_2$ *and* $y>0$,
(9) $\mathfrak{L}(\inf[t:|X(x;g(c)t)|\geqq cy])\to 1/v_3(y,\mu)$.

Proof. We can assume $g(c)>0$ for $c>0$. Let $m_c(x)=c(g(c))^{-1}m(cx)$. Then

$$\lim_{c\to+\infty}m_c(x)=m_0(x),\qquad -\infty<x<\infty.$$

If $X(x;t)$ is a general diffusion process, let $X_c(x;t)$ be the conservative general diffusion process with speed measure $m_c(x)$. If $X(x;t)$ is a random walk, let $X_c(x;t)$ be the conservative random walk with speed measure $m_c(x)$ and time parameter $\theta_c=\theta(g(c))^{-1}$.

LEMMA. *Let* $c>0$ *and* $cx\in E$. *Then* $X_c(x;t)\equiv c^{-1}X(cx;g(c)t)$.

Proof. (For definition of $\equiv$ see [8]). We prove the lemma for any fixed positive function $g(c)$, not necessarily the function defined above. If $X(x;t)$ is a birth and death process, the lemma follows immediately from the equations:

$$P\{X(\alpha_i\,;t)=\alpha_{i-1}\}=\delta_i\,t+o(t),\qquad P\{X(\alpha_i\,;t)=\alpha_{i+1}\}=\beta_i\,t+o(t),$$

$$((\alpha_i-\alpha_{i-1})m\{\alpha_i\})^{-1}=\delta_i,\qquad((\alpha_{i+1}-\alpha_i)m\{\alpha_i\})^{-1}=\beta_i.$$

If $X(x;t)$ is a random walk, the lemma follows from the analogous equations.

Suppose $X(x;t)$ is a general diffusion process. Let $X^{(n)}(x;t)$ be conservative birth and death processes with speed measure $m^{(n)}(x)$ such that $m^{(n)}(x)\to m(x)$ as $n\to\infty$ at all continuity points of (x). Let $cx\in E$. Then $X_c^{(n)}(x;t)$ is weakly convergent to $X_c(x;t)$ and $c^{-1}X^{(n)}(cx;g(c)t)$ is weakly convergent to $c^{-1}X(cx;g(c)t)$ as $n\to\infty$. But $X_c^{(n)}(x;t)\equiv c^{-1}X^{(n)}(cx;g(c)t)$, and hence $X_c(x;t)\equiv c^{-1}X(cx;g(c)t)$ as desired. Alternatively, this lemma may be proved by a direct application of the construction of Section 1.

Let $cx_c\in E$ and $x_c\to x_0$ as $c\to\infty$. Then, by (5) of Theorem 1, $X_c(x_c\,;t)$ is weakly convergent to $X_0(x_0\,;t)$ *as* $c\to\infty$ ((5)–(7) of Theorem 1 hold for $X_c(x_c\,;t)$ even if $X(x;t)$ is a random walk). Since

$$X_c(x_c\,;\;t)\equiv c^{-1}X(cx_c\,;\;g(c)t),$$

$c^{-1}X(cx_c\ ;\ g(c)t)$ is weakly convergent to $X_0(x_0\ ;\ t)$. This proves (1) of Theorem 3.

For any Borel set A

$$\begin{aligned} M\{\tau \mid 0 \leqq \tau \leqq t \text{ and } c^{-1}X(cx_c\,;g(c)\tau) \in A\} &= (g(c))^{-1}M\{\tau \mid 0 \leqq \tau \leqq g(c)t \text{ and } c^{-1}X(cx_c\,;\tau) \in A\} \\ &= (g(c))^{-1} \int_{y \in cA} L_X(cx_c\,; g(c)t, y)\, dm(y) \\ &= \int_A c^{-1}L_X(cx_c\,; cy, g(c)t)\, dm_c(x). \end{aligned}$$

Let

$$\begin{aligned} V_c(x_c\,;t) &= \int_{-\infty}^{\infty} L_{X_c}(x_c\,;t,y)\, db(cx) \\ &\equiv \int_{-\infty}^{\infty} c^{-1}L_X(cx_c\,; g(c)t, cy)\, db(cx) \\ &= c^{-1}\int_{-\infty}^{\infty} L_X(cx_c\,; g(c)t, y)\, db(x) \\ &= c^{-1}V(cx_c\,; g(c)t). \end{aligned}$$

Since

$$\begin{aligned} \lim_{c\to+\infty} b(cx) &= b(-\infty) \quad \text{if} \quad x < 0, \\ &= b(+\infty) \quad \text{if} \quad x > 0, \end{aligned}$$

(2) and (3) of Theorem 3 now follow from (5) and (6) of Theorem 1 applied to $V_c(x_c\ ;\ t)$.

The remaining statements of Theorem 3 are now consequences of the fact that we know the distribution, or at least its Laplace transform, of the relevant functionals of $X_0(x;\ t)$ (see [8]).

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