SOJOURN TIMES OF DIFFUSION PROCESSES

BY

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For a linear diffusion process $X(t)$, the *sojourn times*

$$\int_0^t V(X(t'), E)\, dt',$$

where $V(x, E)$ is the indicator function of E, form a measure on the Borel sets E of the line, for each path $X(\cdot)$ and for each t. A basic property of diffusion [3], [9] is that for almost every path and for each t, the sojourn time measure has a density $\mathfrak{J}(x, t)$ relative to the speed measure m of the process:

$$\int_0^t V(X(t'), E)\, dt' = \int_E \mathfrak{J}(x, t) m(dx).$$

The density is a continuous function of (x, t), and various estimates have been made of its modulus of continuity [6], [9].

The main result of this paper is the following property of the sojourn time density: Let T be a stopping time for the diffusion, which depends only on position (for a precise definition see Section 1). Then, conditional on $X(0) = x$ and $X(T) = y$, *$\mathfrak{J}(x', T)$ is a Markov process with parameter x'.* In fact, a change of variable transforms $\mathfrak{J}(x', T)$ into the radial process of a Brownian motion in two dimensions if x' is between x and y, in four dimensions otherwise (Section 4).

A number of properties follow immediately from this representation. For instance, for almost every path and for each t, the set of x' for which $\mathfrak{J}(x', t)$ is positive is an open interval; that is, the sojourn time density vanishes only on the closed complement of the range of the path. Also, one can write down the *precise local and global moduli of continuity of the density*: If the scale on the line is the natural scale for the diffusion, then with probability one, for each $t > 0$,

$$\limsup_{\Delta \to 0} \frac{| \mathfrak{J}(x' + \Delta, t) - \mathfrak{J}(x', t) |}{\sqrt{| \Delta | \log | \log | \Delta | |}} = 2\sqrt{\mathfrak{J}(x', t)}$$

if $\mathfrak{J}(x', t) > 0$;

$$\lim_{\Delta \to 0} \sup_{x' \in I} \frac{| \mathfrak{J}(x' + \Delta, t) - \mathfrak{J}(x', t) |}{\sqrt{\mathfrak{J}(x', t) \, | \Delta | \, | \log | \Delta | |}} = 2$$

if $\mathfrak{J}(x', t) > 0$ for x' in the closed bounded interval I;

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$$\limsup_{\Delta \to 0} \frac{\mathfrak{I}(x' + \Delta, t)}{|\Delta| \log |\log |\Delta||} = 1$$

if x' is an endpoint of the range of the path $X(t')$, $0 \leqq t' \leqq t$. The last relation holds if x' is an absorbing barrier; if x' is a finite natural barrier, the same relation holds for $t = \infty$. If the diffusion starts at an entrance point, say $x = +\infty$, then for each $t > 0$,

$$\limsup_{x' \to \infty} \frac{\mathfrak{I}(x', t)}{x' \log \log x'} = 1.$$

A word is in order about why one should suspect that $\mathfrak{I}(x', T)$ is Markovian, since the proof given here is completely unmotivated. Consider as a special case Brownian motion starting at the origin and stopped at the passage time P across the point 1. For each positive integer n, let $x_{k,n} = k2^{-n}$, $k = 0, 1, \cdots, 2^n$; and for each path, let $N_{k,n}$ be the number of returns from $x_{k+1,n}$ to $x_{k,n}$ that the path makes before it reaches 1. By the strong Markov property, the successive portions of the path after a return to $x_{k,n}$ until the next passage of $x_{k+1,n}$ are independent copies of a Brownian motion; and this property remains in force conditional on the value of $N_{k,n}$. Hence the sojourn time in the interval $(x_{k-1,n}, x_{k,n})$ is written as the sum of $N_{k,n}$ independent variables, and only Tchebycheff's inequality is needed to show that as $n \to \infty$ and $x_{k,n} \to x'$, the sojourn time, divided by $2^{-2n}N_{k,n}$, converges to 1 with probability one. The same reasoning can be applied to $N_{k,n}$; if $m > k$, $N_{k,n}$ is written as the sum of $N_{m,n}$ independent variables, and so with probability one, $2^{-n}N_{k,n}$ converges to a random variable as $n \to \infty$. This reasoning actually says that for each n, $N_{k,n}$ is a Markov chain with parameter k; hence in the limit one has the Markov property for the sojourn time density.

Note added in proof. The method outlined above has been used independently by F. B. Knight to prove the Markov property of the sojourn time density for certain Brownian motion processes. His work will appear in the Transactions of the American Mathematical Society.

The above technique was used to investigate the asymptotic behavior of the sojourn time of planar Brownian motion in small circles in [7]. It is probable that the Markov property of the sojourn time density should be a powerful tool in treating such problems. However, since the speed measure of the particular diffusion enters, rather than just the density, it seems that each problem must still be given individual study.

In this paper we have chosen to use an analytic approach, which consists of writing down a set of linear equations satisfied by the Laplace transform of the joint distribution of sojourn time densities at finitely many points, and inferring the Markov property from this. This approach has the advantage that the most general diffusion can be handled as easily as Brownian motion. In fact, we allow birth and death processes as well, since these have exactly the same analytic properties [1].

1. Diffusion processes

This brief discussion of diffusion processes is intended only to make the terminology and scope of the paper precise, and to introduce certain conventions. It is essentially a sketch of the presentation in [3], to which we refer the reader for further details.

The *state space* of the diffusion process will be a closed subset $\mathfrak{X}$ of the line; the *left* and *right boundaries* x_- and x_+ are respectively the infimum and supremum of the points of $\mathfrak{X}$. The usual case is that $\mathfrak{X}$ is a finite or infinite interval, and we have a diffusion with continuous paths; or that $\mathfrak{X}$ is discrete, with x_- and x_+ the only possible limit points, and we have a birth and death process.

A Markov process on $\mathfrak{X}$ with stationary transitions is a *diffusion* if the process is *strongly Markovian* and if the paths $X(t)$ are *right continuous*, have limits from the left, and have the *intermediate value property*: If $x \in \mathfrak{X}$ and $X(t) < x < X(t')$, then $X(t'') = x$ for some t'' with $t < t'' < t'$ or $t' < t'' < t$.

The conditional probability measures $\mathcal{P}_x\{\cdot\} = \mathcal{P}\{\cdot|X(0) = x\}$ are uniquely determined for each x in $\mathfrak{X}$ by the transition function, and in what follows we shall assume that each point x in $\mathfrak{X}$ is a possible initial value. We assume further that whenever $x_- < X(0) < x_+$, each point y in $\mathfrak{X}$, $x_- < y < x_+$, is a possible value of the path; in other words, the diffusion is not a translation, and the state space $\mathfrak{X}$ is minimal.

In this paper we reserve the name diffusion for such a process when the paths are defined and in $\mathfrak{X}$ for all $t \geqq 0$. A positive random variable T is a *stopping time depending only on position* if the event $T > t$ is independent of the future path $X(t')$, $t' > t$, conditional on the present value $X(t)$, and if the *stopped process* $X(t)$, $0 \leqq t < T$, is Markovian with the stationary transition function

$$\mathcal{P}\{T > t + t', X(t + t') \in E \mid T > t', X(t''), t'' \leqq t'\} = \mathcal{P}_{X(t')}\{T > t, X(t) \in E\}.$$

Except for the use of passage times in this section, we shall assume that $\mathfrak{X}$ is the minimal state space for the stopped process; this is equivalent to the assumption $\mathcal{P}_x\{T = 0\} = 0$, $x_- < x < x_+$.

There seems to be no way to avoid some special discussion of boundary points. For instance, a boundary point $x = x_\pm$ is a *trap* if $X(0) = x$ implies $X(t) = x$, $t \geqq 0$. Under our assumptions, a boundary point must be a trap unless every point x' in $\mathfrak{X}$, $x_- < x' < x_+$, can be reached from it. For this paper, there is no loss of generality in assuming that for a stopping time T, $\mathcal{P}_x\{T = 0\} = 1$ when x is a trap; and for a stopped diffusion we shall use the word trap to mean any boundary point x for which $\mathcal{P}_x\{T = 0\} = 1$. A trap may be accessible or inaccessible; in the former case it is an *absorbing barrier*; in the latter a *natural barrier*. We call a natural barrier x *finite* if with positive probability $X(t) \to x$ as $t \to \infty$. An *entrance boundary* is an inaccessible boundary point which is not a trap.

Of the two types of stopping times depending only on position which occur most frequently, the one is an *exponentially distributed time* S independent of the paths of the process:

$$\mathcal{P}_x\{S > t, X(t) \in E\} = e^{-st}\mathcal{P}_x\{X(t) \in E\}.$$

The other is the *passage time* across a point of $\mathfrak{X}$ from the left or from the right:

$$P_+(b) = \operatorname{Inf}\{t : X(t) \geqq b\}, \qquad P_-(a) = \operatorname{Inf}\{t : X(t) \leqq a\},$$

with the usual convention that the value is $+\infty$ if the corresponding set is empty. Because of the intermediate value property,

$$\begin{aligned} X(P_+(b)) &= b \quad \text{if} \quad X(0) \leqq b \quad \text{and} \quad P_+(b) < \infty, \\ X(P_-(a)) &= a \quad \text{if} \quad X(0) \geqq a \quad \text{and} \quad P_-(a) < \infty. \end{aligned}$$

According to the strong Markov property, when T is a stopping time depending only on position, the *renewed process* $X_T^+(t) = X(T + t)$, $t \geqq 0$, is a diffusion with the original transition function and with the initial point $X_T^+(0) = X(T)$; and conditional on the value of $X(T)$, the stopped and renewed processes are independent. In particular, a functional Φ which depends only on the renewed process will satisfy

$$E_x\{\Phi; X(T) \in dy\} = E_y\{\Phi\}\,\mathcal{P}_x\{X(T) \in dy\}.$$

Using this type of equation one can easily fill in the proofs of the following statements:

First, the *scale* x of the line may be chosen so that whenever $x_- < a < x < b < x_+$,

$$\begin{aligned} \mathcal{P}_x\{P_+(b) < P_-(a)\} &= 1 - \mathcal{P}_x\{P_-(a) < P_+(b)\} \\ &= (x - a)/(b - a). \end{aligned} \tag{1.1}$$

In this scale, a boundary point is finite if it is an absorbing barrier or a finite natural barrier, infinite if it is an entrance boundary.

Second, let T be a stopping time which depends only on position. Then there corresponds a pair of positive continuous functions h_+ and h_-, respectively increasing and decreasing, and strictly positive on (x_-, x_+), such that

$$\begin{aligned} \mathcal{P}_x\{P_+(b) \leqq T\} &= h_+(x)/h_+(b), \qquad x < b, \\ \mathcal{P}_x\{P_-(a) \leqq T\} &= h_-(x)/h_-(a), \qquad a < x. \end{aligned} \tag{1.2}$$

It is important to think of $h_+(x)$ as defined for all $x < x_+$, and $h_-(x)$ for all $x > x_-$, so as to be positive and linear on the complementary intervals of $\mathfrak{X}$.

Third, either h_+ and h_- are both constant, the process is recurrent with $T \equiv \infty$; or h_+ and h_- are linearly independent on every interval, the stopped process is transient, and for a path with $T = \infty$, $X(T) = \lim_{t\to\infty} X(t)$ exists and is a finite natural barrier. We will naturally assume the latter situation.

Fourth,

$$\Delta(x, y) = h_+(x)h_-(y) - h_-(x)h_+(y)$$

never vanishes for $x < y$, and is a convex function in the natural scale in each variable. It follows that there is a measure k on $\mathfrak{X}$, the *killing measure*, such that

$$dh'_+(x) = h_+(x)k(dx), \qquad dh'_-(x) = h_-(x)k(dx), \tag{1.3}$$

in the sense that for $x < y$

$$h'(y) - h'(x) = \int_{x<x'\leqq y} h(x')k(dx'),$$

for $h = h_+$ or h_-, where h' denotes the right-hand derivative of h. The measure k is finite on every compact subinterval of (x_-, x_+); if x is a finite boundary point every neighborhood of which has infinite measure, then x is a trap. If x is a trap, then $k(\{x\}) = \infty$, while the corresponding function $h = h_+$ or h_- vanishes at x; thus the equations (1.3) remain valid at such a point with the convention that $h(x)k(dx)$ is defined by the left side:

$$h_+(x)k(\{x\}) = \lim_{\delta\to 0} (h'(x + \delta) - h'(x - \delta)).$$

In general the first equation (1.3) holds for $x < x_+$ and the second for $x > x_-$, and the functions h_+ and h_- are determined up to a constant factor by the killing measure k.

Fifth, the Wronskian $W = h'_+(x)h_-(x) - h'_-(x)h_+(x)$ is constant, and we assume h_+ and h_- normalized so that $W \equiv 1$. Set

$$h(x, y) = h_+(\mathrm{Min}(x, y))h_-(\mathrm{Max}(x, y)). \tag{1.4}$$

Then for every bounded continuous function f on the line and point x in $\mathfrak{X}$,

$$E_x\{f(X(T))\} = \int h(x, y)f(y)k(dy). \tag{1.5}$$

Here $X(T) = \lim_{t\nearrow T} X(t)$ is a finite natural barrier if $T = \infty$, as before; while $h(x, y)k(dy)$ is to be interpreted by (1.4) and the end of the preceding paragraph if y is a trap.

Sixth, there is a measure m on $\mathfrak{X}$, the *speed measure*, such that for every bounded continuous function f vanishing at traps, and for each point x in $\mathfrak{X}$,

$$E_x\left\{\int_0^T f(X(t))\,dt\right\} = \int h(x, y)f(y)m(dy). \tag{1.6}$$

The speed measure has closed carrier $\mathfrak{X}$ and is finite on every compact subinterval of (x_1, x_2); we set $m(\{x\}) = \infty$ when x is a trap for the unstopped diffusion. Then m depends only on the unstopped diffusion.

Seventh, the original diffusion may be reconstructed from the speed measure m, by taking the killing measure $k = sm$ for an arbitrary positive value of the parameter s. For then the corresponding stopping time is exponentially distributed with parameter s and independent of the paths. The kernel $h(x, y)$ is uniquely determined by (1.3), and $h(x, y)m(dy)$ is the Laplace transform, with parameter s, of the transition function of the diffusion. Since this stopping time does not satisfy our convention that $\mathcal{P}_y\{T = 0\} = 1$ when y is a trap, it is necessary to define $h(x, y)m(dy)$ at such a point; this is done simply by noting that if $k = sm$, then $h(x, y)m(dy) = (1/s)h(x, y)k(dy)$ can be interpreted as in (1.5).

Similarly, if k is the killing measure corresponding to a stopping time T, one may construct the transition function of the stopped process by considering the new killing measure $k_s = k + sm$.

2. Sojourn time densities

Let $X(t)$ be a diffusion with state space $\mathfrak{X}$, and T a stopping time depending only on position. We assume the conventions and notation of Section 1; in particular we have the speed measure m, the killing measure k, and the density $h(x, \cdot)$ relative to k of the distribution of the place of stopping for initial point x.

For each path define the *sojourn time* of the stopped path in the Borel set E to be

$$\int_0^T V(X(t), E)\, dt,$$

$V(x, E)$ being the indicator function of E. For each path, the sojourn times define a measure on the line.

For almost every path, the sojourn time measure has a density $\mathfrak{Z}(x, T)$ relative to the speed measure m:

$$\int_0^T V(X(t), E)\, dt = \int_E \mathfrak{Z}(x', T) m(dx').$$

The density can be taken to be continuous in probability in the spatial parameter x'; and for this version the joint distribution of $\mathfrak{Z}(x_k, T)$, $1 \leqq k \leqq n$, and of $X(T)$ is determined by

$$\begin{aligned} \phi_n(x) &= \phi_n(x; y; x_1, \cdots, x_n; z_1, \cdots, z_n) \\ &= E_x\{\exp\{-\textstyle\sum_1^n z_k \mathfrak{Z}(x_k, T)\} \mid X(T) = y\} \\ &= 1 - \sum_1^n z_k \frac{h(x, x_k)h(x_k, y)}{h(x, y)} \phi_n(y_k), \end{aligned} \tag{2.1}$$

for positive numbers $z_1, \cdots, z_n$ and for points $x, y, x_1, \cdots, x_n$ in $\mathfrak{X}$ with y in the closed carrier of the killing measure k.

In (2.1) the ratio $h(x, x_k)h(x_k, y)/h(x, y)$ is to be interpreted by (1.4) so that its value may be determined even when numerator and denominator both vanish. For instance, if $y = x_+$,

$$h(x, x_k)h(x_k, y)/h(x, y) = h(x, x_k)h_+(x_k)/h_+(x).$$

Let $\mathfrak{I}(x, T)$ be provisionally the density of the absolutely continuous part of the sojourn time measure, relative to m: Let $\Delta_{p,k}$ be the dyadic interval $[k2^{-p}, (k + 1)2^{-p})$ and say $\Delta_{p,k} \to x$ if as $n \to \infty$, $k = k_n$ is such that $x \in \Delta_{p,k}$. Then [8] for each path

$$\mathfrak{I}(x, T) = \lim_{\Delta_{p,k} \to x} \frac{1}{m(\Delta_{p,k})} \int_0^T V(X(t), \Delta_{p,k})\, dt$$

exists for almost every point x in $\mathfrak{X}$, relative to m. Since $\mathfrak{I}(x, T)$ is the limit of a fixed sequence of functions each measurable jointly on the sample space of the diffusion and on $\mathfrak{X}$, $\mathfrak{I}(x, T)$ is jointly measurable, and by Fubini's theorem there is a subset $\mathfrak{X}'$ of $\mathfrak{X}$ with $m(\mathfrak{X} - \mathfrak{X}') = 0$ such that for each x in $\mathfrak{X}'$, $\mathfrak{I}(x, T)$ is defined as the limit above for almost every path.

We use a technique developed by Kac [4] to show that (2.1) is satisfied by $\mathfrak{I}(x, T)$ when $x_1, \cdots, x_n$ are in $\mathfrak{X}'$. Let E be a Borel set, and let V be a positive Borel measurable function. Then since the stopped process is Markovian with a stationary transition function,

$$\begin{aligned}
F(x, E) &= E_x\left\{ \exp\left\{ -\int_0^T V(X(t'))\, dt' \right\}; X(T) \in E \right\} \\
&= E_x\left\{\left[1 - \int_0^T V(X(t)) \exp\left\{-\int_t^T V(x(t'))\, dt'\right\} dt\right]; X(T) \in E\right\} \\
&= \mathcal{P}_x\{X(T) \in E\} \\
&\quad - E_x\left\{\int_0^T dt\, V(X(t)) E_{X(t)}\left\{\exp\left\{-\int_0^T V(X(t'))\, dt'\right\}; X(T) \in E\right\}\right\} \\
&= \int_E h(x, y)k(dy) - \int h(x, x')V(x')F(x', E)m(dx')
\end{aligned}$$

by (1.5) and (1.6). In particular, as a measure of the set E, $F(x, E)$ has a continuous density relative to k, and

$$\begin{aligned}
f(x, y) &\equiv E_x\left\{\exp\left\{-\int_0^T V(X(t'))\, dt'\right\} \middle| X(T) = y\right\} \\
&= 1 - \int V(x') \frac{h(x, x')h(x', y)}{h(x, y)} f(x', y)m(dx'),
\end{aligned}$$

for y in the closed carrier of k.

Note that $f(x, y)$ is continuous in x, and for any collection of functions V with $\int V(x')m(dx')$ bounded, the corresponding functions f are uniformly equicontinuous in x. Indeed, using (1.4) and the fact that $(h_+/h_-)' = h_-^{-2}$,

$$\frac{\partial}{\partial x} f(x, y) = \int_{x'<x} V(x') \frac{h_+^2(x')}{h_+^2(x)} f(x', y) m(dx), \qquad x < y,$$

$$= -\int_{x<x'} V(x') \frac{h_-^2(x')}{h_-^2(x)} f(x', y) m(dx'), \qquad y < x,$$

$$\left| \frac{\partial}{\partial x} f(x, y) \right| \leqq \int V(x') m(dx').$$

Now let $x_1, \cdots, x_n$ be points in $\mathfrak{X}'$. Let

$$V_p(x; x_k) = V(x, \Delta_{p,j})/m(\Delta_{p,j}),$$

where j is chosen so that $x_k \in \Delta_{p,j}$, and let

$$V(x) = V_p(x) = \textstyle\sum_1^n z_k V_p(x; x_k).$$

As p becomes infinite, the corresponding functions $f_p(x, y)$ approach $\phi_n(x; y; x_1, \cdots, x_n; z_1, \cdots, z_n)$, and are uniformly equicontinuous in x. Thus

$$\lim_{p\to\infty} \int V_p(x; x_k) \frac{h(x, x')h(x', y)}{h(x, y)} f_p(x', y) m(dx')$$

$$= \lim_{p\to\infty} \frac{1}{m(\Delta_{p,j})} \int_{\Delta_{p,j}} \frac{h(x, x')h(x', y)}{h(x, y)} f_p(x', y) m(dx')$$

$$= \frac{h(x, x_k)h(x_k, y)}{h(x, y)} \phi_n(x_k),$$

and this implies (2.1) whenever $x_1, \cdots, x_n$ are in $\mathfrak{X}'$.

It is now quite easy to show that the sojourn times are absolutely continuous relative to m. By putting $n = 1$ in (2.1), for z_1 small,

$$E_x\{e^{-z_1\mathfrak{Z}(x_1, T)} \mid X(T) = y\} = 1 - z_1 \frac{h(x, x_1)h(x_1, y)}{h(x, y)} + O(z_1),$$

$$E_x\{\mathfrak{Z}(x_1, T) \mid X(T) = y\} = \frac{h(x, x_1)h(x_1, y)}{h(x, y)},$$

$$E_x\{\mathfrak{Z}(x_1, T)\} = h(x, x_1),$$

$$E_x\left\{ \int_a^b \mathfrak{Z}(x_1, T) m(dx_1) \right\} = \int_a^b E_x\{\mathfrak{Z}(x, T)\} m(dx_1)$$

$$= \int_a^b h(x, x_1) m(dx_1)$$

$$= E_x\left\{ \int_0^T V(X(t); (a, b))\, dt \right\}$$

by (1.6), when (a, b) is an open interval containing no traps. But for almost every path, by definition

$$\int_a^b \mathfrak{J}(x_1, T)m(dx_1) \leqq \int_0^T V(X(t); (a,b))\, dt.$$

Because of the above, equality must hold with probability one. If x_1 is a trap, the sojourn time at x_1 vanishes for all paths, and $\mathfrak{J}(x_1, T) \equiv 0$, in agreement with (2.1). Thus $\mathfrak{J}(x_1, T)$ is indeed a version of the density of the sojourn time measure.

To prove the continuity in probability of $\mathfrak{J}(x_1, T)$, consider (2.1) with $n = 2$, x_1 and x_2 points in $\mathfrak{X}'$. Multiplying by $h(x, y)k(dy)$ and integrating, we obtain

$$\begin{aligned} \psi(x) &= E_x\{e^{-z_1\mathfrak{J}(x_1,T)-z_2\mathfrak{J}(x_2,T)}\} \\ &= 1 - z_1 h(x, x_1)\psi(x_1) - z_2 h(x, x_2)\psi(x_2). \end{aligned}$$

Setting first $x = x_1$, then $x = x_2$, and finally letting x be arbitrary, for small z_1 and z_2, we have

$$\begin{aligned} \psi(x_1) &= 1 - z_1 h(x_1, x_1) - z_2 h(x_1, x_2) + O(z_1, z_2), \\ \psi(x_2) &= 1 - z_1 h(x_1, x_2) - z_2 h(x_2, x_2) + O(z_1, z_2), \\ \psi(x) &= 1 - z_1 h(x, x_1) - z_2 h(x, x_2) \\ &\qquad + z_1^2 h(x, x_1)h(x_1, x_1) + z_2^2 h(x, x_2)h(x_2, x_2) \\ &\qquad + z_1 z_2(h(x, x_1) + h(x, x_2))h(x_1, x_2) + O(z_1^2, z_2^2); \end{aligned}$$

it follows that

$$\begin{aligned} E_x\{(\mathfrak{J}(x_k, T))^2\} &= 2h(x, x_k)h(x_k, x_k), \\ E_x\{\mathfrak{J}(x_1, T)\mathfrak{J}(x_2, T)\} &= (h(x, x_1) + h(x, x_2))h(x_1, x_2), \\ E_x\{(\mathfrak{J}(x_1, T) - \mathfrak{J}(x_2, T))^2\} &= 2[h(x, x_1)(h(x_1, x_1) - h(x_1, x_2)) \\ &\qquad - h(x, x_2)(h(x_1, x_2) - h(x_2, x_2))] \\ &\to 0, \qquad \text{as } x_1 - x_2 \to 0. \end{aligned}$$

Thus $\mathfrak{J}(x_1, T)$ is continuous in probability, and this enables us finally to define $\mathfrak{J}(x_1, T)$ for all x_1 in $\mathfrak{X}$ so that (2.1) holds.

3. The Markov property of the sojourn time density

Since the kernel $h(x, y)$ is defined for all x and y in $[x_-, x_+]$, we need not restrict $x_1, \cdots, x_n$ to the state space $\mathfrak{X}$ in the equations (2.1). The solutions ϕ_n are defined for all values of $x_1, \cdots, x_n$ in $[x_-, x_+]$ and for fixed x and y determine the joint distribution of a process $\mathfrak{J}(x', T)$ with parameter x' ranging over $x_- \leqq x' \leqq x_+$.

We will show in this section that *conditional on* $X(0) = x$ *and* $X(T) = y$,

$\mathfrak{Z}(x', T)$ *is a Markov process.* The *absolute probabilities* are given by

$$\mathcal{P}_x\{\mathfrak{Z}(x', T) > t \mid X(T) = y\} = \frac{h(x, x')h(x', y)}{h(x', x')h(x, y)} e^{-t/h(x',x')}; \tag{3.1}$$

the *transition probabilities* by

$$\begin{aligned} \phi(x_1, t; x_2, z) &= E_x\{e^{-z\mathfrak{Z}(x_2,T)} \mid \mathfrak{Z}(x_1, T) = t, X(T) = y\} \\ &= A \exp\left\{-\frac{w_2 t}{h_-^2(x_1)(1 + w_2(\xi_2 - \xi_1))}\right\}, \end{aligned} \tag{3.2}$$

for $x_1 < x_2$, where

$$w_2 = z\, h_-^2(x_2), \qquad \xi_n = h_+(x_n)/h_-(x_n);$$

we set also

$$x^* = \text{Max}(x, y), \qquad x_* = \text{Min}(x, y),$$

$$\xi^* = \frac{h_+(x)h_+(y)}{h(x, y)} = \text{Max}\left(\frac{h_+(x)}{h_-(x)}, \frac{h_+(y)}{h_-(y)}\right),$$

$$\xi_* = \frac{h(x, y)}{h_-(x)h_-(y)} = \text{Min}\left(\frac{h_+(x)}{h_-(x)}, \frac{h_+(y)}{h_-(y)}\right);$$

then

$$\begin{aligned} A &= 1, & x^* &\leqq x_1 < x_2, \\ &= (1 + w_2(\xi_2 - \xi_1))^{-1}, & x_* &\leqq x_1 < x_2 \leqq x^*, \\ &= (1 + w_2(\xi_2 - \xi_1))^{-2}, & x_1 &< x_2 \leqq x_*, t > 0, \\ &= \frac{1}{\xi_* - \xi_1}\left(\xi_* - \xi_2 + \frac{\xi_2 - \xi_1}{1 + w_2(\xi_2 - \xi_1)}\right), & x_1 &< x_2 \leqq x_*, t = 0. \end{aligned} \tag{3.3}$$

The formula (3.1) for the absolute probabilities is easily proved by solving the equations (2.1) for $n = 1$. The statement that $\mathfrak{Z}(x', T)$ is Markovian with the transition probability given by (3.2) is equivalent to

$$\begin{aligned} \phi_{n+1}(x) &= E_x\{\exp\{-\textstyle\sum_1^{n+1} z_k\, \mathfrak{Z}(x_k, T)\}\,|X(T) = y\} \\ &= E_x\{\exp\{-\textstyle\sum_1^n z_k\, \mathfrak{Z}(x_k, T)\}\phi(x_n, \mathfrak{Z}(x_n, T); x_{n+1}, z_{n+1})\,|X(T) = y\} \end{aligned} \tag{3.4}$$

for $n \geqq 1$, and we will prove (3.4) by solving the equations (2.1) more or less explicitly for arbitrary n. We fix points $x_1 < \cdots < x_{n+1}$ in $[x_-, x_+]$ for the remainder of this section.

First suppose $x_n \leqq y$. In this case $h(x_k, y) = h_+(x_k)h_-(y)$ by (1.4), and by letting $x = x_j$, $j = 1, \cdots, n$ in (2.1),

$$h_+(x_j)\phi_n(x_j) = h_+(x_j) - \textstyle\sum_1^n h(x_j, x_k)z_k\, h_+(x_k)\phi_n(x_k). \tag{3.5}$$

This system of equations has the solution

$$\phi_n(x_j) = \alpha_j/\beta_n,$$

where α_j and β_n are determined by

$$(3.6)\qquad \beta_n = \beta_{n-1} + z_n h_+(x_n)h_-(x_n)\alpha_n ,$$

$$(3.7)\qquad \begin{aligned}\alpha_n &= \beta_n - (h_-(x_n)/h_+(x_n))\textstyle\sum_1^n z_k h_+^2(x_k)\alpha_k \\ &= \beta_{n-1} - (h_-(x_n)/h_+(x_n))\textstyle\sum_1^{n-1} z_k h_+^2(x_k)\alpha_k ,\end{aligned}$$

with $\beta_0 = \alpha_1 = 1$. We set $\xi_n = h_+(x_n)/h_-(x_n)$ as before, and

$$(3.8)\qquad \delta_n = 1 + z_n h_-^2(x_n)(\xi_n - \xi_{n-1}),$$

$$(3.9)\qquad \gamma_n = z_n h_-^2(x_n)/\delta_n h_-^2(x_{n-1}).$$

Then from (3.6) and (3.7),

$$(3.10)\qquad \xi_{n+1}\alpha_{n+1} = \xi_n\alpha_n + \beta_n(\xi_{n+1} - \xi_n),$$

$$(3.11)\qquad \xi_{n+1}\alpha_{n+1}\delta_{n+1} = \xi_n\alpha_n + \beta_{n+1}(\xi_{n+1} - \xi_n),$$

$$(3.12)\qquad \beta_{n+1} = \delta_{n+1}(\beta_{n-1} + (z_n + \gamma_{n+1})h_+(x_n)h_-(x_n)\alpha_n).$$

If $x \leqq x_n < x_{n+1} \leqq y$, there is no loss of generality in assuming that $x = x_j$ for some $j \leqq n$. Then $\phi_n(x) = \alpha_j/\beta_n$, and by (3.6)

$$E_x\{\exp\{-\textstyle\sum_1^{n-1} z_k\mathfrak{J}(x_k, T)\}; \mathfrak{J}(x_n, T)\in dt\,|\,X(T) = y\}$$
$$= \frac{\alpha_j}{h_+(x_n)h_-(x_n)\alpha_n}\exp\left\{-\frac{\beta_{n-1}t}{h_+(x_n)h_-(x_n)\alpha_n}\right\}dt.$$

By (3.12)

$$\begin{aligned}\phi_{n+1}(x) &= \alpha_j/\beta_{n+1} \\ &= (1/\delta_{n+1})E_x\{\exp\{-\textstyle\sum_1^n z_k\,\mathfrak{J}(x_k, T) - \gamma_{n+1}\,\mathfrak{J}(x_n, T)\}\,|X(T) = y\}.\end{aligned}$$

In view of (3.8) and (3.9), this is just the desired result (3.4) in the specified range.

If $x_n < x_{n+1} \leqq x, y$,

$$\begin{aligned}\phi_n(x) &= 1 - \frac{h_-(x)h_-(y)}{h(x, y)}\sum_1^n z_k h_-^2(x_k)\phi_n(x_k) \\ &= 1 - \frac{h_-(x)h_-(y)}{h(x, y)}\frac{h_+(x_n)}{h_-(x_n)}(1 - \phi_n(x_n)) \\ &= 1 - \frac{\xi_n}{\xi_*} + \frac{\xi_n\alpha_n}{\xi_*\beta_n};\end{aligned}$$

$$\begin{aligned}\phi_{n+1}(x) &= 1 - \frac{\xi_{n+1}}{\xi_*} + \frac{\xi_{n+1}\alpha_{n+1}}{\xi_*\beta_{n+1}} \\ &= 1 - \frac{\xi_{n+1}}{\xi_*} + \frac{\xi_{n+1} - \xi_n}{\xi_*\delta_{n+1}} + \frac{\xi_n\alpha_n}{\xi_*\delta_{n+1}\beta_{n+1}},\end{aligned}$$

by (3.11). As in the preceding paragraph, (3.6) and (3.12) again show that (3.4) holds.

Now suppose $y \leqq x_n$. In this case, (2.1) becomes

$$\phi_n(x_j) = 1 - z_n h_-^2(x_n) \frac{h_+(x_j)h_+(y)}{h(x_j, y)} \phi_n(x_n) - \sum_1^{n-1} z_k \frac{h(x_j, x_k)h(x_k, y)}{h(x_j, y)} \phi_n(x_k).$$

Considering $\phi_n(x_n)$ a parameter for the moment, this is a system of $n - 1$ equations which is a combination of (2.1) and (3.5). Again we can write down the solution:

$$\phi_n(x_j) = \phi_{n-1}(x_j) - z_n h_-^2(x_n)\phi_n(x_n) \frac{h_+(x_j)h_+(y)}{h(x_j, y)} \frac{\alpha_j}{\beta_{n-1}}$$

for $j = 1, \cdots, n - 1$; by substitution one sees that this formula holds also when $j = n$. But when $x \geqq x_n$,

$$\phi_n(x) = 1 - \sum_1^n \frac{h_+(x_k)h(x_k, y)}{h_+(y)} z_k \phi_n(x_k)$$

is independent of x. Thus putting $j = n$ above and using (3.6), we obtain

$$\beta_n \phi_n(x_n) = \beta_{n-1} \phi_{n-1}(x_{n-1}) = a,$$

where a is independent of z_n.

If $x = x_j \leqq x_n < x_{n+1}$ and $y \leqq x_n$,

$$\phi_n(x) = \phi_{n-1}(x) - z_n h_-^2(x_n)a\, \xi^* \frac{\alpha_j}{\beta_n \beta_{n-1}}$$
$$= \phi_{n-1}(x) - \frac{a\, \xi^* \alpha_j}{\xi_n \alpha_n \beta_{n-1}} + \frac{a\, \xi^* \alpha_j}{\xi_n \alpha_n \beta_n}$$

by (3.6). And by using (3.10) and (3.6),

$$\phi_{n+1}(x) = \phi_n(x) - z_{n+1} h_-^2(x_n)a\, \xi^* \frac{\alpha_j}{\beta_n \beta_{n+1}}$$
$$= \phi_{n-1}(x) - \frac{a\, \xi^* \alpha_j}{\xi_n \alpha_n \beta_{n-1}} + \frac{a\, \xi^* \alpha_j}{\xi_n \alpha_n \beta_{n+1}} \delta_{n+1}.$$

Again, these equations, along with (3.6) and (3.12), imply (3.4).

Finally, if $y \leqq x_n < x_{n+1} \leqq x$,

$$\phi_{n+1}(x) = \phi_{n+1}(x_{n+1}) = a/\beta_{n+1}, \qquad \phi_n(x) = \phi_n(x_n) = a/\beta_n,$$

and (3.4) follows.

4. The sojourn time density as a diffusion

To describe the properties of the Markov process $\mathfrak{I}(x', T)$ most effectively, we make the change of variable

$$\xi = h_+(x')/h_-(x'), \qquad \mathfrak{S}(\xi) = 2\sqrt{\mathfrak{I}(x', T)}/h_-(x'). \tag{4.1}$$

The formula (3.2) becomes

$$E_x\{e^{-(1/4)w_2 \mathcal{S}^2(\xi_2)} \mid \mathcal{S}(\xi_1) = s_1, X(T) = y\} = A \exp\left\{-\frac{1}{4}\frac{w_2 s_1^2}{1 + w_2(\xi_2 - \xi_1)}\right\},$$

with A given by (3.3). This and the absolute probabilities (3.1) determine $\mathcal{S}(\xi)$ for $0 \leqq \xi < \infty$, although the range $[x_-, x_+]$ of x' may correspond to a smaller interval. It is an easy matter to invert the Laplace transform and compute the transition density

$$f(\xi_1, s_1; \xi_2, s_2)\, ds_2 = \mathcal{P}_x\{\mathcal{S}(\xi_2) \in ds_2 \mid \mathcal{S}(\xi_1) = s_1, X(T) = y\}.$$

Set

$$\begin{aligned} f_2(s_1, s_2, \xi) &= \frac{s^2}{2\xi} \exp\left\{-\frac{s_1^2 + s_2^2}{4\xi}\right\} I_0\left(\frac{s_1 s_2}{2\xi}\right) \\ &= 2s_2 \int_0^\pi d\theta \frac{1}{4\pi\xi} \exp\left\{-\frac{s_1^2 + s_2^2 - 2s_1 s_2 \cos\theta}{4\xi}\right\}, \\ f_4(s_1, s_2, \xi) &= \frac{s_2^2}{2\xi s_1} \exp\left\{-\frac{s_1^2 + s_2^2}{4\xi}\right\} I_1\left(\frac{s_1 s_2}{2\xi}\right) \\ &= 4\pi s_2^3 \int_0^\pi d\theta \frac{\sin^2\theta}{(4\pi\xi)^2} \exp\left\{-\frac{s_1^2 + s_2^2 - 2s_1 s_2 \cos\theta}{4\xi}\right\}. \end{aligned}$$

Then when $s_1 > 0$ and ξ_1 and ξ_2 are in the ranges indicated

$$\begin{aligned} f(\xi_1, s_1; \xi_2, s_2) &= f_4(s_1, s_2, \xi_2 - \xi_1), & \xi_1 < \xi_2 \leqq \xi_*, \\ &= f_2(s_1, s_2, \xi_2 - \xi_1), & \xi_* \leqq \xi_1 < \xi_2 \leqq \xi^*, \\ &= (s_1/s_2)^2 f_4(s_1, s_2, \xi_2 - \xi_1), & \xi^* \leqq \xi_1 < \xi_2. \end{aligned} \tag{4.2}$$

Recall that ξ_* is the point corresponding to $\mathrm{Min}(x, y)$ under (4.1), ξ^* to $\mathrm{Max}(x, y)$.

But f_N, $N = 2, 4$, as defined above, is the transition density of the radial process $R_N(\xi) = |W_N(\xi)|$ of a Brownian motion W_N in dimension N. And applying the change of variable (4.1) to the formula (3.1), we see that in the range $\xi_* \leqq \xi \leqq \xi^*$, the absolute probabilities of $\mathcal{S}(\xi)$ are those of the radial process $R_2(\xi)$ of planar Brownian motion with initial point $R_2(0) = 0$.

In the range $\xi \leqq \xi_*$, it is convenient to describe the process in terms of a random initial parameter value [2]. Since the origin is an entrance boundary for R_4, $\mathcal{S}(\xi) > 0$ for some $\xi < \xi_*$ implies $\mathcal{S}(\xi') > 0$, $\xi \leqq \xi' \leqq \xi_*$. Define

$$\Xi_1 = \mathrm{Inf}\,\{\xi : \mathcal{S}(\xi) > 0\}; \tag{4.3}$$

then by (3.1)

$$\mathcal{P}_x\{\Xi_1 \leqq \xi \mid X(T) = y\} = \mathcal{P}_x\{\mathcal{S}(\xi) > 0 \mid X(T) = y\} = \xi/\xi_*, \quad \xi \leqq \xi_*.$$

We have

$$\begin{aligned} \mathcal{S}(\xi) &= 0, & \xi \leqq \Xi_1, \\ \mathcal{S}(\xi) &= R_4(\xi - \Xi_1), & \Xi_1 \leqq \xi \leqq \xi_*, \end{aligned} \tag{4.4}$$

where R_4 is the radial process of Brownian motion in four dimensions, with initial point $R_4(0) = 0$; and Ξ_1 is uniformly distributed on $[0, \xi_*]$ and independent of R_4.

$$\mathcal{S}(\xi) = R_2(\xi), \qquad \xi_* \leqq \xi \leqq \xi^*, \tag{4.4'}$$

where R_2 is the radial process of planar Brownian motion, with initial point $R_2(\xi_*) = R_4(\xi_* - \Xi_1)$, but otherwise independent. As noted above, the distribution of $R_2(\xi_*)$ is the same as if $R_2(0) = 0$.

Finally, let R_4^* be a copy of R_4, and let Ξ_2 be uniformly distributed on the interval $[0, 1/\xi^*]$, and independent. Then

$$\begin{aligned} \mathcal{S}(\xi) &= \xi R_4^*((1/\xi) - \Xi_2), & \xi^* \leqq \xi \leqq 1/\Xi_2, \\ \mathcal{S}(\xi) &= 0, & 1/\Xi_2 \leqq \xi < \infty. \end{aligned} \tag{4.4''}$$

We have the matching condition

$$\xi^* R_4^*((1/\xi^*) - \Xi_2) = R_2(\xi^*),$$

but the distribution of this variable is the same as if R_4^* and Ξ_2 were completely independent of $\mathcal{S}(\xi)$, $\xi \leqq \xi^*$, as well as of each other.

It is not hard to compute the absolute probabilities and transition function of $\xi R_4^*((1/\xi) - \Xi_2)$ and check that they agree with (3.1) and (4.2). The pertinent formulas are

$$\frac{1}{\xi_2} f_4\left(\frac{s_2}{\xi_2}, \frac{s_1}{\xi_1}, \frac{1}{\xi_1} - \frac{1}{\xi_2}\right) = \frac{s_1^3 \xi_2^2}{s_2^3 \xi_1^2} \exp\left\{\frac{s_2^2}{4\xi_2} - \frac{s_1^2}{4\xi_1}\right\} f_4(s_1, s_2, \xi_2 - \xi_1),$$

$$\begin{aligned} \mathcal{P}\left\{\xi R_4^*\left(\frac{1}{\xi} - \Xi_2\right) \epsilon\, ds;\ \Xi_2 < \frac{1}{\xi}\right\} &= \xi^* \int_0^{1/\xi} d\eta\, f_4\left(0, \frac{s}{\xi}, \frac{1}{\xi} - \eta\right) \frac{ds}{\xi} \\ &= \frac{s}{2\xi} \exp\left\{-\frac{s^2}{4\xi}\right\} ds. \end{aligned}$$

Of course, it is possible that $\xi_* = 0$, $\xi^* = \infty$, or both. In this case the corresponding range of ξ disappears, but the description of $\mathcal{S}(\xi)$ is otherwise unchanged. In general, the values of x' corresponding to the initial and final values Ξ_1 and Ξ_2^{-1} are respectively $\operatorname{Min} X(t)$, $0 \leqq t \leqq T$, and $\operatorname{Max} X(t)$, $0 \leqq t \leqq T$. To see this, note that with probability one, Ξ_1 is not less than the value of ξ corresponding to the minimum of the path, and that the two variables have the same distribution.

Because of this representation, the known properties of Brownian motion are reflected in those of the sojourn time density $\mathfrak{J}(x', T)$. For instance $\mathcal{S}(\xi)$ may be assumed continuous, and strictly positive on $(\Xi_1, 1/\Xi_2)$; and the corresponding version of $\mathfrak{J}(x', T)$ still serves as a density of the sojourn time measures, since the latter is continuous in probability. Hence conditional on $X(T) = y$, $\mathfrak{J}(x', T)$ is a continuous function of x' and is strictly positive on any interval interior to the range of the path $X(t)$. But this property is independent of y, and so by Fubini's theorem it holds with probability one unconditionally. Similarly, we may take T to be exponentially distributed and independent of the path, and it follows that with probability one, $\mathfrak{I}(x', t)$ is continuous in x' and strictly positive on the interior of the range of the path $X(t')$, $0 \leqq t' \leqq t$, for almost every t. Finally, $\mathfrak{I}(x', t)$ is an increasing function of t, for each path, and $\mathfrak{I}(x', t + \Delta) = \mathfrak{I}(x', t)$ unless for the corresponding path $X(t') = x'$ for some t', $t \leqq t' \leqq t + \Delta$. It follows that with probability one, $\mathfrak{I}(x', t)$ is jointly continuous in (x', t).

We also have the precise Lipschitz condition for Brownian motion [5]: With probability one,

$$\limsup_{\Delta\to 0} \frac{|\mathcal{S}(\xi + \Delta) - \mathcal{S}(\xi)|}{\sqrt{|\Delta| \log|\log|\Delta||}} = 2, \tag{4.5}$$

$$\lim_{\Delta\to 0} \sup_{|\xi-\xi'|<\Delta} \frac{|\mathcal{S}(\xi') - \mathcal{S}(\xi)|}{\sqrt{|\xi - \xi'| |\log|\xi - \xi'||}} = 2, \tag{4.6}$$

$$\limsup_{\xi\to\infty} \frac{\mathcal{S}(\xi)}{\xi \log\log \xi} = 2. \tag{4.7}$$

Note that (4.5) and (4.6) are invariant under the change of variable relating $\mathcal{S}$ and R_4^* in the range $\xi \geqq \xi^*$.

Under the change of variable (4.1), (4.5) and (4.6) become the first two of the corresponding statements given in the introduction, with $t = T$ and with the result holding conditional on the value of $X(T)$. But exactly as in the preceding case, we may remove the conditions; the Lipschitz conditions given in the introduction hold with probability one, simultaneously for each $t \geqq 0$.

The third and fourth statements in the introduction come from (4.5) when $\mathfrak{I}(x', T) = 0$; that is, when x' is an endpoint of the range of the path $X(t')$, $0 \leqq t' \leqq T$. If the endpoint is not a trap or an entrance boundary, then the preceding reasoning applies without further difficulty.

Finally, if x' is a boundary point and $\xi = h_+(x')/h_-(x')$ is zero or infinite, then (4.5), or respectively (4.7), becomes

$$\limsup_{x''\to x'} \frac{\mathfrak{I}(x'', T)}{h_+(x'')h_-(x'') \log|\log h_+(x'')/h_-(x'')|} = 1 \tag{4.8}$$

under (4.1). If x' is a trap, we may take T to be the passage time of x', and $h_+(x'')h_-(x'') = |x' - x''|$, $h_+(x'')/h_-(x'') = |x' - x''|$ or $|x' - x''|^{-1}$. Also the condition that x' is in the range of the path up to time t is the same as the condition that $t \geqq T$, or alternatively $\mathfrak{I}(x'', T) = \mathfrak{I}(x'', t)$. Hence in this case (4.8) translates exactly into the third statement of the introduction. If x' is an entrance boundary, then x' is infinite, and the diffusion must start at x' if x' is in the range of the path. If x' is, say, $+\infty$, take T to be the passage time of a finite point a. Then $h_-(x'') = 1$, $h_+(x'') = x - a$.

Since a may be arbitrary and since the result does not depend on a or $P_{-}(a) = T$, the previous reasoning implies the result as stated in the introduction.

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