

REPRESENTATION OF MOMENTS BY INTEGRALS¹

BY
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1. Introduction

The purpose of this note is to indicate how the moment-problem technique, particularly as presented in the book of Shohat and Tamarkin [4], gives integral representations for linear functionals on linear spaces of continuous functions, even in cases where the functions are unbounded.

Although Daniell's procedure is not confined to bounded functions, still, boundedness is usually assumed when the procedure is applied to continuous-function spaces, in order to help establish Daniell's limit condition [3, 16A, p. 43].

In our theorem (2.1 below), we do not assume compactness for the space on which the functions are defined, nor boundedness of the functions. The main application (3.3 below) is to give a shorter proof of a known generalization of Jensen's formula.

2. The main theorem

Let X be any topological space. Let $\mathcal{C}(X, \mathbf{R})$ be the space of continuous real-valued functions on X . For $f, g \in \mathcal{C}(X, \mathbf{R})$ we say f is *little o of g* ($f \in o(g)$) if for every $\varepsilon > 0$ there is a compact set K in X such that $|f(x)| < \varepsilon g(x)$ for all x outside K . If P is any subset of $\mathcal{C}(X, \mathbf{R})$, we denote by $o(P)$ the union of all $o(g)$, $g \in P$. It is not hard to see that $o(P)$ is a vector lattice [3, p. 29] whenever P is a linear subspace.

2.1 THEOREM. *Let X be a topological space, and let P be a linear subspace of $\mathcal{C}(X, \mathbf{R})$ such that for each x_0 in X there is a nonnegative p in P such that $p(x_0) = 1$. Let L be a nonnegative linear functional on P . Then there exists a regular Baire measure m on X such that $m \geq 0$ and*

$$2.2 \quad L(f) = \int_X f(x) m(dx)$$

for every f in $P \cap o(P)$. Every function in P is measurable by m .

Proof. Let $Q = P + o(P)$. Then Q is a linear subspace of $\mathcal{C}(X, \mathbf{R})$. We wish to extend L linearly and nonnegatively from P to Q . Suppose L_1 is a nonnegative linear extension of L to some subspace Q_1 , $P \subset Q_1 \subset Q$. Let $f \in Q_1$, and define

$$L_{1*}(f) = \sup_{g \leq f, g \in Q_1} L_1(g), \quad L_1^*(f) = \inf_{f \leq h, h \in Q_1} L_1(h).$$

Received June 14, 1962.

¹ This work was supported in part by the National Science Foundation.

Then $L_{1*}(f) \leq L_1^*(f)$. There is a compact set K and a $p \in P$ such that $|f| < p$ outside K . Another nonnegative q in P can be found such that $|f| < p + q$ on X . From this we see that

$$L_1^*(f) \leq L(p) + L(q) < \infty, \quad \text{and} \quad L_{1*}(f) = -L_1^*(-f) > -\infty.$$

It is not hard to see that if we choose λ such that

$$L_{1*}(f) \leq \lambda \leq L_1^*(f)$$

and define $L_2(\mu f + g) = \mu\lambda + L_1(g)$, we obtain an extension of L_1 to a larger subspace. Thus the extension L_Q to Q is possible. It is unique (obviously) if and only if

$$2.3 \quad L_*(f) = L^*(f) \quad (= \inf_{f \leq h, h \in P} L(h))$$

for every f in Q .

We form the restriction L_0 of L_Q to $o(P)$. Now $o(P)$ is a vector lattice. Let us show that $f_n \downarrow 0 \Rightarrow L_0(f_n) \downarrow 0$ [3, p. 29, (4)]. Let $f_1 \in o(g)$, $g \in P$. We may assume that $g \geq 0$. Let ε_0 be given. Let $\varepsilon = \varepsilon_0/(1 + 2L_0(g))$. Find a compact set K such that $f_1(x) < \varepsilon g(x)$ for all $x \notin K$. Find a nonnegative $p \in P$ such that $p \geq 1$ on K . Let $\delta = \varepsilon_0/(1 + 2L_0(p))$. There is an N such that $n \geq N \Rightarrow f_n(x) < \delta p(x)$ for all $x \in K$. Then for $n \geq N$ we have $f_n(x) < \delta p(x) + \varepsilon g(x)$ for all x . Hence $L(f_n) \leq \delta L(p) + \varepsilon L(g) < \varepsilon_0$.

Obviously, the condition of Stone [3, 13B, p. 34] is also satisfied. The only one of Loomis's conditions that we cannot meet is that the functions of the vector lattice [3, 12A] be bounded. (Indeed, the functions in $o(P)$ need not be bounded.) Daniell himself imposed no such condition, nor is it used in [3] prior to [3, p. 43], and our references are at most to [3, p. 36]. Nevertheless we refer to [3] rather than to Daniell because Loomis discusses the important consequences of Stone's condition. (The very object of the present paper is to see what happens when unbounded functions actually are allowed in the domain of the functional I of [3, 16A, p. 44].) We may therefore apply the theorems of [3, p. 35], with our $o(P)$ as his L , and our L_0 as his I , and conclude that there is a regular Baire measure m on X such that for each nonnegative f in the monotonic closure of $o(P)$, we have

$$2.4 \quad L_0(f) = \int f(x) m(dx).$$

In particular, 2.4 holds for every f in $o(P)$. If in addition $f \in P$, then $L_0(f) = L(f)$. Thus 2.1 is proved.

The formula of Herglotz is one consequence of 2.1. It involves the Poisson kernel

$$P_{\zeta}(u) = \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} \quad (u = e^{i\theta}, \zeta = re^{it}).$$

2.5 THEOREM (Herglotz). *Let h be harmonic and nonnegative on the disc $\{|z| < 1\}$. Then there is a regular Borel measure m on $\{|z| = 1\}$ such that*

$$h(\zeta) = \int_{|u|=1} P_{\zeta}(u) m(du) \quad (|\zeta| < 1).$$

Proof. Let E be the linear hull of the P_{ζ} , regarded as functions on the unit circle (not the disc) X . Then $E \subset o(E)$. Define L linearly on E by setting

$$2.5.1 \quad L\left(\sum_1^n \lambda_i P_{\zeta_i}\right) = \sum_1^n \lambda_i h(\zeta_i).$$

The only chore is to show that this is nonnegative if $\sum \lambda_i P_{\zeta_i} > 0$. One does this by observing that

$$2\pi h(\zeta_i) = \int_0^{2\pi} P_{\zeta_i/r}(e^{i\theta}) h(re^{i\theta}) d\theta$$

for $|\zeta_i| < r < 1$. The reader will be able to fill in the details, and complete the argument.

Henceforth we shall usually abbreviate

$$2.5.2 \quad \int_X f(x) m(dx) \quad \text{by} \quad m(f).$$

We did not do so in 2.5 for fear of giving the classical formula a strange appearance.

Another consequence, for *algebras* of functions, is the following (cf. [1, 4.4]).

2.6 THEOREM. *Let X be a locally compact space, and A , a subalgebra of $\mathcal{C}_0(X, \mathbf{R})$ (the continuous functions that vanish at ∞). Suppose each element of A is the difference of nonnegative elements. Let L be a nonnegative linear functional on A . Then there exists a regular Baire measure m on X such that for every pair f, g in A*

$$2.6.1 \quad m(fg) = L(fg)$$

and for $f \geq 0, f$ in A ,

$$2.6.2 \quad m(f) = \sup_{0 \leq u \leq 1, u \in A} L(uf).$$

Proof. Delete from X , if necessary, all common zeros of the functions in A , and call the residue Y . Then Y, A, L satisfy the conditions of 2.1, and we obtain our m . The concept $o(A)$ is connected with Y , not X ; but it is clear that every product fg is the sum of four functions in $o(A)$, so that 2.6.1 holds. In particular for f and u as in 2.6.2, we have $L(uf) = m(uf) \leq m(f)$, so that $m(f) \geq \sup m(uf) = \sup L(uf)$. On the other hand, making use of the boundedness of f one can easily prove (see [1, 4.7]) that $m(f) \leq \sup m(uf)$. This completes the proof of 2.6. Incidentally, it shows that

(hypotheses as in 2.6)

2.6.3. *there exists an m such that $m(f) = L(f)$ for every f in A if and only if*

$$L(f) = \sup_{0 \leq u \leq 1, u \in A} L(uf)$$

for every nonnegative f in A .

The “classical” moment problems deal with linear functionals L defined in the algebra A of polynomials, regarded as functions on a closed, but by no means bounded, subset X of \mathbb{R}^n . The presence of unbounded functions such as $F = x_1^2 + \cdots + x_n^2$ in the domain of L is exploited as follows (cf. [4, p. 1]).

2.7 THEOREM. *Let X be locally compact. Let A be a subalgebra of $\mathcal{C}(X, \mathbb{R})$ (all continuous real functions on X). Let A contain 1 and also at least one function F such that the sets*

$$\{|F| \leq n\}, \quad n = 1, 2, \dots,$$

are compact. Let L be a nonnegative linear functional on A . Then there is a regular Baire measure m on X such that

$$m(f) = L(f)$$

for every f in A .

Proof. Let $f \in A$. Form $g = f^2 + F^2$. It is not hard to see that $g \in o(1 + g^2)$. Hence $L(g) = m(g)$, which is to say,

$$L(f^2 + F^2) = m(f^2 + F^2).$$

Let $f = 0$, subtract, and we have $L(f^2) = m(f^2)$. In particular $L(1) = m(1)$. Combining these with $L((1 + f)^2) = m((1 + f)^2)$ yields $L(f) = m(f)$.

It would certainly be absurd to say that this is the essence of the classical moment problem (see [4]). There, one is more concerned with the question of whether the L is really nonnegative. Usually the L is defined only on some basis for P or A , and the elegance of the solution in each case depends on how easily one can tell whether such an assignment of “moments” leads to a nonnegative linear extension.

3. The value of $m(\phi)$ for $\phi \in o(P)$.

Consider the example in which $X = (0, 1]$, $P =$ all polynomials, regarded as functions on X , $L(p) = p(0)$. Then 2.2 provides an m , but it is 0. Hence $1 = L(1) \neq m(1) = 0$. The connection is re-established by observing that

$$m(1) = \sup_{p \in o(1), p \leq 1} L(p).$$

A similar example is obtained by taking X as before, and $P =$ those polynomials which vanish at 0, while $L(p) = p'(0)$. If x is the cartesian co-ordinate, then $1 = L(x) \neq m(x) = 0$, since $m = 0$.

The general situation is similar, as we now show.

3.1 THEOREM. *Let X be a topological space. Let E be a linear subspace of $\mathcal{C}(X, \mathbf{R})$ containing at least one positive function u . Let $\phi \in \mathcal{C}(X, \mathbf{R})$, and suppose $E \subset o(\phi)$. Let J be a nonnegative linear functional defined on E . Then a regular Baire measure m exists on X such that $m = J$ on E , and (see 2.5.2)*

$$3.2 \quad m(\phi) = \sup_{f \in E, f \leq \phi} J(f).$$

Proof. Since $u \in o(\phi)$, there is an N such that $\phi + Nu \geq 0$. Let

$$j = \sup \{J(f) : f \leq \phi, f \in E\},$$

and suppose that $j < \infty$. Let E_0 be the linear hull of ϕ and E . We can surely define on E_0 a J_0 such that $J_0(\phi) = j$, such that J_0 is a nonnegative linear extension of J . Now let $\phi_n = \phi \wedge nu$ ($n = 1, 2, \dots$), and let E_n be the linear hull of $E_0, \phi_1, \dots, \phi_n$. Let J_n be a nonnegative linear extension of J_0 to E_n . Then

$$-NJ(u) \leq \sup_{f \leq \phi_{n+1}, f \in E_n} J_n(f) \leq \inf_{\phi_{n+1} \leq f \in E_n} J_n(f) \leq j < \infty.$$

Therefore we can extend J_n to a J_{n+1} on E_{n+1} , with $J_{n+1}(\phi_{n+1}) \leq j$. Ultimately, we have a nonnegative linear L defined on P , the direct sum of the E_n , where $L|E_n = J_n$ ($n = 1, 2, \dots$). To $\{X, P, L\}$ we may apply 2.1 obtaining an m as therein described. Each $f \in E$ is $o(\phi)$ and $\phi \in P$, so $m = J$ on E . Moreover, $\phi_n \in o(\phi)$ whence $m(\phi_n)$ exists and $\leq j$; and $\phi_n \uparrow \phi$, whence $m(\phi) \leq j$. On the other hand, $f \leq \phi$ ($f \in E$) implies $m(f) \leq m(\phi)$, and $m(f) = J(f)$. Therefore $j \leq m(\phi)$, and 3.1 is proved.

3.3 THEOREM (cf. [2, 7.1]). *Let A be a commutative Banach algebra with unit. Let S be the space of maximal ideals. Let B be the Shilov boundary. Regard A as an algebra of functions on S . Let s be any point of S . Let a_0 be a particular element of A . Then there exists a regular Baire probability-measure m_s on B such that*

$$3.4 \quad m_s(a) = a(s) \quad (\text{all } a \in A)$$

and

$$3.5 \quad m_s(\log |a|) = \log |a(s)| \quad (\text{all } a \in A^{-1}),$$

where A^{-1} is the set of invertible elements of A , and

$$3.6 \quad m_s(\log |a_0|) \geq \log |a_0(s)|.$$

Proof. Let E be the linear hull of $\log |A^{-1}|$ (the set of logarithms of the moduli of elements of A^{-1}). For $f = \log |a|$, $a \in A^{-1}$, define $J(f) = \log |a(s)|$. J has a nonnegative linear extension to E [2, 5.1] which we also denote by J .

Now let X be the subset of B on which $\phi = -\log |a_0| \neq \infty$. Then $E \subset o(\phi)$, and all the hypotheses of 3.1 are fulfilled. The ensuing m is the desired measure. First of all, 3.5 holds. By replacing a in 3.5 by $e^{\lambda a}$, $a \in A$,

we get 3.4, and $m(1) = 1$. Now we consider 3.6. Suppose $f \in E$ and $f \leq \phi$ (on X). Then $\exp(-f) \geq \exp(-\phi) = |a_0|$ on B . Suppose f is of the form $\log |a|$, $a \in A^{-1}$. For such f we have $|a_0| |a| \leq 1$ on B , so $|a_0(s)a(s)| \leq 1$, and $f(s) \leq \phi(s)$, and by a limiting argument [2, 5.1] the same is true for all $f \leq \phi$. Thus

$$\sup_{f \leq \phi} J(f) \leq \phi(s) = -\log |a_0(s)|,$$

whence (3.1)

$$m(-\log |a_0|) \leq -\log |a_0(s)|.$$

This proves 3.3.

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